

# L07: Power and Velocities in EM Waves

EE3109 Applied Electromagnetics

Lecture Notes

Dr. Alex M. H. Wong

Jan – Apr 2025

# Review and Preview

- Up till now, we talked about
  - Plane waves and their properties
  - Plane wave propagation, reflection and transmission
- This class we will examine
  - Power transfer in plane waves
  - Phase and group velocities

# Outline

- Power in plane waves
- Velocities in plane waves

# Poynting Vector

Energy is transported through space to distant receiving points by electromagnetic waves.

Maxwell:

$$\nabla \times \mathbf{E}(t) = -\frac{\partial \mathbf{B}(t)}{\partial t} \quad \nabla \times \mathbf{H}(t) = \frac{\partial \mathbf{D}(t)}{\partial t}$$

Consider: ( $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{B}$  and  $\mathbf{D}$  are dependent on time)

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \\ &= -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J} \\ &= -\mathbf{H} \cdot \frac{\partial(\mu \mathbf{H})}{\partial t} - \mathbf{E} \cdot \frac{\partial(\epsilon \mathbf{E})}{\partial t} - \mathbf{E} \cdot (\sigma \mathbf{E}) \\ &= -\frac{\partial}{\partial t} \left( \frac{1}{2} \mu \mathbf{H}^2 \right) - \frac{\partial}{\partial t} \left( \frac{1}{2} \epsilon \mathbf{E}^2 \right) - \sigma \mathbf{E}^2 \end{aligned}$$

Integrate over a volume:

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dV$$
$$= - \underbrace{\frac{\partial}{\partial t} \int_V \left( \frac{1}{2} \mu \mathbf{H}^2 + \frac{1}{2} \epsilon \mathbf{E}^2 \right) dV}_{\text{Time rate of change of the energy stored in the electric and magnetic fields}} - \underbrace{\int_V \sigma \mathbf{E}^2 dV}_{\text{Power dissipated in the volume as a result of the flow of conduction current}}$$

Poynting Theorem: The term  $\oint_S \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \cdot d\mathbf{s}$  represents the power leaving the enclosed volume through its surface S.

Poynting Vector:

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \text{ (W/m}^2\text{)}$$

# Power Density

Poynting vector:

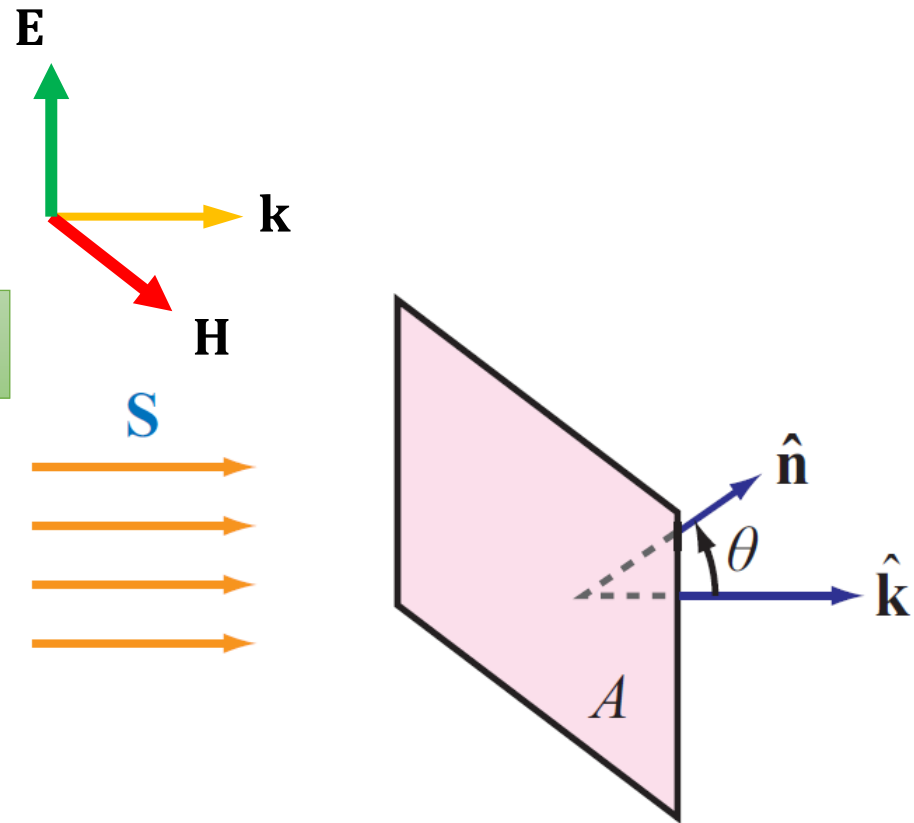
$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \quad (\text{W/m}^2)$$

