

## MA1201 Calculus and Basic Linear Algebra II

## Solution of Problem Set 4 Vector Algebra

## Problem 1

- (a) The position vectors of
- $A$
- ,
- $B$
- and
- $C$
- are given by

$$\overrightarrow{OA} = \vec{i} + \vec{j}, \quad \overrightarrow{OB} = 2\vec{j} + 3\vec{k} \quad \text{and} \quad \overrightarrow{OC} = 2\vec{i} - \vec{j}$$

$$(b) \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\vec{j} + 3\vec{k}) - (\vec{i} + \vec{j}) = -\vec{i} + \vec{j} + 3\vec{k}.$$

$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = (\vec{i} + \vec{j}) - (2\vec{i} - \vec{j}) = -\vec{i} + 2\vec{j}.$$

- (c) Note that

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (2\vec{i} - \vec{j}) - (2\vec{j} + 3\vec{k}) = 2\vec{i} - 3\vec{j} - 3\vec{k}.$$

Since the  $\vec{i}$ -component of  $\overrightarrow{AB}$  and  $\overrightarrow{BC}$  are not the same, thus  $\overrightarrow{AB} \neq \overrightarrow{BC}$ .

$$(d) \quad \widehat{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{-\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{(-1)^2 + 1^2 + 3^2}} = -\frac{1}{\sqrt{11}}\vec{i} + \frac{1}{\sqrt{11}}\vec{j} + \frac{3}{\sqrt{11}}\vec{k}$$

$$\widehat{BC} = \frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} = \frac{2\vec{i} - 3\vec{j} - 3\vec{k}}{\sqrt{(2)^2 + (-3)^2 + (-3)^2}} = \frac{2}{\sqrt{22}}\vec{i} - \frac{3}{\sqrt{22}}\vec{j} - \frac{3}{\sqrt{22}}\vec{k}$$

$$(e) \quad (i) \quad \vec{a} = 3 \times (-\widehat{AB}) = \frac{3}{\sqrt{11}}\vec{i} - \frac{3}{\sqrt{11}}\vec{j} - \frac{9}{\sqrt{11}}\vec{k}$$

$$(ii) \quad \vec{b} = 5 \times (\widehat{BC}) = \frac{10}{\sqrt{22}}\vec{i} - \frac{15}{\sqrt{22}}\vec{j} - \frac{15}{\sqrt{22}}\vec{k}.$$

## Problem 2

$$(a) \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (\vec{i} + 2\vec{j}) - (\vec{j} - \vec{k}) = \vec{i} + \vec{j} + \vec{k}.$$

$$\overrightarrow{AX} = \underbrace{\frac{2}{3}|\overrightarrow{AB}|}_{\text{magnitude}} \times \underbrace{(\widehat{AB})}_{\text{direction}} = \frac{2}{3}|\overrightarrow{AB}| \times \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{2}{3}(\vec{i} + \vec{j} + \vec{k}) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

- (b) Using the fact that
- $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$
- , we have

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = (\vec{j} - \vec{k}) + \left(\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right) = \frac{2}{3}\vec{i} + \frac{5}{3}\vec{j} - \frac{1}{3}\vec{k}.$$

## Problem 3

$$(a) \quad |\vec{a}| = \sqrt{2^2 + (-3)^2 + 5^2} = \sqrt{38}$$

$$|\vec{a} - 2\vec{b}| = |-9\vec{j} + 5\vec{k}| = \sqrt{(-9)^2 + 5^2} = \sqrt{106}.$$

$$(b) \quad \hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{i} + 3\vec{j}}{\sqrt{1^2 + 3^2}} = \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}.$$

- (c) Note that

$$|2\vec{a} + \vec{b}| = |5\vec{i} - 3\vec{j} + 10\vec{k}| = \sqrt{5^2 + (-3)^2 + 10^2} = \sqrt{134}.$$

Thus the vector  $\vec{c}$  is given by

$$\vec{c} = \underbrace{|2\vec{a} + \vec{b}|}_{\text{magnitude}} \times \underbrace{\hat{b}}_{\text{direction}} = \sqrt{134} \left( \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j} \right) = \frac{\sqrt{134}}{\sqrt{10}}\vec{i} + \frac{3\sqrt{134}}{\sqrt{10}}\vec{j}.$$

**Problem 4**

(a)  $\vec{a} \cdot \vec{b} = 1(-2) + 3(1) - 2(3) = -5.$

(b) Let  $\theta$  be the angle between the vectors  $\vec{a}$  and  $\vec{b}$ . Then we have

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{-5}{\sqrt{1^2 + 3^2 + (-2)^2} \sqrt{(-2)^2 + 1^2 + 3^2}} = \frac{-5}{\sqrt{14}\sqrt{14}} \Rightarrow \theta \approx 110.92^\circ.$$

