

Suggested Solution

Version 1 = P.1 - P.5

Version 2 = p.6 - p.10

Version 3 = p.11 - p.15

①

MA1201 Calculus and Basic Algebra II (Section: CE1)

Quiz 2: Vector, Complex Number and Matrix

Version 1

Name:

SID:

Score:

Time allowed: 50 minutes

Instruction:

Write all your answer in the spaces provided. You need to show all your working steps and method used CLEARLY. Marks will be deducted for any unclear solution.

In your solution, you MUST include

- The formula and method that you are using. Don't just give me a set of messy numbers.
- FULL detail about the calculation of determinant, scalar product, vector product.

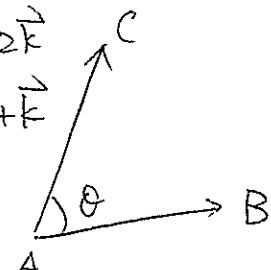
Problem 1

- (a) Let $A = (4, -2, 1)$, $B = (0, 1, 2)$, $C = (1, -3, 3)$ be three points in 3D-space.
- (i) Find the $\angle CAB$ (11 marks)
- (ii) Find a vector which is perpendicular to both $\vec{AC}$ and $\vec{AB}$. (11 marks)
- (b) We let $\vec{a}$ and $\vec{b}$ be two vectors so that $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 3$. Find the value of $|\vec{a} + 3\vec{b}|$. (11 marks)
- (Hint: You need to use the property of scalar product)

(a)(i) Note that $\begin{cases} \vec{AC} = \vec{OC} - \vec{OA} = -3\vec{i} - \vec{j} + 2\vec{k} \\ \vec{AB} = \vec{OB} - \vec{OA} = -4\vec{i} + 3\vec{j} + \vec{k} \end{cases}$

Then $\cos \theta = \frac{\vec{AC} \cdot \vec{AB}}{|\vec{AC}| |\vec{AB}|}$

$= \frac{(-3)(-4) + (-1)(3) + (2)(1)}{\sqrt{(-3)^2 + (-1)^2 + 2^2} \sqrt{(-4)^2 + 3^2 + 1^2}} = \frac{11}{\sqrt{14} \sqrt{26}} \Rightarrow \theta =$



(ii) According to the definition of vector product, the vector $\vec{AC} \times \vec{AB}$ is the vector perpendicular to $\vec{AB}$ and $\vec{AC}$. So the required vector is

$$\begin{aligned} \vec{AC} \times \vec{AB} &= (-3\vec{i} - \vec{j} + 2\vec{k}) \times (-4\vec{i} + 3\vec{j} + \vec{k}) \\ &= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -1 & 2 \\ -4 & 3 & 1 \end{pmatrix} = (-1)^{1+1} \det \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \vec{i} + (-1)^{1+2} \det \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \vec{j} \\ &\quad + (-1)^{1+3} \det \begin{pmatrix} -3 & -1 \\ -4 & 3 \end{pmatrix} \vec{k} \end{aligned}$$

Recall that $|\vec{x}|^2 = \vec{x} \cdot \vec{x}$

$$= -7\vec{i} - 5\vec{j} - 13\vec{k}.$$

(b) $|\vec{a} + 3\vec{b}| = \sqrt{(\vec{a} + 3\vec{b}) \cdot (\vec{a} + 3\vec{b})} = \sqrt{\vec{a} \cdot \vec{a} + 6(\vec{a} \cdot \vec{b}) + 9(\vec{b} \cdot \vec{b})}$

$$= \sqrt{|\vec{a}|^2 + 6(\vec{a} \cdot \vec{b}) + 9|\vec{b}|^2} = \sqrt{2^2 + 6(3) + 9(3)^2}$$

$$= \sqrt{103}.$$

Problem 2

(a) Let $z_1 = \sqrt{2} - \sqrt{6}i$ and $z_2 = -\sqrt{6} + \sqrt{2}i$ be two complex numbers. Find all values of

(i) $(z_1)^4$, (ii) $\sqrt[6]{\frac{z_1}{z_2}}$, (iii) $(\bar{z}_2)^{\frac{1}{2}}$.

Express your answer in Euler form.