Total power intercepted by A:

$$P(\mathbf{r}, t) = \int_A \mathbf{S}(\mathbf{r}, t) \cdot \hat{\mathbf{n}} \, dA \quad (\text{W})$$

Time-average power density:

$$\mathbf{s}_{av} = \frac{1}{T} \int_{t_0}^{t_0+T} \mathbf{S}(\mathbf{r}, t) \, dt = \frac{1}{2} \text{Re}\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*\} \quad (\text{W/m}^2)$$



# Power Density Carried by Plane Wave

For a plane wave with **E** field :

$$\begin{aligned}\tilde{\mathbf{E}}(z) &= \hat{\mathbf{x}} \tilde{E}_x(z) + \hat{\mathbf{y}} \tilde{E}_y(z) \\ &= (\hat{\mathbf{x}} E_{x0} + \hat{\mathbf{y}} E_{y0})e^{-jkz},\end{aligned}$$

the average power density carried by the wave is:

$$\begin{aligned}\mathbf{S}_{\text{av}} &= \hat{\mathbf{z}} \frac{1}{2\eta} (|E_{x0}|^2 + |E_{y0}|^2) \\ &= \hat{\mathbf{z}} \frac{|\tilde{\mathbf{E}}|^2}{2\eta} \quad (\text{W/m}^2),\end{aligned}$$

# Plane Wave in Lossy Medium

For a plane wave travelling in a lossy medium:

$$\begin{aligned}\tilde{\mathbf{E}}(z) &= \hat{\mathbf{x}} \tilde{E}_x(z) + \hat{\mathbf{y}} \tilde{E}_y(z) \\ &= (\hat{\mathbf{x}} E_{x0} + \hat{\mathbf{y}} E_{y0}) e^{-\alpha z} e^{-j\beta z}, \\ \tilde{\mathbf{H}}(z) &= \frac{1}{\eta_c} (-\hat{\mathbf{x}} E_{y0} + \hat{\mathbf{y}} E_{x0}) e^{-\alpha z} e^{-j\beta z},\end{aligned}$$

the power density is :

$$\begin{aligned}\mathbf{S}_{\text{av}}(z) &= \frac{1}{2} \Re \left[ \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \right] \\ &= \frac{\hat{\mathbf{z}} (|E_{x0}|^2 + |E_{y0}|^2)}{2} e^{-2\alpha z} \Re \left( \frac{1}{\eta_c^*} \right).\end{aligned}$$

$$\mathbf{S}_{\text{av}}(z) = \hat{\mathbf{z}} \frac{|\tilde{E}(0)|^2}{2|\eta_c|} e^{-2\alpha z} \cos \theta_\eta \quad (\text{W/m}^2)$$

where  $\eta_c = |\eta_c| e^{j\theta_\eta}$

Whereas  $\mathbf{E}$  and  $\mathbf{H}$  decay with  $z$  as  $e^{-\alpha z}$ ,  $\mathbf{S}$  decay with  $z$  as  $e^{-2\alpha z}$ .