(c)  $\vec{c}$  and  $\vec{b}$  are perpendicular, then  $\vec{c} \cdot \vec{b} = |\vec{c}||\vec{b}| \cos 90^\circ = 0$

$$\Rightarrow (3\vec{i} + x\vec{j} - 2\vec{k}) \cdot (-2\vec{i} + \vec{j} + 3\vec{k}) = 0$$

$$\Rightarrow 3(-2) + x(1) + (-2)(3) = 0 \Rightarrow x = 12.$$

(d)  $\vec{d} = y\vec{a} + 3\vec{b}$  and  $\vec{a} - \vec{b}$  are perpendicular, then  $\vec{d} \cdot (\vec{a} - \vec{b}) = |\vec{d}||\vec{a} - \vec{b}| \cos 90^\circ = 0$

$$\Rightarrow (y\vec{a} + 3\vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$\Rightarrow y(\vec{a} \cdot \vec{a}) + (3 - y)(\vec{a} \cdot \vec{b}) - 3(\vec{b} \cdot \vec{b}) = 0$$

$$\Rightarrow y|\vec{a}|^2 + (3 - y)(\vec{a} \cdot \vec{b}) - 3|\vec{b}|^2 = 0$$

$$\Rightarrow y(\sqrt{14})^2 + (3 - y)(-5) - 3(\sqrt{14})^2 = 0$$

$$\Rightarrow y = 3.$$

**Problem 5**

(a) Let  $\theta = \angle ABC$ , then  $\theta$  satisfies

$$\cos \theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} \dots \dots (*)$$

Since  $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \vec{i} - 2\vec{k}$ ,  $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 2\vec{i} - 2\vec{k}$  and  $\overrightarrow{BA} \cdot \overrightarrow{BC} = (1)(2) + (0)(0) + (-2)(-2) = 6$ , then we can deduce from (\*) that

$$\cos \theta = \frac{6}{\sqrt{(1)^2 + (-2)^2} \sqrt{(2)^2 + (-2)^2}} = \frac{6}{\sqrt{5}\sqrt{8}} \Rightarrow \theta \approx 18.43^\circ.$$

(b) Given that  $DE$  is perpendicular to  $EF$ , we then have  $\overrightarrow{ED} \cdot \overrightarrow{EF} = |\overrightarrow{ED}||\overrightarrow{EF}| \cos 90^\circ = 0.$

Since  $\begin{cases} \overrightarrow{ED} = \overrightarrow{OD} - \overrightarrow{OE} = (x-1)\vec{i} - \vec{j} \\ \overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = 3\vec{i} - 6\vec{j} - 2\vec{k} \end{cases}$  then we have

$$\overrightarrow{ED} \cdot \overrightarrow{EF} = 0 \Rightarrow (x-1)(3) + (-1)(-6) + (0)(-2) = 0 \Rightarrow x = -1.$$

**Problem 6**

(a) Let  $\theta = \angle ABC$ ,  $\theta$  is found to be

$$\cos \theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} \dots \dots (*)$$

Note that  $\begin{cases} \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = 3\vec{i} + \vec{j} \\ \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{i} + 2\vec{j} \end{cases}$  then we obtain from (\*) that

$$\cos \theta = \frac{3(1) + (1)(2)}{\sqrt{3^2 + (1)^2} \sqrt{1^2 + 2^2}} = \frac{5}{\sqrt{10}\sqrt{5}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ.$$

(b) We need to check whether  $\angle ABD = \frac{1}{2}\angle ABC.$

Let  $\phi = \angle ABD$ , then we have

$$\cos \phi = \frac{\overrightarrow{BA} \cdot \overrightarrow{BD}}{|\overrightarrow{BA}| |\overrightarrow{BD}|} \dots \dots (**)$$

Since  $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = 2\vec{i} + 2\vec{j}$ , then we can obtain from (\*\*) that

$$\cos \phi = \frac{3(2) + (1)(2)}{\sqrt{10}\sqrt{8}} = \frac{8}{\sqrt{80}} = \frac{2}{\sqrt{5}} \Rightarrow \phi \approx 26.57^\circ.$$

It is obvious that  $\angle ABD \neq \frac{1}{2}\angle ABC$ . Therefore  $BD$  is not an angle bisector of  $\angle ABC$ .

### Problem 7

#### (Method 1)

Let  $\theta = \angle BAC$ , then

$$\cos \theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \frac{2}{(4)(4)} = \frac{1}{8}.$$

Using cosine law, we have

$$BC^2 = AB^2 + AC^2 - 2(AB)(AC) \cos \theta = 4^2 + 4^2 - 2(4)(4) \left(\frac{1}{8}\right) = 28 \Rightarrow BC = \sqrt{28}.$$

#### (Method 2)

Using the fact that  $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{AC} - \overrightarrow{AB}$  and  $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$ , we get

$$\begin{aligned} BC &= |\overrightarrow{BC}| = \sqrt{\overrightarrow{BC} \cdot \overrightarrow{BC}} = \sqrt{(\overrightarrow{AC} - \overrightarrow{AB}) \cdot (\overrightarrow{AC} - \overrightarrow{AB})} \\ &= \sqrt{(\overrightarrow{AC} \cdot \overrightarrow{AC}) - 2(\overrightarrow{AC} \cdot \overrightarrow{AB}) + (\overrightarrow{AB} \cdot \overrightarrow{AB})} = \sqrt{|\overrightarrow{AC}|^2 - 2(\overrightarrow{AB} \cdot \overrightarrow{AC}) + |\overrightarrow{AB}|^2} = \sqrt{4^2 - 2(2) + 4^2} = \sqrt{28}. \end{aligned}$$

### Problem 8

(a) Let  $\theta$  be the angle between  $\vec{a}$  and  $\vec{b}$ , then we have

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{1}{1(2)} = \frac{1}{2} \Rightarrow \theta = 30^\circ.$$