(18 marks)

(b) Solve the equation for z :

$$z^4 - 4z^3 + 4z^2 + 8z + 39 = 0$$

given that $z = 3 + 2i$ is one of the root.

(15 marks)

(Hint: Recall if $3 + 2i$ is a root, then ?? is also the root.)

(a) $z_1 = \sqrt{2} - \sqrt{6}i$
 $= \sqrt{8} \left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) \right)$

$z_2 = -\sqrt{6} + \sqrt{2}i$
 $= \sqrt{8} \left(\cos\frac{\sqrt{\pi}}{6} + i\sin\frac{\sqrt{\pi}}{6} \right)$

$r_2 = \sqrt{(-\sqrt{6})^2 + (\sqrt{2})^2} = \sqrt{8}$
 $\theta_2 = \pi - \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{6}}\right) = \frac{5\pi}{6}$

$r_1 = \sqrt{(\sqrt{2})^2 + (-\sqrt{6})^2} = \sqrt{8}$
 $\theta_1 = -\tan^{-1}\left(\frac{\sqrt{6}}{\sqrt{2}}\right) = -\frac{\pi}{3}$

Then we have

$$\begin{aligned} (i) (z_1)^4 &= (\sqrt{8})^4 \left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right) \right) \\ &= 64 e^{-i\left(\frac{4\pi}{3}\right)}. \end{aligned}$$

$$\begin{aligned} (ii) \sqrt[6]{\frac{z_1}{z_2}} &= \sqrt[6]{\cos\left(-\frac{\pi}{3} - \frac{5\pi}{6}\right) + i\sin\left(-\frac{\pi}{3} - \frac{5\pi}{6}\right)} = \sqrt[6]{\cos\left(-\frac{7\pi}{6}\right) + i\sin\left(-\frac{7\pi}{6}\right)} \\ &= \cos\frac{2k\pi - \frac{7\pi}{6}}{6} + i\sin\frac{2k\pi - \frac{7\pi}{6}}{6} = e^{i\left(\frac{k\pi}{3} - \frac{7\pi}{36}\right)} \text{ for } k=0,1,2,3,4. \end{aligned}$$

$$\begin{aligned} (iii) (\bar{z}_2)^{\frac{1}{2}} &= \left[\sqrt{8} \left(\cos\left(-\frac{\sqrt{\pi}}{6}\right) + i\sin\left(-\frac{\sqrt{\pi}}{6}\right) \right) \right]^{\frac{1}{2}} = 8^{\frac{1}{4}} \left(\cos\frac{2k\pi - \frac{\sqrt{\pi}}{6}}{2} + i\sin\frac{2k\pi - \frac{\sqrt{\pi}}{6}}{2} \right) \\ &= 8^{\frac{1}{4}} e^{i\left(k\pi - \frac{\sqrt{\pi}}{12}\right)} \text{ for } k=0,1. \end{aligned}$$

(b). If $3+2i$ is a root, then $\overline{3+2i} = 3-2i$ is also the root. (4)

By factor theorem, the linear functions $(z-(3+2i))$ and $(z-(3-2i))$ are factors of $z^4 - 4z^3 + 4z^2 + 8z + 19$.

This implies that $(z-(3+2i))(z-(3-2i)) = z^2 - 6z + 13$ is also factor of the same function.

By long division, we have

$$\begin{aligned} z^4 - 4z^3 + 4z^2 + 8z + 19 &= 0 \\ \Rightarrow (z^2 - 6z + 13)(z^2 + 2z + 3) &= 0 \\ \Rightarrow z^2 - 6z + 13 = 0 \text{ or } z^2 + 2z + 3 &= 0 \\ \Rightarrow z = 3 \pm 2i \text{ or } z = \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2} \\ &= -1 \pm \sqrt{2}i. \end{aligned}$$

$$\begin{array}{r} z^2 + 2z + 3 \\ z^2 - 6z + 13 \overline{) z^4 - 4z^3 + 4z^2 + 8z + 19} \\ \underline{z^4 - 6z^3 + 13z^2} \\ 2z^3 - 9z^2 + 8z \\ \underline{2z^3 - 12z^2 + 26z} \\ 3z^2 - 18z + 39 \\ \underline{3z^2 - 18z + 39} \\ 0 \end{array}$$

Problem 3

(a) Let $C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$ be two matrices, find the value of $\det(CD)$.

