

EF2452: Mathematics for Economics and Finance

Lecture 2

Part II of II

Department of Economics and Finance
City University of Hong Kong
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Main Topics

- Interest, capital growth and the meaning of present value
- Recurrence equations in annuity problems; present value of annuity
- Cobweb model: deriving the recurrence equation and solution
- Analyzing the stability of the cobweb model; when $q^S(p) = -a + bp$ and $q^D(p) = c - dp$, stability requires $b < d$

Capital Growth

- Suppose we invest principal, P for N years at a constant annual interest rate of r .
- If y_t is the capital at the end of the t th year, we have $y_0 = P$ and the first-order recurrence

$$y_t = (1 + r)y_{t-1}$$

- This is in the standard form, with $a = (1 + r)$ and $b = 0$. We have

$$y_t = (1 + r)y_{t-1} = (1 + r)^2y_{t-2} = (1 + r)^3y_{t-3} = \cdots = (1 + r)^tP$$

- So, the capital C after N years is given by

$$C = y_N = (1 + r)^N P$$

Present Value

- Now, suppose we ask the reverse question: if we want the final capital to be C , what should be the principal, P ?
- We can simply rearrange the previously obtained solution $C = (1 + r)^N P$ to write P as a function of C (inverse function)

$$P(C) = \frac{C}{(1 + r)^N}$$

This is known as calculating the present value of a capital sum C , due in N years, given a fixed annual interest rate r .

Annuity

- An annuity is a financial product that provides a guaranteed income at regular intervals.
- Suppose we invest P , and withdraw an amount I at the end of each year for N years, at which time the capital is used up.
- What income can be generated from the principal P , or (mathematically speaking) what is I as a function of P ?

$$y_t = (1 + r)y_{t-1} - I, \quad \text{where} \quad y_0 = P \quad \& \quad y_N = 0$$

- The above equation is a first-order recurrence, in standard form with $a = (1 + r)$, $b = -I$.

Annuity

- The time-independent solution is

$$y^* = \frac{b}{1-a} = \frac{-I}{-r} = \frac{I}{r}$$

- The general solution is

$$y_t = y^* + (y_0 - y^*)a^t = \frac{I}{r} + \left(P - \frac{I}{r}\right)(1+r)^t$$

- In order to determine I as a function of P , we must use the condition that nothing is left after N years, that is, $y_N = 0$. This condition is

$$y_N = \frac{I}{r} + \left(P - \frac{I}{r}\right)(1+r)^N = 0$$

Annuity

- Rearranging, we get

$$\frac{I}{r} \left((1+r)^N - 1 \right) = P(1+r)^N$$

$$\implies I(P) = \left(\frac{r(1+r)^N}{(1+r)^N - 1} \right) P$$

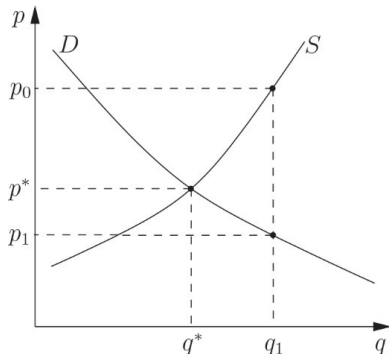
- One can also ask: what principal P is required to provide an annual income I for the next N years? We can write the inverse function $P(I)$ by rearranging the above equation

$$P(I) = \frac{I}{r} \left(1 - \frac{1}{(1+r)^N} \right)$$

This gives us the present value of an annuity generating I each year guaranteed for the next N years.

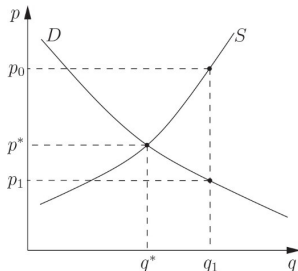
Cobweb Model

- In this model, there is a time lag in the actions of the suppliers.
- You can think of the good as an agricultural product for which there is a yearly 'crop'. The supply and demand sets are as shown in the figure:



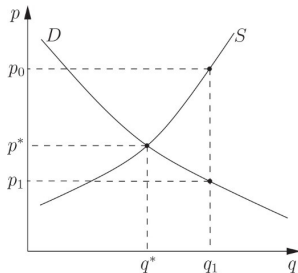
Cobweb Model

- If there are no disturbances, the equilibrium price is p^* , and the equilibrium quantity is q^* .
- Suppose that one year, for some external reason such as drought, there is a shortage in the market so that the quantity traded falls and the price rises to p_0 .