(b) Using properties of scalar product, we get

$$\begin{aligned} (3\vec{a} - 2\vec{b}) \cdot (\vec{a} + 3\vec{b}) &= 3(\vec{a} \cdot \vec{a}) + 7(\vec{a} \cdot \vec{b}) - 6(\vec{b} \cdot \vec{b}) = 3|\vec{a}|^2 + 7(\vec{a} \cdot \vec{b}) - 6|\vec{b}|^2 \\ &= 3(1)^2 + 7(1) - 6(2)^2 = -14. \\ |\vec{a} - 2\vec{b}| &= \sqrt{(\vec{a} - 2\vec{b}) \cdot (\vec{a} - 2\vec{b})} = \sqrt{(\vec{a} \cdot \vec{a}) - 4(\vec{a} \cdot \vec{b}) + 4(\vec{b} \cdot \vec{b})} = \sqrt{|\vec{a}|^2 - 4(\vec{a} \cdot \vec{b}) + 4|\vec{b}|^2} \\ &= \sqrt{1^2 - 4(1) + 4(2)^2} = \sqrt{13}. \end{aligned}$$

(c) Let  $\phi$  be the angle between the vectors  $\vec{a} - 2\vec{b}$  and  $2\vec{a} + 3\vec{b}$ . We have

$$\cos \phi = \frac{(\vec{a} - 2\vec{b}) \cdot (2\vec{a} + 3\vec{b})}{|\vec{a} - 2\vec{b}| |2\vec{a} + 3\vec{b}|} \dots \dots (*)$$

Since

$$\begin{aligned} (\vec{a} - 2\vec{b}) \cdot (2\vec{a} + 3\vec{b}) &= 2(\vec{a} \cdot \vec{a}) - (\vec{a} \cdot \vec{b}) - 6(\vec{b} \cdot \vec{b}) = 2|\vec{a}|^2 - (\vec{a} \cdot \vec{b}) - 6|\vec{b}|^2 \\ &= 2(1)^2 - (1) - 6(2)^2 = -23. \end{aligned}$$

and

$$\begin{aligned} |2\vec{a} + 3\vec{b}| &= \sqrt{(2\vec{a} + 3\vec{b}) \cdot (2\vec{a} + 3\vec{b})} = \sqrt{4(\vec{a} \cdot \vec{a}) + 12(\vec{a} \cdot \vec{b}) + 9(\vec{b} \cdot \vec{b})} \\ &= \sqrt{4|\vec{a}|^2 + 12(\vec{a} \cdot \vec{b}) + 9|\vec{b}|^2} = \sqrt{4(1)^2 + 12(1) + 9(2)^2} = \sqrt{52}, \end{aligned}$$

then we can conclude from (\*) that

$$\cos \phi = \frac{-23}{\sqrt{13}\sqrt{52}} = -\frac{23}{26} \Rightarrow \phi \approx 152.2^\circ.$$

**Problem 9**

(a) Let  $\theta$  be the angle between  $\vec{a} - 2\vec{b}$  and  $-9\vec{a} + 2\vec{b}$ , then we have

$$\cos \theta = \frac{(\vec{a} - 2\vec{b}) \cdot (-9\vec{a} + 2\vec{b})}{|\vec{a} - 2\vec{b}||-9\vec{a} + 2\vec{b}|} \dots \dots (*)$$

We first note that

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \left( \cos^{-1} \frac{3}{5} \right) = 2(3) \left( \frac{3}{5} \right) = \frac{18}{5}.$$

Then we have

$$\begin{aligned} (\vec{a} - 2\vec{b}) \cdot (-9\vec{a} + 2\vec{b}) &= -9(\vec{a} \cdot \vec{a}) + 20(\vec{a} \cdot \vec{b}) - 4(\vec{b} \cdot \vec{b}) \\ &= -9|\vec{a}|^2 + 20(\vec{a} \cdot \vec{b}) - 4|\vec{b}|^2 = -9(2)^2 + 20\left(\frac{18}{5}\right) - 4(3)^2 = 0. \end{aligned}$$

Then we obtain from (\*) that

$$\cos \theta = \frac{0}{|\vec{a} - 2\vec{b}||-9\vec{a} + 2\vec{b}|} = 0 \Rightarrow \theta = 90^\circ.$$

This shows that the two vectors are perpendicular to each other.

(b) Let  $\phi$  be the angle between  $\vec{a}$  and  $\vec{a} + k\vec{b}$ , then we have

$$\cos \underset{=60^\circ}{\phi} = \frac{\vec{a} \cdot (\vec{a} + k\vec{b})}{|\vec{a}||\vec{a} + k\vec{b}|} \Rightarrow \frac{\vec{a} \cdot (\vec{a} + k\vec{b})}{|\vec{a}||\vec{a} + k\vec{b}|} = \frac{1}{2} \dots \dots (**).$$

Note that

$$\vec{a} \cdot (\vec{a} + k\vec{b}) = |\vec{a}|^2 + k(\vec{a} \cdot \vec{b}) = 4 + \frac{18}{5}k,$$

$$|\vec{a} + k\vec{b}| = \sqrt{(\vec{a} + k\vec{b}) \cdot (\vec{a} + k\vec{b})} = \sqrt{|\vec{a}|^2 + 2k(\vec{a} \cdot \vec{b}) + k^2|\vec{b}|^2} = \sqrt{4 + \frac{36}{5}k + 9k^2},$$

then we obtain from (\*\*) that

$$\begin{aligned} \frac{4 + \frac{18}{5}k}{2\sqrt{4 + \frac{36}{5}k + 9k^2}} &= \frac{1}{2} \Rightarrow \dots \Rightarrow 99k^2 + 540k + 300 = 0 \\ \Rightarrow k &= -\frac{40\sqrt{3}}{33} - \frac{30}{11} \text{ or } k = \frac{40\sqrt{3}}{33} - \frac{30}{11} \end{aligned}$$