(10 marks)

(b) Let $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 5 & -2 & 4 \end{pmatrix}$ be a matrix. Show that the matrix A is invertible and find its inverse.

Show all calculation.

(15 marks)

(c) If B be another 3×3 matrix such that $\det(A^2 B) = 10$, find $\det B$.

(8 marks)

$$(a) \quad CD = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \times 0 + (-1)(1) + 0(1) & (1)(1) + (-1)(-1) + 0(0) \\ 0 \times 0 + (1)(1) + 0(1) & 0(1) + 1(-1) + 0(0) \end{pmatrix}$$

$2 \times 3 \qquad 3 \times 2 \qquad 2 \times 2$

$$\Rightarrow CD = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \Rightarrow \det CD = (-1)(-1) - (1)(2) = -1.$$

expand along first row

$$(b) \quad \det A = \begin{matrix} \downarrow = 2 \\ a_{11} \end{matrix} A_{11} + \begin{matrix} \downarrow = 1 \\ a_{12} \end{matrix} A_{12} + \begin{matrix} \downarrow = 0 \\ a_{13} \end{matrix} A_{13} = 2(-1)^{1+1} \det \begin{pmatrix} 0 & 1 \\ -2 & 4 \end{pmatrix} + 1(-1)^{1+2} \det \begin{pmatrix} -1 & 1 \\ 5 & 4 \end{pmatrix}$$

$$= 2(2) - (-9) = 13.$$

Since $\det A \neq 0$, so A is invertible.

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \frac{1}{13} \begin{pmatrix} 2 & 9 & 2 \\ -4 & 8 & 9 \\ 1 & -2 & 1 \end{pmatrix}^T = \begin{pmatrix} \frac{2}{13} & -\frac{4}{13} & \frac{1}{13} \\ \frac{9}{13} & \frac{8}{13} & \frac{-2}{13} \\ \frac{2}{13} & \frac{9}{13} & \frac{1}{13} \end{pmatrix}.$$

$$A_{11} = (-1)^{1+1} \det \begin{pmatrix} 0 & 1 \\ -2 & 4 \end{pmatrix} = 2 \quad A_{21} = (-1)^{2+1} \det \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix} = -4 \quad A_{31} = (-1)^{3+1} \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$A_{12} = (-1)^{1+2} \det \begin{pmatrix} -1 & 1 \\ 5 & 4 \end{pmatrix} = 9 \quad A_{22} = (-1)^{2+2} \det \begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix} = 8 \quad A_{32} = (-1)^{3+2} \det \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = -2$$

$$A_{13} = (-1)^{1+3} \det \begin{pmatrix} -1 & 0 \\ 5 & -2 \end{pmatrix} = 2 \quad A_{23} = (-1)^{2+3} \det \begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix} = 9 \quad A_{33} = (-1)^{3+3} \det \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = -1$$

$$(c) \quad \because \det(A^2 B) = 10 \Rightarrow (\det A)^2 \det B = 10 \Rightarrow \det B = \frac{10}{169}.$$

$$\text{Since } \det(CD) = (\det C)(\det D)$$

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- FULL detail about the calculation of determinant, scalar product, vector product.

Problem 1

- (a) Let $A = (1, 3, 0)$, $B = (-1, 1, 2)$, $C = (3, 3, 0)$ be three points in 3D-space.
- (i) Find the area of $\triangle ABC$. (11 marks)
- (ii) Determine if the line AB is perpendicular to BC . (11 marks)
- (b) Let $D = (2, 1, x)$ be another point in the same space such that A, B, C, D are coplanar. Find the value of x . (11 marks)

(a)(i) Note that
$$\begin{cases} \vec{AB} = \vec{OB} - \vec{OA} = -2\vec{i} - 2\vec{j} + 2\vec{k} \\ \vec{AC} = \vec{OC} - \vec{OA} = 2\vec{i} \end{cases}$$