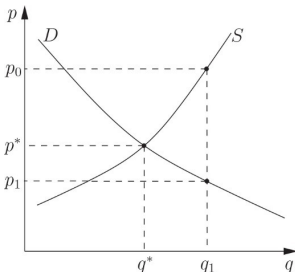
Cobweb Model

- The farmers plan their production for next year on the basis of this higher price, and so an increased quantity appears on the market in the next year; specifically, $q_1 = q^S(p_0)$.
- TODAY'S PRICE DETERMINES TOMORROW'S QUANTITY SUPPLIED/PRODUCED BY SELLERS



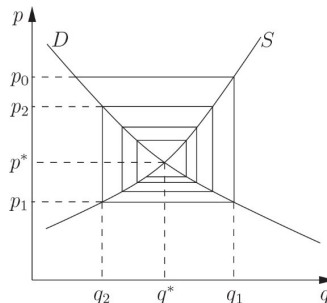
Cobweb Model

- Because the quantity is greater, the price consumers pay falls to the value $p_1 = P^D(q_1)$. In other words, the market has excess supply, and prices fall.
- TODAY'S QUANTITY DETERMINES TODAY'S PRICE (USING THE INVERSE DEMAND FUNCTION: MAXIMUM PRICE CONSUMERS ARE WILLING TO PAY AT THE GREATER QUANTITY)



Cobweb Model

- The process is repeated again in the following year: this time, the lower price p_1 leads to a decrease in the production q_2 , and that, in turn, means a higher price p_2 due to lower production (or shortage). And so on.
- The sequences $p_0, p_1, p_2, \dots$ and $q_1, q_2, \dots$ can be plotted as follows (looks like a 'cobweb' of a spider):



Cobweb Model: The General Linear Case

- Consider the following demand and supply functions (with $a, b, c, d > 0$)

$$q^S(p) = -a + bp \qquad p^D(q) = \frac{c - q}{d}$$

Recall that, in a cobweb model, p_{t-1} determines q^S , which in turn, gives the quantity traded in the market. Therefore,

$$q_t = -a + bp_{t-1}$$

- Also, recall that, this q_t determines p_t from the demand function ($p^D(q)$):

$$p_t = \frac{c - q_t}{d} = \frac{c - (-a + bp_{t-1})}{d} = \frac{bp_{t-1} + c + a}{d}$$

Cobweb Model: The General Linear Case

$$\implies p_t = \left(-\frac{b}{d}\right)p_{t-1} + \left(\frac{c+a}{d}\right)$$

- This is a first-order recurrence, and the time-independent solution is

$$p^* = \left(-\frac{b}{d}\right)p^* + \left(\frac{c+a}{d}\right)$$

$$(b+d)p^* = c+a \implies p^* = \frac{c+a}{b+d}$$

This is the equilibrium value of the price.

Cobweb Model: The General Linear Case

- The general solution is:

$$p_t = p^* + (p_0 - p^*) \left(-\frac{b}{d} \right)^t$$

- Stability requires $p_t \rightarrow p^*$ as $t \rightarrow \infty$. Given that b and d are both positive, $p_t \rightarrow p^*$ if $b < d$. Because then $-1 < -\frac{b}{d} < 0$, and

$$\left(-\frac{b}{d} \right)^t \rightarrow 0$$

- Economic Interpretation: note that $1/b$ is the slope of the supply line, and $1/d$ is the slope of the demand line. So, $b < d$ means the supply is steeper than the demand. [p on the y-axis and q on the x-axis]

References

Following chapter of the textbook:

4 The Elements of Finance

5 The Cobweb Model