### Example 7-6: Power Received by a Submarine Antenna

A submarine at a depth of 200 m below the sea surface uses a wire antenna to receive signal transmissions at 1 kHz. Determine the power density incident upon the submarine antenna due to the EM wave of Example 7-4.

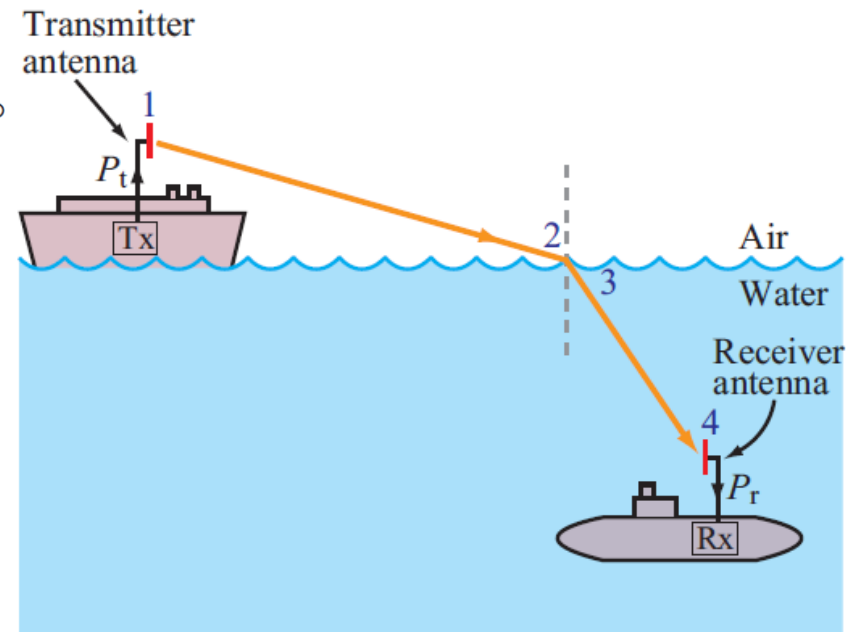
$$\left( \begin{array}{l} |E(z=0)| = 4.44 [\text{mV/m}]; \\ \alpha = 0.126 [\text{Np/m}]; \\ \eta_c = 0.044 \angle 45^\circ [\Omega] \end{array} \right)$$

**Solution:** From Example 7-4,  $|\vec{E}(0)| = |E_{x0}| = 4.44$  (mV/m),  $\alpha = 0.126$  (Np/m), and  $\eta_c = 0.044 \angle 45^\circ$  ( $\Omega$ ). Application of Eq. (7.109) gives

$$\begin{aligned} \mathbf{S}_{\text{av}}(z) &= \hat{\mathbf{z}} \frac{|E_0|^2}{2|\eta_c|} e^{-2\alpha z} \cos \theta_\eta \\ &= \hat{\mathbf{z}} \frac{(4.44 \times 10^{-3})^2}{2 \times 0.044} e^{-0.252z} \cos 45^\circ \\ &= \hat{\mathbf{z}} 0.16 e^{-0.252z} \quad (\text{mW/m}^2). \end{aligned}$$

At  $z = 200$  m, the incident power density is

$$\begin{aligned} \mathbf{S}_{\text{av}} &= \hat{\mathbf{z}} (0.16 \times 10^{-3} e^{-0.252 \times 200}) \\ &= 2.1 \times 10^{-26} \quad (\text{W/m}^2). \end{aligned}$$



### Example 8-6: Wave Incident Obliquely on a Soil Surface

Using the coordinate system of Fig. 8-15, a plane wave radiated by a distant antenna is incident in air upon a plane soil surface located at  $z = 0$ . The electric field of the incident wave is given by

$$\mathbf{E}^i = \hat{\mathbf{y}}100 \cos(\omega t - \pi x - 1.73\pi z) \quad (\text{V/m}), \quad (8.61)$$

and the soil medium may be assumed to be a lossless dielectric with a relative permittivity of 4.