The area of $\triangle ABC$

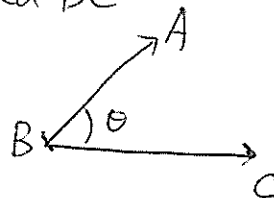
$$\begin{aligned} &= \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} |(-2\vec{i} - 2\vec{j} + 2\vec{k}) \times 2\vec{i}| \\ &= \frac{1}{2} | -4(\vec{j} \times \vec{i}) - 4(\vec{k} \times \vec{i}) + 4(\vec{i} \times \vec{i}) | = \frac{1}{2} | 4\vec{j} + 4\vec{k} | = \frac{1}{2} \sqrt{4^2 + 4^2} = \sqrt{8}. \end{aligned}$$

(ii) $\because \vec{BC} = \vec{OC} - \vec{OB} = 4\vec{i} + 2\vec{j} - 2\vec{k}$, $\vec{BA} = 2\vec{i} + 2\vec{j} - 2\vec{k}$.

Let θ be the angle between $\vec{AB}$ and $\vec{BC}$

$$\cos \theta = \frac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}| |\vec{BC}|} = \frac{2(4) + 2(2) + (-2)(-2)}{\sqrt{4^2 + 2^2 + (-2)^2} \sqrt{2^2 + 2^2 + (-2)^2}}$$

$$= \frac{16}{\sqrt{24} \sqrt{12}} \Rightarrow \theta \neq 90^\circ.$$



$\Rightarrow AB$ and BC are NOT perpendicular.

(b) $\vec{AD} = \vec{OD} - \vec{OA} = \vec{i} - 2\vec{j} + x\vec{k}$

If A, B, C, D are coplanar, then $\vec{AD} \cdot (\vec{AB} \times \vec{AC}) = 0$

$$\Rightarrow (\vec{i} - 2\vec{j} + x\vec{k}) \cdot (4\vec{j} + 4\vec{k}) = (1)(0) + (-2)(4) + x(4) = 0$$

$$\Rightarrow -8 + 4x = 0 \Rightarrow x = 2.$$

Problem 2

(a) Let $z_1 = 3 + 3i$ and $z_2 = -\sqrt{2} - \sqrt{2}i$ be two complex numbers. Find all values of

(i) $(z_1)^5$, (ii) $\sqrt[4]{-iz_2}$, (iii) $(\bar{z}_2)^{-3}$.

Express your answer in Euler form.

(18 marks)

(b) Solve the equation for z :

$$(3z - 5)^4 = -81(z + 5)^4.$$

(15 marks)

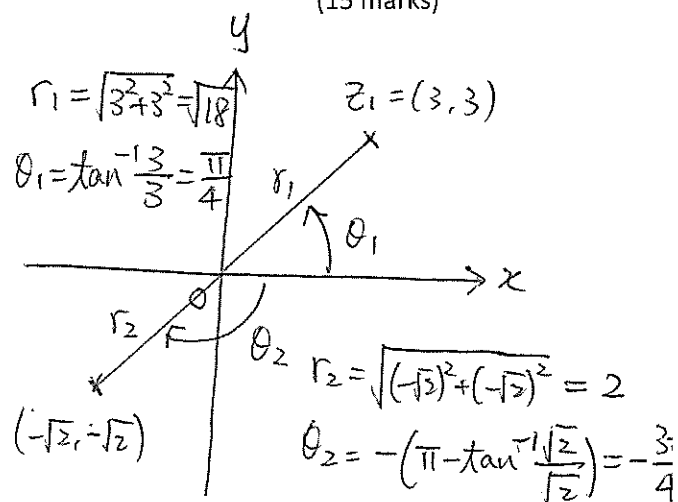
(a) Note that

$$z_1 = \sqrt{18} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$z_2 = 2 \left(\cos \left(-\frac{3\pi}{4} \right) + i \sin \left(-\frac{3\pi}{4} \right) \right)$$

Then we have

$$\begin{aligned} (i) z_1^4 &= (\sqrt{18})^4 \left(\cos \frac{4\pi}{4} + i \sin \frac{4\pi}{4} \right) \\ &= 324 e^{i\pi}, \end{aligned}$$