**Problem 10**

In the following,  $\theta$  refers to the angle between the vectors  $\vec{a}$  and  $\vec{b}$

(a) Note that

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{3(1) + (-4)(-18)}{5\sqrt{325}} = \frac{3}{\sqrt{13}} \Rightarrow \theta < 90^\circ.$$

Then the required projection vector is given by

$$proj_{\vec{b}} \vec{a} = \underbrace{(|\vec{a}| \cos \theta)}_{\text{magnitude}} \times \underbrace{(\hat{b})}_{\text{direction}} \stackrel{\hat{b} = \frac{\vec{b}}{|\vec{b}|}}{=} 5 \left( \frac{3}{\sqrt{13}} \right) \times \frac{\vec{i} - 18\vec{j}}{\sqrt{325}} = \frac{3}{13}\vec{i} - \frac{54}{13}\vec{j}.$$

(b) Note that

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{2(6) + (-3)(-2) + (-6)(11)}{\sqrt{49}\sqrt{161}} = \frac{-48}{7\sqrt{161}} \Rightarrow \theta > 90^\circ.$$

Then the required projection vector is given by

$$\begin{aligned} \text{proj}_{\vec{b}} \vec{a} &= \underbrace{(|\vec{a}| \cos(180^\circ - \theta))}_{\text{magnitude}} \times \underbrace{\left(\frac{\vec{b}}{|\vec{b}|}\right)}_{\text{direction}} \stackrel{\vec{b}=|\vec{b}|}{=} -|\vec{a}| \cos \theta \times \left(-\frac{6\vec{i} - 2\vec{j} + 11\vec{k}}{\sqrt{161}}\right) \\ &= -7 \left(\frac{-48}{7\sqrt{161}}\right) \times \frac{-6\vec{i} + 2\vec{j} - 11\vec{k}}{\sqrt{161}} = -\frac{288}{161}\vec{i} + \frac{96}{161}\vec{j} - \frac{528}{161}\vec{k}. \end{aligned}$$

(c) Note that

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(-1)(1) + (2)(7) + (2)(7)}{3\sqrt{99}} = \frac{9}{\sqrt{99}} \Rightarrow \theta < 90^\circ.$$

Then the required projection vector is given by

$$\text{proj}_{\vec{b}} \vec{a} = \underbrace{(|\vec{a}| \cos \theta)}_{\text{magnitude}} \times \underbrace{\left(\frac{\vec{b}}{|\vec{b}|}\right)}_{\text{direction}} \stackrel{\vec{b}=|\vec{b}|}{=} 3 \left(\frac{9}{\sqrt{99}}\right) \times \frac{\vec{i} + 7\vec{j} + 7\vec{k}}{\sqrt{99}} = \frac{3}{11}\vec{i} + \frac{21}{11}\vec{j} + \frac{21}{11}\vec{k}.$$

### Problem 11

(a) Using the same method as in Example 10, we need to find  $|\overrightarrow{AC}|$  and  $|\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC}|$ .

First, we note that

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -\vec{i}, \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{i} + \vec{j} + 3\vec{k}$$

It remains to find the magnitude of  $\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC}$ .

Let  $\theta$  be the angle between  $\overrightarrow{AC}$  and  $\overrightarrow{AB}$

$$\cos \theta = \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{|\overrightarrow{AC}||\overrightarrow{AB}|} = \frac{(-1)(-2) + 0(1) + 0(3)}{(1)\sqrt{14}} = \frac{2}{\sqrt{14}} \Rightarrow \theta < 90^\circ.$$

Thus the magnitude of  $\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC}$  is given by

$$|\overrightarrow{AC}| \cos \theta = \sqrt{1} \left(\frac{2}{\sqrt{14}}\right) = \frac{2}{\sqrt{14}}.$$

Therefore the required distance  $d$  is given by

$$d = \sqrt{|\overrightarrow{AC}|^2 - |\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC}|^2} = \sqrt{(1)^2 - \left(\frac{2}{\sqrt{14}}\right)^2} = \sqrt{\frac{10}{14}}.$$

(b) Using the same method as in Example 10, we need to find  $|\overrightarrow{DF}|$  and  $|\text{proj}_{\overrightarrow{DE}} \overrightarrow{DF}|$ .

First, we note that

$$\overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD} = -\vec{i} + 4\vec{j} - 2\vec{k}, \quad \overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = -2\vec{i} + \vec{j}$$

It remains to find the magnitude of  $\text{proj}_{\overrightarrow{DE}} \overrightarrow{DF}$ .

Let  $\phi$  be the angle between  $\overrightarrow{DF}$  and  $\overrightarrow{DE}$

$$\cos \phi = \frac{\overrightarrow{DF} \cdot \overrightarrow{DE}}{|\overrightarrow{DF}||\overrightarrow{DE}|} = \frac{(-1)(-2) + 4(1) + (-2)(0)}{\sqrt{21}\sqrt{5}} = \frac{6}{\sqrt{105}} \Rightarrow \phi < 90^\circ.$$

Thus the magnitude of  $\text{proj}_{\overrightarrow{DE}} \overrightarrow{DF}$  is given by

$$|\overrightarrow{DF}| \cos \phi = \sqrt{21} \left(\frac{6}{\sqrt{105}}\right) = \frac{6}{\sqrt{5}}.$$