- (a) Determine  $k_1$ ,  $k_2$ , and the incidence angle  $\theta_i$ .
- (b) Obtain expressions for the total electric fields in air and in the soil.
- (c) Determine the average power density carried by the wave traveling in soil.

(Space for working on this question)

Plane soil surface at  $z = 0$ .

$$E^i = \hat{\mathbf{y}} 100 \cos(\omega t - \pi x - 1.73\pi z) \quad (\text{V/m})$$

(a) Determine  $k_1, k_2$  and  $\theta_i$

(Space for working on this question)

Plane soil surface at  $z = 0$ .

$$E^i = \hat{y} 100 \cos(\omega t - \pi x - 1.73\pi z) \quad (\text{V/m})$$

(b) Obtain expressions of the total electric fields in air and in the soil.

(Space for working on this question)

Plane soil surface at  $z = 0$ .

$$E^i = \hat{y} 100 \cos(\omega t - \pi x - 1.73\pi z) \quad (\text{V/m})$$

(c) Determine the average power density carried by the wave in the soil.

## Example 8-6 (cont.)

First, we write the electric field phasor, which is

$$\mathbf{E}^i = 100e^{-j(\pi x + 1.73\pi z)} \hat{\mathbf{y}} \text{ [V/m]} .$$

(a) The indices are:  $n_1 = 1$  (air);  $n_2 = \sqrt{\epsilon_{r2}} = 2$ .

From the electric field, we see that

$$-\mathbf{k}^i \cdot \mathbf{r} = -\pi x - 1.73\pi z \Rightarrow \mathbf{k}^i = \pi \hat{\mathbf{x}} + 1.73\pi \hat{\mathbf{z}}$$

$$\Rightarrow k_{1x} = \pi; k_{1z} = 1.73\pi; k_1 = \sqrt{k_{1x}^2 + k_{1z}^2} = 2\pi.$$

The incidence angle is  $\tan \theta_i = \frac{k_{1x}}{k_{1z}} \Rightarrow \theta_1 = \theta_i = 30^\circ$

The wave number in the soil is:  $\frac{k_2}{k_1} = \frac{n_2}{n_1} = 2 \Rightarrow k_2 = 2k_1 = 4\pi$ .

## Example 8-6 (cont.)

(b) The components of  $\mathbf{k}_2$  are:

$$k_{2x} = k_{1x} = \pi \text{ [rad/m]}; \quad k_{2z} = \sqrt{k_2^2 - k_{2x}^2} = \sqrt{15}\pi = 3.87\pi \text{ [rad/m]}.$$

The transmitted wave's angle is  $\tan \theta_2 = \frac{k_{2x}}{k_{2z}} \Rightarrow \theta_2 = 14.48^\circ$

Further, the intrinsic impedances of the two media are:

$$\eta_1 = \eta_0 = 120\pi; \quad \eta_2 = \frac{\eta_0}{\sqrt{\epsilon_{r2}}} = \frac{\eta_1}{2} = 60\pi.$$

We note that this is a case of TE wave incidence.

$$\Gamma_{\text{TE}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_2}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_2} = \frac{(0.5)(0.8660) - (1)(0.9681)}{(0.5)(0.8660) + (1)(0.9681)} = -0.382$$

$$\tau_{\text{TE}} = 1 + \Gamma_{\text{TE}} = 0.618$$

We can now write the reflected and transmitted electric fields as

$$\begin{aligned} \mathbf{E}^r &= \Gamma_{\text{TE}}(100)e^{-j(\pi x - 1.73\pi z)} \hat{\mathbf{y}} = -38.2e^{-j(\pi x - 1.73\pi z)} \hat{\mathbf{y}} \text{ [V/m]}; \\ \mathbf{E}^t &= \tau_{\text{TE}}(100)e^{-j(k_{2x}x + k_{2z}z)} \hat{\mathbf{y}} = 61.8e^{-j(\pi x + 3.87\pi z)} \hat{\mathbf{y}} \text{ [V/m]}. \end{aligned}$$