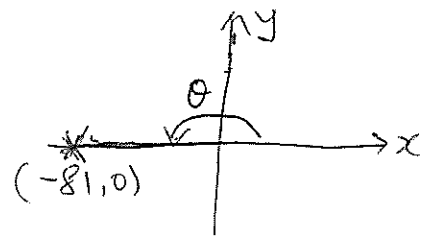
$$\begin{aligned} (ii) \sqrt[4]{-iz_2} &= \sqrt[4]{2 \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right) \left(\cos \left(-\frac{3\pi}{4} \right) + i \sin \left(-\frac{3\pi}{4} \right) \right)} \\ &= \sqrt[4]{2 \left(\cos \left(-\frac{5\pi}{4} \right) + i \sin \left(-\frac{5\pi}{4} \right) \right)} \quad \left[= -\frac{\pi}{2} + \left(-\frac{3\pi}{4} \right) \right] \\ &= \sqrt[4]{2} \left(\cos \frac{2k\pi - \frac{5\pi}{4}}{4} + i \sin \frac{2k\pi - \frac{5\pi}{4}}{4} \right) \\ &= \sqrt[4]{2} e^{i \left(\frac{k\pi}{2} - \frac{5\pi}{16} \right)} \quad \text{for } k=0, 1, 2, 3. \end{aligned}$$

$$\begin{aligned} (iii) (\bar{z}_2)^{-3} &= \left[2 \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \right]^{-3} \\ &= 2^{-3} \left(\cos \left(-\frac{9\pi}{4} \right) + i \sin \left(-\frac{9\pi}{4} \right) \right) = \frac{1}{8} e^{-i \left(\frac{9\pi}{4} \right)}. \end{aligned}$$

(9)

$$(b) \quad (3z-5)^4 = -81(z+5)^4$$

$$\Rightarrow \left(\frac{3z-5}{z+5} \right)^4 = -81$$



$$\begin{aligned} \Rightarrow \left(\frac{3z-5}{z+5} \right) &= \sqrt[4]{-81} = \sqrt[4]{81(\cos \pi + i \sin \pi)} \\ &= \sqrt[4]{81} \left(\cos \frac{2k\pi + \pi}{4} + i \sin \frac{2k\pi + \pi}{4} \right) \text{ for } k=0,1,2,3. \\ &= \underbrace{3 \left(\cos \frac{(2k+1)\pi}{4} + i \sin \frac{(2k+1)\pi}{4} \right)}_{\omega_k} \end{aligned}$$

$$\Rightarrow (3z-5) = \omega_k(z+5)$$

$$\Rightarrow (3-\omega_k)z = 5+5\omega_k$$

$$\Rightarrow z = \frac{5-5\omega_k}{3-\omega_k} \text{ where } \omega_k = 3 \left(\cos \frac{(2k+1)\pi}{4} + i \sin \frac{(2k+1)\pi}{4} \right).$$

Problem 3

Let $A = \begin{pmatrix} a+2 & 0 & 3 \\ 0 & a & 0 \\ 5 & 0 & a-2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 & 0 \\ -1 & x & 1 \\ 1 & -2 & x \end{pmatrix}$ be two matrices.

(a) Find the value of a so that A is NOT invertible.

(11 marks)

(b) Take $a = 1$, find the inverse of A .

(11 marks)

(c) Determine if the matrix B^{2013} is invertible.

(12 marks)

(a) A is not invertible, then $\det A = 0$

$$\begin{aligned} &\Rightarrow \overset{=0}{a_{21}}A_{21} + \overset{=a}{a_{22}}A_{22} + \overset{=0}{a_{23}}A_{23} = 0 \\ \text{expand } \det A &\Rightarrow a \left[(-1)^{2+2} \det \begin{pmatrix} a+2 & 3 \\ 5 & a-2 \end{pmatrix} \right] = 0 \\ \text{along 2nd row} &\Rightarrow a \left[(a+2)(a-2) - 15 \right] = 0 \Rightarrow a(a^2 - 19) = 0 \\ &\Rightarrow a = 0 \text{ or } a = \pm\sqrt{19} \end{aligned}$$