Therefore the required distance  $d$  is given by

$$d = \sqrt{|\overrightarrow{DF}|^2 - |\text{proj}_{\overrightarrow{DE}} \overrightarrow{DF}|^2} = \sqrt{(\sqrt{21})^2 - \left(\frac{6}{\sqrt{5}}\right)^2} = \sqrt{\frac{69}{5}}.$$

**Problem 12**

- (a)  $\vec{a} \times \vec{b} = (\vec{i} + 3\vec{j}) \times (-2\vec{j} + 5\vec{k}) = -2(\vec{i} \times \vec{j}) + 5(\vec{i} \times \vec{k}) - 6(\vec{j} \times \vec{j}) + 15(\vec{j} \times \vec{k})$   
 $= -2\vec{k} + 5(-\vec{j}) - 6(\vec{0}) + 15\vec{i} = 15\vec{i} - 5\vec{j} - 2\vec{k}.$
- (b)  $\vec{a} \times \vec{b} = (\vec{i} + \vec{j} - 2\vec{k}) \times (-3\vec{i} + 2\vec{j} + 5\vec{k})$   
 $= -3(\vec{i} \times \vec{i}) + 2(\vec{i} \times \vec{j}) + 5(\vec{i} \times \vec{k}) - 3(\vec{j} \times \vec{i}) + 2(\vec{j} \times \vec{j}) + 5(\vec{j} \times \vec{k}) + 6(\vec{k} \times \vec{i}) - 4(\vec{k} \times \vec{j})$   
 $\quad - 10(\vec{k} \times \vec{k})$   
 $= -3(\vec{0}) + 2\vec{k} + 5(-\vec{j}) - 3(-\vec{k}) + 2(\vec{0}) + 5\vec{i} + 6\vec{j} - 4(-\vec{i}) - 10(\vec{0}) = 9\vec{i} + \vec{j} + 5\vec{k}.$
- (c)  $\vec{a} \times \vec{b} = (-3\vec{i} + \vec{j} + 3\vec{k}) \times (6\vec{j} + \vec{k})$   
 $= -18(\vec{i} \times \vec{j}) - 3(\vec{i} \times \vec{k}) + 6(\vec{j} \times \vec{j}) + (\vec{j} \times \vec{k}) + 18(\vec{k} \times \vec{j}) + 3(\vec{k} \times \vec{k})$   
 $= -18\vec{k} - 3(-\vec{j}) + 6(\vec{0}) + \vec{i} + 18(-\vec{i}) + 3(\vec{0}) = -17\vec{i} + 3\vec{j} - 18\vec{k}.$
- (d)  $\vec{a} \times \vec{b} = (\vec{j} + \vec{k}) \times (3\vec{i} - \vec{j} + 2\vec{k})$   
 $= 3(\vec{j} \times \vec{i}) - (\vec{j} \times \vec{j}) + 2(\vec{j} \times \vec{k}) + 3(\vec{k} \times \vec{i}) - (\vec{k} \times \vec{j}) + 2(\vec{k} \times \vec{k})$   
 $= 3(-\vec{k}) - (\vec{0}) + 2\vec{i} + 3\vec{j} - (-\vec{i}) + 2(\vec{0}) = 3\vec{i} + 3\vec{j} + 3\vec{k}.$

**Problem 13**

According to the definition of vector product, the vector  $\vec{a} \times \vec{b}$  is perpendicular to both  $\vec{a}$  and  $\vec{b}$ . Then we conclude that

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = |\vec{a}| |\vec{a} \times \vec{b}| \cos 90^\circ = 0.$$

**Problem 14**

- (a) We first note that  $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\vec{i} - 3\vec{j} - 2\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -3\vec{i} - 2\vec{j} + \vec{k} \end{cases}$ . According to the definition of vector product, the vector  $\overrightarrow{AB} \times \overrightarrow{AC}$  is perpendicular to both  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$ . Thus the required vector is found to be
- $$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= (2\vec{i} - 3\vec{j} - 2\vec{k}) \times (-3\vec{i} - 2\vec{j} + \vec{k}) \\ &= -6(\vec{i} \times \vec{i}) - 4(\vec{i} \times \vec{j}) + 2(\vec{i} \times \vec{k}) + 9(\vec{j} \times \vec{i}) + 6(\vec{j} \times \vec{j}) - 3(\vec{j} \times \vec{k}) + 6(\vec{k} \times \vec{i}) + 4(\vec{k} \times \vec{j}) \\ &\quad - 2(\vec{k} \times \vec{k}) \\ &= -6(\vec{0}) - 4(\vec{k}) + 2(-\vec{j}) + 9(-\vec{k}) + 6(\vec{0}) - 3(\vec{i}) + 6(\vec{j}) + 4(-\vec{i}) - 2(\vec{0}) \\ &= -7\vec{i} + 4\vec{j} - 13\vec{k}. \end{aligned}$$
- (b) Note that  $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -5\vec{i} + \vec{j} + 3\vec{k}$ , the required vector is given by
- $$\begin{aligned} \vec{a} &= |\overrightarrow{BC}| \times (\widehat{AB \times AC}) = \sqrt{35} \times \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|} = \sqrt{35} \times \frac{-7\vec{i} + 4\vec{j} - 13\vec{k}}{\sqrt{234}} \\ &= -\frac{7\sqrt{35}}{\sqrt{234}}\vec{i} + \frac{4\sqrt{35}}{\sqrt{234}}\vec{j} - \frac{13\sqrt{35}}{\sqrt{234}}\vec{k}. \end{aligned}$$

- (c) For any point  $P = (x, y, z)$  in the plane, the vector  $\overrightarrow{AP}$  lies on the same plane and its perpendicular to the vector  $\overrightarrow{AB} \times \overrightarrow{AC}$ . Note that  $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x-1)\vec{i} + (y-2)\vec{j} + z\vec{k}$ , then we have

$$\begin{aligned}\overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) &= 0 \\ \Rightarrow -7(x-1) + 4(y-2) - 13(z) &= 0 \\ \Rightarrow 7x - 4y + 13z &= -1.\end{aligned}$$

Thus the equation of plane is  $7x - 4y + 13z = -1$ .