## Example 8-6 (cont.)

(b) Cont'd

The total electric field in Medium 1 (in phasor form) is:

$$\mathbf{E}_1 = \mathbf{E}^i + \mathbf{E}^r = 100e^{-j(\pi x + 1.73\pi z)} \hat{\mathbf{y}} - 38e^{-j(\pi x - 1.73\pi z)} \hat{\mathbf{y}} \text{ [V/m]} .$$

The total electric field in Medium 2 (in phasor form) is:

$$\mathbf{E}_2 = \mathbf{E}^t = 61.8e^{-j(\pi x + 3.87\pi z)} \hat{\mathbf{y}} \text{ [V/m]} .$$

If it is necessary, we can convert into time domain:

$$\begin{aligned} \mathbf{E}_1(\mathbf{r}, t) &= 100 \cos(\omega t - \pi x - 1.73\pi z) \hat{\mathbf{y}} \\ &\quad - 38 \cos(\omega t - \pi x + 1.73\pi z) \hat{\mathbf{y}} \text{ [V/m]} , \\ \mathbf{E}_2(\mathbf{r}, t) &= 61.8 \cos(\omega t - \pi x - 3.87\pi z) \hat{\mathbf{y}} \text{ [V/m]} . \end{aligned}$$

(c) The power density carried by the transmitted wave is

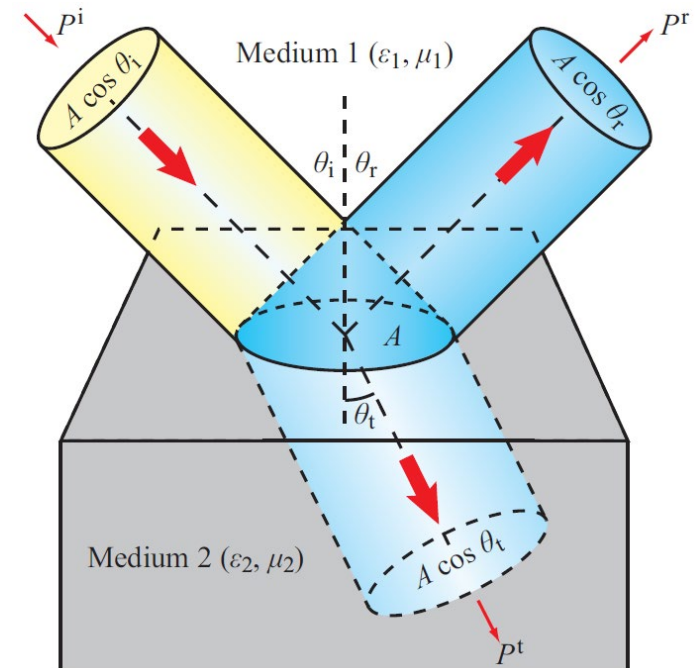
$$S_{av}^t = \frac{|E_0^t|^2}{2\eta_2} = \frac{(61.8)^2}{2(60\pi)} = 10.1 \text{ [W/m}^2\text{]} .$$

# Power Reflectivity and Transmissivity

$$R_{\perp} = \frac{P_{\perp}^r}{P_{\perp}^i} = |r_{\perp}|^2 \quad ; \quad R_{\parallel} = \frac{P_{\parallel}^r}{P_{\parallel}^i} = |r_{\parallel}|^2$$

$$\begin{aligned} T_{\perp} &= \frac{P_{\perp}^t}{P_{\perp}^i} = \frac{|E_{\perp 0}^t|^2}{|E_{\perp 0}^i|^2} \frac{\eta_1}{\eta_2} \frac{A \cos \theta_t}{A \cos \theta_i} \\ &= |\tau_{\perp}|^2 \left( \frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i} \right), \end{aligned} \quad (8.79a)$$

$$T_{\parallel} = \frac{P_{\parallel}^t}{P_{\parallel}^i} = |\tau_{\parallel}|^2 \left( \frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i} \right). \quad (8.79b)$$