(b) Take $a = 1$, $\det A = 1(-18) = -18 \neq 0$ and $A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & 0 \\ 5 & 0 & -1 \end{pmatrix}$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = -\frac{1}{18} \begin{pmatrix} -1 & 0 & -5 \\ 0 & -18 & 0 \\ -3 & 0 & 3 \end{pmatrix}^T = \begin{pmatrix} \frac{1}{18} & 0 & \frac{5}{18} \\ 0 & 1 & 0 \\ \frac{3}{18} & 0 & -\frac{3}{18} \end{pmatrix}$$

$$\begin{aligned} A_{11} &= (-1)^{1+1} \det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1, \quad A_{21} = (-1)^{2+1} \det \begin{pmatrix} 3 & 3 \\ 0 & -1 \end{pmatrix} = 0, \quad A_{31} = (-1)^{3+1} \det \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} = -3 \\ A_{12} &= (-1)^{1+2} \det \begin{pmatrix} 0 & 0 \\ 5 & -1 \end{pmatrix} = 0, \quad A_{22} = (-1)^{2+2} \det \begin{pmatrix} 3 & 3 \\ 5 & -1 \end{pmatrix} = -18, \quad A_{32} = (-1)^{3+2} \det \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} = 0 \\ A_{13} &= (-1)^{1+3} \det \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} = -5, \quad A_{23} = (-1)^{2+3} \det \begin{pmatrix} 3 & 0 \\ 5 & 0 \end{pmatrix} = 0, \quad A_{33} = (-1)^{3+3} \det \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = 3 \end{aligned}$$

(c) Note that $\det B = 1 \times \det \begin{pmatrix} x & 1 \\ -2 & x \end{pmatrix} - 4 \times \det \begin{pmatrix} -1 & 1 \\ 1 & x \end{pmatrix} = x^2 + 2 - 4(-x-1) = x^2 + 4x + 6$.

$$\text{Then } \det(B^{2013}) = (\det B)^{2013} = (x^2 + 4x + 6)^{2013}$$

$$= [(x+2)^2 + 2]^{2013} \geq 2^{2013} > 0$$

$$\Rightarrow \det(B^{2013}) \neq 0 \Rightarrow B^{2013} \text{ is invertible.}$$

☺End of the Quiz☺

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- The formula and method that you are using. Don't just give me a set of messy numbers.
- FULL detail about the calculation of determinant, scalar product, vector product.

Problem 1

- (a) Let $A = (1,3,0)$, $B = (-1,1,2)$, $C = (2,3,0)$ be three points in 3D-space.
- (i) Determine if A, B and C are collinear. (11 marks)
- (ii) Find the projection vector of $\vec{AB}$ onto $\vec{AC}$ ($\text{proj}_{\vec{AC}} \vec{AB}$) (11 marks)
- (b) Let $\vec{a}$ be a vector having magnitude 2 and it is always perpendicular to both $\vec{AB}$ and $\vec{AC}$. Find $\vec{a}$. (11 marks)

(a)(i) Note that
$$\begin{cases} \vec{AB} = \vec{OB} - \vec{OA} = -2\vec{i} - 2\vec{j} + 2\vec{k} \\ \vec{AC} = \vec{OC} - \vec{OA} = \vec{i} \end{cases}$$

Then
$$\vec{AB} \times \vec{AC} = (-2\vec{i} - 2\vec{j} + 2\vec{k}) \times \vec{i} = -2(\vec{j} \times \vec{i}) - 2(\vec{k} \times \vec{i}) + 2(\vec{i} \times \vec{i})$$

$$= 2\vec{j} + 2\vec{k}.$$

and the area of $\triangle ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} \sqrt{2^2 + 2^2} = \sqrt{2} \neq 0.$

So A, B, C are not collinear.

(ii) Let θ be the angle between $\vec{AB}$ and $\vec{AC}$. Note that

$$\cos \theta = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} = \frac{(-2)(1) + (-2)(0) + 2(0)}{\sqrt{(-2)^2 + (-2)^2 + 2^2} \sqrt{1^2 + 0^2 + 0^2}} = \frac{-2}{\sqrt{12}} \Rightarrow \theta > 90^\circ.$$

$\Rightarrow$ Magnitude of $\text{proj}_{\vec{AC}} \vec{AB} = |\vec{AB}| \cos(180^\circ - \theta)$

Since $\cos(180^\circ - \theta) = -\cos \theta$, $\rightarrow = \sqrt{12} \frac{2}{\sqrt{12}} = 2.$