### Problem 15

- (a) According to the definition, the vector  $\vec{a} \times \vec{b}$  is perpendicular to both  $\vec{a}$  and  $\vec{b}$ . The required vector is found to be

$$\begin{aligned}\vec{c} &= \vec{a} \times \vec{b} = (2\vec{i} - \vec{j} + 2\vec{k}) \times (4\vec{i} - 4\vec{j} + 3\vec{k}) \\ &= 8(\vec{i} \times \vec{i}) - 8(\vec{i} \times \vec{j}) + 6(\vec{i} \times \vec{k}) - 4(\vec{j} \times \vec{i}) + 4(\vec{j} \times \vec{j}) - 3(\vec{j} \times \vec{k}) + 8(\vec{k} \times \vec{i}) - 8(\vec{k} \times \vec{j}) \\ &\quad + 6(\vec{k} \times \vec{k}) \\ &= 8(\vec{0}) - 8(\vec{k}) + 6(-\vec{j}) - 4(-\vec{k}) + 4(\vec{0}) - 3(\vec{i}) + 8(\vec{j}) - 8(-\vec{i}) + 6(\vec{0}) \\ &= 5\vec{i} + 2\vec{j} - 4\vec{k}.\end{aligned}$$

- (b) The area of the triangle is

$$= \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} \sqrt{5^2 + 2^2 + (-4)^2} = \frac{\sqrt{45}}{2}.$$

- (c) For any point  $P = (x, y, z)$  in the plane, the vector  $\overrightarrow{AP}$  lies on the same plane and its perpendicular to the vector  $\vec{c} = \vec{a} \times \vec{b}$ . Note that  $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x-1)\vec{i} + (y-1)\vec{j} + (z-1)\vec{k}$ , then we have

$$\begin{aligned}\overrightarrow{AP} \cdot \vec{c} &= 0 \\ \Rightarrow 5(x-1) + 2(y-1) - 4(z-1) &= 0 \\ \Rightarrow 5x + 2y - 4z &= 3.\end{aligned}$$

Thus the equation of plane is  $5x + 2y - 4z = 3$ .

- (d) The volume of parallelepiped is given by

$$V = |\vec{d} \cdot (\vec{a} \times \vec{b})| = |(\vec{i} + 2\vec{k}) \cdot (5\vec{i} + 2\vec{j} - 4\vec{k})| = |(1)(5) + (0)(2) + 2(-4)| = |-3| = 3.$$

Since the volume is non-zero, thus the vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{d}$  are not coplanar.

### Problem 16

Note that  $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -3\vec{i} + 8\vec{j} - 5\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -12\vec{i} + 4\vec{j} - 6\vec{k} \end{cases}$  and

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (-3\vec{i} + 8\vec{j} - 5\vec{k}) \times (-12\vec{i} + 4\vec{j} - 6\vec{k}) \\ &= 36(\vec{i} \times \vec{i}) - 12(\vec{i} \times \vec{j}) + 18(\vec{i} \times \vec{k}) - 96(\vec{j} \times \vec{i}) + 32(\vec{j} \times \vec{j}) - 48(\vec{j} \times \vec{k}) + 60(\vec{k} \times \vec{i}) - 20(\vec{k} \times \vec{j}) \\ &\quad + 30(\vec{k} \times \vec{k}) \\ &= 36(\vec{0}) - 12(\vec{k}) + 18(-\vec{j}) - 96(-\vec{k}) + 32(\vec{0}) - 48(\vec{i}) + 60(\vec{j}) - 20(-\vec{i}) - 30(\vec{0}) \\ &= -28\vec{i} + 42\vec{j} + 84\vec{k}.\end{aligned}$$

Then

- the area of  $\Delta ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{(-28)^2 + 42^2 + 84^2} = 49$ .
- the area of parallelogram  $= |\overrightarrow{AB} \times \overrightarrow{AC}| = 98$ .

**Problem 17**

Recall that the points  $A$ ,  $B$  and  $C$  are coplanar if and only if  $|\overrightarrow{AB} \times \overrightarrow{AC}| = 0$

(a) We note that  $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\vec{i} + 4\vec{j} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\vec{i} + \vec{j} - \vec{k} \end{cases}$  and  $\overrightarrow{AB} \times \overrightarrow{AC} = \dots = -4\vec{i} + 3\vec{j} - 5\vec{k}$ . Since  $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-4)^2 + 3^2 + (-5)^2} = \sqrt{50} \neq 0$ , so  $A$ ,  $B$  and  $C$  are not coplanar.

(b) We note that  $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{i} - \vec{j} + 3\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\vec{i} + \vec{j} - 3\vec{k} \end{cases}$  and  $\overrightarrow{AB} \times \overrightarrow{AC} = \dots = 0\vec{i} + 0\vec{j} + 0\vec{k}$ . Since  $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{0^2 + 0^2 + 0^2} = 0$ , so  $A$ ,  $B$  and  $C$  are coplanar.

