

EE6617: Tutorial 7

- 7.1** Consider a hypothesis testing problem with observation Y . We have two hypothesis $\mathcal{H}_0$ (null hypothesis) and $\mathcal{H}_1$ (alternative hypothesis).

$\mathcal{H}_0$: Y is of distribution $P_Y \sim e^{-2|y-1|}$

$\mathcal{H}_1$: Y is of distribution $Q_Y \sim e^{-\pi(y+1)^2}$.

We want to distinguish two distribution types.

- (a) Write down the ML detector in the form of a log-likelihood-ratio test.
 - (b) Find out the decision for observation $Y = 0.5$.
 - (c) Find out the decision for observation $Y = -0.5$.
 - (d) Find out the decision for observation $Y = -3$.
- 7.2** Suppose X takes values in $\{0, 1\}$ and $P_X(0) = 1/3$ and $P_X(1) = 2/3$. The conditional probability of $P_{Y|X}(y|x)$ is Laplacian distributed with probability density function $p_{Y|X}(y|x) = \frac{1}{2} \exp(-|y - (-1)^x|)$.
- (a) Find the maximum a posteriori probability (MAP) detector as a function of y .
 - (b) Find the maximum likelihood (ML) detector as a function of y .
 - (c) Find out the mis-detection probability of the MAP detector.
 - (d) Find out the error probability of the MAP detector.

- 7.3** Consider the following inference problem specified by the joint probability mass function of X and Y

P(X,Y)	X=0	X = 1
Y=0	$\frac{5}{36}$	$\frac{5}{18}$
Y=1	$\frac{4}{18}$	$\frac{7}{36}$
Y=2	$\frac{1}{48}$	$\frac{1}{16}$
Y=3	$\frac{1}{16}$	$\frac{1}{48}$

- (a) Find $\hat{X}_{MAP}(Y)$ (i.e., MAP detector of X given the observation Y).
- (b) What is the corresponding error probability P_e of the MAP detector?

Sol. 7.1 (a) The ML detector is

$$\frac{P(y|\mathcal{H}_1)}{P(y|\mathcal{H}_0)} = \frac{e^{-\pi(y+1)^2}}{e^{-2|y-1|}} = e^{2|y-1| - \pi(y+1)^2} \stackrel{1}{\geq}_0 1$$

$$\text{take } \ln(\cdot) \Rightarrow 2|y-1| - \pi(y+1)^2 \stackrel{1}{\geq}_0 0.$$

(b) For $Y = 0.5 \Rightarrow 2(0.5) - \pi(\frac{9}{4}) < 0$, decides $\mathcal{H}_0$.

(c) For $Y = -0.5 \Rightarrow 2(1.5) - \pi(\frac{1}{4}) > 0$, decides $\mathcal{H}_1$.

(d) For $Y = -3 \Rightarrow 2(4) - \pi(4) < 0$, decides $\mathcal{H}_0$.

Sol. 7.2 (a) Given $p_{Y|X}(y|x) = \frac{1}{2} \exp(-|y - (-1)^x|)$, we formulate the hypothesis testing

$$\mathcal{H}_0 : p(y|x=0) = \frac{1}{2} \exp(-|y-1|)$$

$$\mathcal{H}_1 : p(y|x=1) = \frac{1}{2} \exp(-|y+1|).$$

$$L(y) = \frac{p(y|x=0)}{p(y|x=1)} = \exp(-|y-1| + |y+1|) \stackrel{0}{\geq}_1 2$$

$$\Rightarrow -|y-1| + |y+1| \stackrel{0}{\geq}_1 \ln(2).$$

We need to divide the decision region

$$y < -1 \Rightarrow y-1 - y-1 = -2 \stackrel{0}{\geq}_1 \ln(2) \Rightarrow \hat{x} = 1$$

$$-1 < y < 1 \Rightarrow y-1 + y+1 = 2y \stackrel{0}{\geq}_1 \ln(2) \Rightarrow y \stackrel{0}{\geq}_1 \ln(2)/2$$

$$1 < y \Rightarrow -y+1 + y+1 = 2 \stackrel{0}{\geq}_1 \ln(2) \Rightarrow \hat{x} = 0$$

(b) Since we assume $P_X(0) = P_X(1) = 1/2$,

$$-|y-1| + |y+1| \stackrel{0}{\geq}_1 \ln(1) = 0.$$

Similarly,

$$y < -1 \Rightarrow y-1 - y-1 = -2 \stackrel{0}{\geq}_1 0 \Rightarrow \hat{x} = 1$$

$$-1 < y < 1 \Rightarrow y-1 + y+1 = 2y \stackrel{0}{\geq}_1 0 \Rightarrow y \stackrel{0}{\geq}_1 0$$

$$y > 1 \Rightarrow -y+1 + y+1 = 2 \stackrel{0}{\geq}_1 0 \Rightarrow \hat{x} = 0$$

(c) P_{MD} of MAP is

$$P(\hat{x} = 0|x=1) = p(y > \ln(2)/2|x=1)$$

$$= \int_{\ln(2)/2}^{\infty} \frac{1}{2} \exp(-|y+1|) dy$$

$$= \frac{1}{2} e^{-1} \int_{\ln(2)/2}^{\infty} e^{-y} dy = \frac{e^{-1}}{2} (-e^{-y}) \Big|_{\ln(2)/2}^{\infty} = e^{-1}/2\sqrt{2}.$$

(1)

(d) P_{FA} of MAP is

$$\begin{aligned}
P(\hat{x} = 1|x = 0) &= p(y < \ln(2)/2|x = 0) \\
&= \int_{-\infty}^{\ln(2)/2} \frac{1}{2} \exp(-|y - 1|) dy \\
&= \frac{e^{-1}}{2} \int_{-\infty}^{\ln(2)/2} e^y dy = \frac{e^{-1}}{2} (e^y) \Big|_{-\infty}^{\ln(2)/2} = e^{-1}/\sqrt{2}.
\end{aligned}$$

Thus,

$$P_e = P(\hat{x} = 1|x = 0)P_X(x = 0) + P(\hat{x} = 0|x = 1)P_X(x = 1) = e^{-1} \frac{1}{3\sqrt{2}} + e^{-1} \frac{1}{3\sqrt{2}} = e^{-1} \frac{\sqrt{2}}{3}.$$

Sol. 7.3 (a) The MAP detector determines $X \in \{0, 1\}$ such that

$$\hat{X}_{MAP}(Y) = \operatorname{argmax}_{X \in \{0,1\}} P(X|Y) = \operatorname{argmax}_{X \in \{0,1\}} \frac{P(X, Y)}{P(Y)} = \operatorname{argmax}_{X \in \{0,1\}} P(X, Y).$$

For instance of $Y = 0$, the detector decides

$$\hat{X}_{MAP}(Y = 0) = \operatorname{argmax}_{X \in \{0,1\}} P(X, Y = 0) = \operatorname{argmax}_X \left\{ \frac{5}{36}_{X=0}, \frac{5}{18}_{X=1} \right\} = 1.$$

Following the same criterion, the detector decides $\hat{X}_{MAP}(Y = 1) = 0$, $\hat{X}_{MAP}(Y = 2) = 1$, $\hat{X}_{MAP}(Y = 3) = 0$.

(b) $P_e = P(X = 0, Y = 0) + P(X = 1, Y = 1) + P(X = 0, Y = 2) + P(X = 1, Y = 3) = 5/36 + 7/36 + 1/48 + 1/48 = 12/36 + 2/48 = 18/48 = 3/8$.