Direction of $\text{proj}_{\vec{AC}} \vec{AB} = -\frac{\vec{AC}}{|\vec{AC}|} = -\vec{i}.$

Hence $\text{proj}_{\vec{AC}} \vec{AB} = 2 \times (-\vec{i}) = -2\vec{i}.$

(iii)
$$\vec{a} = \underbrace{2}_{\text{magnitude}} \times \underbrace{\frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|}}_{\text{direction}} = 2 \times \frac{(2\vec{j} + 2\vec{k})}{\sqrt{2^2 + 2^2}} = \sqrt{2}\vec{j} + \sqrt{2}\vec{k}.$$

Recall that $\vec{AB} \times \vec{AC}$ is the vector which is perpendicular to both $\vec{AB}$ and $\vec{AC}$ by the definition of vector product.

Recall that $\vec{AB} \times \vec{AC}$ is the vector which is perpendicular to both $\vec{AB}$ and $\vec{AC}$ by the definition of vector product.

Problem 2

(a) Let $z_1 = 3 - \sqrt{3}i$ and $z_2 = -1 + \sqrt{3}i$ be two complex numbers. Find all values of

(i) $\left(\frac{z_2}{z_1}\right)^5$, (ii) $\sqrt[9]{z_2}$, (iii) $(\bar{z}_1)^{-3}$.

Express your answer in Euler form.

(18 marks)

(b) Solve the equation for z :

$$\frac{z^3 - 1}{z^3 + 1} = \sqrt{3}i$$

(15 marks)

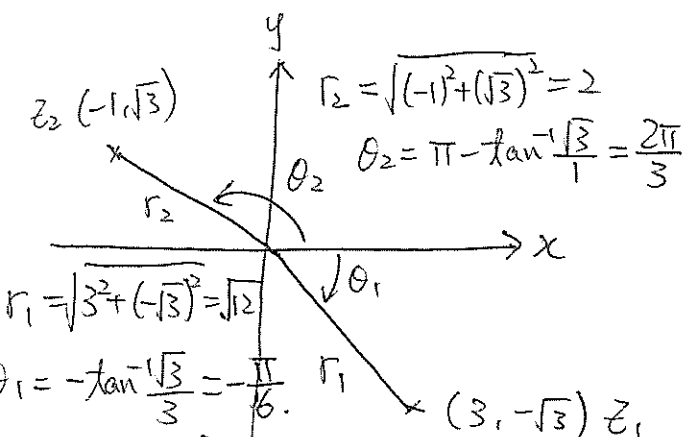
(a) Note that

$$z_1 = \sqrt{12} \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right)$$

$$z_2 = 2 \left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} \right)$$

Then we have

$$\begin{aligned} \text{(i)} \quad \left(\frac{z_2}{z_1}\right)^5 &= \left[\frac{2}{\sqrt{12}} \left(\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6} \right) \right]^5 \quad \theta_1 = -\tan^{-1}\frac{\sqrt{3}}{3} = -\frac{\pi}{6} \\ &= \frac{32}{12^{5/2}} \left(\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6} \right) = \frac{1}{3^{5/2}} e^{i\left(\frac{5\pi}{6}\right)} \quad \left(\text{or } \frac{1}{3^{5/2}} e^{i\frac{\pi}{6}} \right) \end{aligned}$$



$$\begin{aligned} \text{(ii)} \quad \sqrt[9]{z_2} &= \sqrt[9]{2 \left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} \right)} = 2^{1/9} \left(\cos\frac{2k\pi + 2\pi}{9} + i \sin\frac{2k\pi + 2\pi}{9} \right) \\ &= 2^{1/9} e^{i\left(\frac{2k\pi}{9} + \frac{2\pi}{9}\right)} \quad \text{for } k=0, 1, 2, \dots, 8. \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad (\bar{z}_1)^{-3} &= \left[\sqrt{12} \left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6} \right) \right]^{-3} = \frac{1}{12^{3/2}} \left(\cos\frac{-3\pi}{6} + i \sin\frac{-3\pi}{6} \right) \\ &= \frac{1}{12^{3/2}} e^{-i\frac{\pi}{2}}. \end{aligned}$$