**Problem 18**

Recall that the points  $A$ ,  $B$ ,  $C$  and  $D$  are coplanar if and only if the volume of parallelepiped with  $A$ ,  $B$ ,  $C$  and  $D$  as adjacent vertices is 0, i.e.

$$|\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})| = 0 \Leftrightarrow \overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 0.$$

(a) Note that  $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{i} + 2\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -4\vec{i} - 2\vec{j} + 6\vec{k} \\ \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 2\vec{j} - 2\vec{k} \end{cases}$ , then  $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \dots = \overrightarrow{AB} \cdot (-8\vec{i} - 8\vec{j} - 8\vec{k}) = (-2\vec{i} + 2\vec{k}) \cdot (-8\vec{i} - 8\vec{j} - 8\vec{k}) = (-2)(-8) + 0(-8) + 2(-8) = 0$ .

Therefore, the four points are coplanar.

(b) Note that  $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{j} + 2\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -2\vec{i} - \vec{j} + \vec{k} \\ \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \vec{i} - 2\vec{j} + \vec{k} \end{cases}$ , then  $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \dots = \overrightarrow{AB} \cdot (\vec{i} + 3\vec{j} + 5\vec{k}) = (-2\vec{j} + 2\vec{k}) \cdot (\vec{i} + 3\vec{j} + 5\vec{k}) = (0)(1) + (-2)(3) + 2(5) = 4$ .

Therefore, the four points are not coplanar.

**Problem 19**

(a) Following the procedure as in Example 15, we need to obtain the vectors  $\overrightarrow{AD}$  and  $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ , one can find that

- $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -2\vec{i} + 2\vec{j} - \vec{k}$ ,
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i} + 2\vec{j} + 3\vec{k}$  and  $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -2\vec{i} + \vec{j} + \vec{k}$   
 $\Rightarrow \vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \dots = -\vec{i} - 5\vec{j} + 3\vec{k}$ .

Let  $\theta$  be the angle between  $\overrightarrow{AD}$  and  $\vec{n}$ , then we have

$$\cos \theta = \frac{\overrightarrow{AD} \cdot \vec{n}}{|\overrightarrow{AD}| |\vec{n}|} = \frac{(-2)(-1) + (2)(-5) + (-1)(3)}{3\sqrt{35}} = \frac{-11}{3\sqrt{35}}.$$

Then the required distance is simply the magnitude of the projection vector  $proj_{\vec{n}} \overrightarrow{AD}$  and is given by

$$|\overrightarrow{AD}| \cos \theta = \left| 3 \left( \frac{-11}{3\sqrt{35}} \right) \right| = \frac{11}{\sqrt{35}}.$$

(b) We first note that  $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \vec{i} + 2\vec{j} - 7\vec{k}$ .

Let  $\theta$  be the angle between  $\overrightarrow{BA}$  and  $\vec{n}$ , then we have

$$\cos \theta = \frac{\overrightarrow{BA} \cdot \vec{n}}{|\overrightarrow{BA}| |\vec{n}|} = \frac{(1)(-1) + 2(-1) + (-7)(-1)}{\sqrt{54}\sqrt{3}} = \frac{4}{\sqrt{162}}$$

The required distance is the magnitude of the projection vector  $\text{proj}_{\vec{n}} \overrightarrow{BA}$  and is given by

$$|\overrightarrow{BA}| \cos \theta = \sqrt{54} \left( \frac{4}{\sqrt{162}} \right) = \frac{4}{\sqrt{3}}$$

### Problem 20

- (a) We let  $A = (5, 0, -1)$  and  $B = (6, 2, -2)$  be two points on the line  $L_1$ . We also let  $C = (2, 4, 0)$  and  $D = (3, 3, 1)$  be two points on the line  $L_2$ .

Following the procedure as in Example 16, we need to obtain the vectors  $\overrightarrow{AD}$  and  $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ .

- $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -2\vec{i} + 3\vec{j} + 2\vec{k}$ .
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{i} + 2\vec{j} - \vec{k}$  and  $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \vec{i} - \vec{j} + \vec{k}$ .

The normal vector  $\vec{n}$  is found to be

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD} = (\vec{i} + 2\vec{j} - \vec{k}) \times (\vec{i} - \vec{j} + \vec{k}) = \vec{i} - 2\vec{j} - 3\vec{k}.$$

Let  $\theta$  be the angle between  $\overrightarrow{AD}$  and  $\vec{n}$ , then we have

$$\cos \theta = \frac{\overrightarrow{AD} \cdot \vec{n}}{|\overrightarrow{AD}| |\vec{n}|} = \frac{(-2)(1) + (3)(-2) + (2)(-3)}{\sqrt{17}\sqrt{14}} = \frac{-14}{\sqrt{17}\sqrt{14}}$$

Then the required distance is simply the magnitude of the projection vector  $\text{proj}_{\vec{n}} \overrightarrow{DA}$  and is given by

$$|\overrightarrow{DA}| \cos \theta = |\sqrt{17} \left( \frac{-14}{\sqrt{17}\sqrt{14}} \right)| = \sqrt{14}.$$

- (b) We let  $A = (1, 1, 1)$  and  $B = (2, 1, 2)$  be two points on the line  $L_1$ . We also let  $C = (2, 1, 0)$  and  $D = (3, 2, 0)$  be two points on the line  $L_2$ .

Following the procedure as in Example 16, we need to obtain the vectors  $\overrightarrow{AD}$  and  $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ .

- $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 2\vec{i} + \vec{j} - \vec{k}$ .
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{i} + \vec{k}$  and  $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \vec{i} + \vec{j}$ .

The normal vector  $\vec{n}$  is found to be

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD} = (\vec{i} + \vec{k}) \times (\vec{i} + \vec{j}) = -\vec{i} + \vec{j} + \vec{k}.$$

Let  $\theta$  be the angle between  $\overrightarrow{AD}$  and  $\vec{n}$ , then we have

$$\cos \theta = \frac{\overrightarrow{AD} \cdot \vec{n}}{|\overrightarrow{AD}| |\vec{n}|} = \frac{(2)(-1) + (1)(1) + (-1)(1)}{\sqrt{6}\sqrt{3}} = \frac{-2}{\sqrt{18}}$$

Then the required distance is simply the magnitude of the projection vector  $\text{proj}_{\vec{n}} \overrightarrow{DA}$  and is given by

$$|\overrightarrow{DA}| \cos \theta = |\sqrt{6} \left( \frac{-2}{\sqrt{18}} \right)| = \frac{2}{\sqrt{3}}$$

### Problem 21

- (a) Since

$$|\vec{a} \times \vec{b}| = |(\vec{i} - 2\vec{j}) \times (2\vec{i} + \vec{j})| = |5\vec{k}| = 5 \neq 0,$$

thus  $\vec{a}$  and  $\vec{b}$  are not collinear and these two vectors are linearly independent.

- (b) Note that

$$\begin{aligned} \vec{a} \cdot (\vec{b} \times \vec{c}) &= \vec{a} \cdot (-17\vec{i} + 9\vec{j} - 11\vec{k}) = (\vec{i} - 2\vec{j} + 3\vec{k}) \cdot (-17\vec{i} + 9\vec{j} - 11\vec{k}) \\ &= (1)(-17) - 2(9) + (3)(-11) = -68 \neq 0. \end{aligned}$$

The vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are not coplanar, these three vectors are linearly independent.

(c) Note that

$$\begin{aligned}\vec{a} \cdot (\vec{b} \times \vec{c}) &= \vec{a} \cdot (-4\vec{i} - 8\vec{j} - 4\vec{k}) = (\vec{i} + 2\vec{j} - 5\vec{k}) \cdot (-4\vec{i} - 8\vec{j} - 4\vec{k}) \\ &= (1)(-4) + 2(-8) + (-5)(-4) = 0.\end{aligned}$$

The vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are coplanar, these three vectors are linearly dependent.

### Problem 22

If  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are linearly dependent, then we may have

$$\begin{aligned}\vec{a} \cdot (\vec{b} \times \vec{c}) &= 0 \Rightarrow \vec{a} \cdot (-5m\vec{i} - 10\vec{j} - 10m\vec{k}) = 0 \\ \Rightarrow ((1-m)\vec{i} + 6\vec{j} + 5\vec{k}) \cdot (-5m\vec{i} - 10\vec{j} - 10m\vec{k}) &= 0 \\ \Rightarrow -5m(1-m) + 6(-10) + 5(-10m) &= 0 \\ \Rightarrow m^2 - 11m - 12 = 0 \Rightarrow (m-12)(m+1) &= 0 \\ \Rightarrow m = 12 \text{ or } m = -1.\end{aligned}$$

### Problem 23

$$\begin{aligned}\text{(a)} \quad (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) &= \underbrace{(\vec{a} \times \vec{a})}_{\vec{0}} + (\vec{a} \times \vec{b}) - (\vec{b} \times \vec{a}) - \underbrace{(\vec{b} \times \vec{b})}_{=\vec{0}} = (\vec{a} \times \vec{b}) - (-\vec{a} \times \vec{b}) \\ &= 2(\vec{a} \times \vec{b}).\end{aligned}$$

(b) Given that  $\vec{a}$  and  $\vec{b}$  are perpendicular, we have  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 90^\circ = 0$ .

$$\begin{aligned}|\vec{a} + \vec{b}| &= \sqrt{(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})} = \sqrt{(\vec{a} \cdot \vec{a}) + 2(\vec{a} \cdot \vec{b}) + (\vec{b} \cdot \vec{b})} \\ &\stackrel{\vec{a} \cdot \vec{b} = 0}{=} \sqrt{(\vec{a} \cdot \vec{a}) - 2(\vec{a} \cdot \vec{b}) + (\vec{b} \cdot \vec{b})} = \sqrt{(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})} = |\vec{a} - \vec{b}|.\end{aligned}$$

(c) If  $\vec{a}$  and  $\vec{b}$  are parallel, the angle between these two vectors is either  $0^\circ$  or  $180^\circ$ . For either case, we always have  $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \underbrace{\sin \theta}_{=0} \hat{n} = \vec{0}$ .

We consider

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) \stackrel{\text{from (a)}}{=} 2(\vec{a} \times \vec{b}) = \vec{0}.$$

Since the cross product is  $\vec{0}$ , the vector  $\vec{a} - \vec{b}$  and  $\vec{a} + \vec{b}$  may be collinear and hence are parallel.

$$\text{(d)} \quad \frac{|\vec{a} \times \vec{b}|}{\vec{a} \cdot \vec{b}} = \frac{|\vec{a}||\vec{b}| \sin \theta}{|\vec{a}||\vec{b}| \cos \theta} = \tan \theta.$$

$$\begin{aligned}\text{(e)} \quad |\vec{a} \times \vec{b}|^2 &= |\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta = |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2 \theta) = |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta \\ &= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2.\end{aligned}$$