$$(b) \frac{z^3 - 1}{z^3 + 1} = \sqrt{3}i \Rightarrow z^3 - 1 = (\sqrt{3}i)z^3 + \sqrt{3}i$$

$$\Rightarrow (1 - \sqrt{3}i)z^3 = 1 + \sqrt{3}i$$

$$\Rightarrow z^3 = \frac{1 + \sqrt{3}i}{1 - \sqrt{3}i} \Rightarrow z = \sqrt[3]{\frac{1 + \sqrt{3}i}{1 - \sqrt{3}i}}$$

$$\Rightarrow z = \sqrt[3]{\frac{2[\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}]}{2[\cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3})]}} = \sqrt[3]{\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}}$$

$$= \cos \frac{2k\pi + \frac{2\pi}{3}}{3} + i \sin \frac{2k\pi + \frac{2\pi}{3}}{3} \quad \text{for } k=0,1,2.$$

Problem 3

Let $A = \begin{pmatrix} 3 & a & 3 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} x^2 & 4 & 0 \\ 0 & x & 1 \\ 1 & 0 & x \end{pmatrix}$ be two matrices.

(a) Find the value of A^2 , $\det A$. Express your answer in terms of a . (11 marks)

(b) Take $a = 1$

(i) Find the inverse of A . (11 marks)

(ii) Suppose that $\det(AB) = 40$, find the value of B . (12 marks)

$$(a) A^2 = \begin{pmatrix} 3 & a & 3 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & a & 3 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 3(3)+a(-2)+3(1) & 3a+2a+3(-1) & 3(3)+a(1)+3(2) \\ -2(3)+2(-2)+1(1) & -2(a)+2(2)+1(-1) & -2(3)+2(1)+1(2) \\ 1(3)-1(-2)+2(1) & 1(a)-1(2)+2(-1) & 1(3)-1(1)+2(2) \end{pmatrix} = \begin{pmatrix} 12-2a & 5a-3 & a+15 \\ -9 & 3-2a & -2 \\ 7 & a-4 & 6 \end{pmatrix}$$

$$\det A = \overset{=3}{a_{11}} A_{11} + \overset{=a}{a_{12}} A_{12} + \overset{=3}{a_{13}} A_{13} \\ = 3 \times (-1)^{1+1} \det \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} + a \times (-1)^{1+2} \det \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} + 3 \times (-1)^{1+3} \det \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \\ = 3(5) - a(-5) + 0 = 15 + 5a.$$

(b)(i) Take $a=1$, $\det A = 20$ and $A = \begin{pmatrix} 3 & 1 & 3 \\ -2 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

$$\Rightarrow A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \frac{1}{20} \begin{pmatrix} 5 & 5 & 0 \\ -5 & 3 & 4 \\ -5 & -9 & 8 \end{pmatrix}^T = \frac{1}{20} \begin{pmatrix} 5 & -5 & -5 \\ 5 & 3 & -9 \\ 0 & 4 & 8 \end{pmatrix}$$

$$A_{11} = (-1)^{1+1} \det \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = 5 \quad A_{21} = (-1)^{2+1} \det \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = -5 \quad A_{31} = (-1)^{3+1} \det \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = -5$$

$$A_{12} = (-1)^{1+2} \det \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} = 5 \quad A_{22} = (-1)^{2+2} \det \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix} = 3 \quad A_{32} = (-1)^{3+2} \det \begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix} = -9$$

$$A_{13} = (-1)^{1+3} \det \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} = 0 \quad A_{23} = (-1)^{2+3} \det \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = 4 \quad A_{33} = (-1)^{3+3} \det \begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix} = 8$$

$$(ii) \det B = \overset{x^2}{b_{11}} B_{11} + \overset{4}{b_{12}} B_{12} + \overset{0}{b_{13}} B_{13} = x^2 \det \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} - 4 \det \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} = x^4 + 4.$$

B_{ij} = cofactor
of b_{ij}

$$\Rightarrow \det(AB) = 40 \Rightarrow (\det A)(\det B) = 40 \Rightarrow 20(x^4 + 4) = 40$$

$$\Rightarrow x^4 = -2 \Rightarrow x \text{ has no real solution.}$$