*The incident, reflected, and transmitted waves do not have to obey any such laws as conservation of electric field, conservation of magnetic field, or conservation of power density, but they do have to obey the law of conservation of power.*

# Summary For Reflection and Transmission

**Table 8-2:** Expressions for  $\Gamma$ ,  $\tau$ ,  $R$ , and  $T$  for wave incidence from a medium with intrinsic impedance  $\eta_1$  onto a medium with intrinsic impedance  $\eta_2$ . Angles  $\theta_i$  and  $\theta_t$  are the angles of incidence and transmission, respectively.

| Property                                                                                                                                                                                                                                 | Normal Incidence<br>$\theta_i = \theta_t = 0$       | Perpendicular Polarization                                                                                         | Parallel Polarization                                                                                                  |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------|--------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------|
| Reflection coefficient                                                                                                                                                                                                                   | $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$  | $\Gamma_{\perp} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$ | $\Gamma_{\parallel} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$ |
| Transmission coefficient                                                                                                                                                                                                                 | $\tau = \frac{2\eta_2}{\eta_2 + \eta_1}$            | $\tau_{\perp} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$                         | $\tau_{\parallel} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$                         |
| Relation of $\Gamma$ to $\tau$                                                                                                                                                                                                           | $\tau = 1 + \Gamma$                                 | $\tau_{\perp} = 1 + \Gamma_{\perp}$                                                                                | $\tau_{\parallel} = (1 + \Gamma_{\parallel}) \frac{\cos \theta_i}{\cos \theta_t}$                                      |
| Reflectivity                                                                                                                                                                                                                             | $R =  \Gamma ^2$                                    | $R_{\perp} =  \Gamma_{\perp} ^2$                                                                                   | $R_{\parallel} =  \Gamma_{\parallel} ^2$                                                                               |
| Transmissivity                                                                                                                                                                                                                           | $T =  \tau ^2 \left( \frac{\eta_1}{\eta_2} \right)$ | $T_{\perp} =  \tau_{\perp} ^2 \frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i}$                                   | $T_{\parallel} =  \tau_{\parallel} ^2 \frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i}$                               |
| Relation of $R$ to $T$                                                                                                                                                                                                                   | $T = 1 - R$                                         | $T_{\perp} = 1 - R_{\perp}$                                                                                        | $T_{\parallel} = 1 - R_{\parallel}$                                                                                    |
| Notes: (1) $\sin \theta_t = \sqrt{\mu_1 \epsilon_1 / \mu_2 \epsilon_2} \sin \theta_i$ ; (2) $\eta_1 = \sqrt{\mu_1 / \epsilon_1}$ ; (3) $\eta_2 = \sqrt{\mu_2 / \epsilon_2}$ ; (4) for nonmagnetic media, $\eta_2 / \eta_1 = n_1 / n_2$ . |                                                     |                                                                                                                    |                                                                                                                        |

# Outline

- Power in plane waves
- Velocities in plane waves

# Phase and Group Velocities

## 1. Phase Velocity

The phase velocity is defined as the velocity of the sinusoidal pattern of the wave

$$v_p = \frac{\omega}{k}$$

## 2. Group Velocity

The velocity with which the **envelope**—or equivalently the wave group—travels through the medium is called the **group velocity**  $v_g$ . As such,  $v_g$  is the velocity of the energy carried by the wave-group, and of the information encoded in it. Depending on whether or not the propagation medium is dispersive,  $v_g$  may or may not be equal to the phase velocity  $v_p$ .

$$v_g = \left( \frac{dk}{d\omega} \right)^{-1}$$

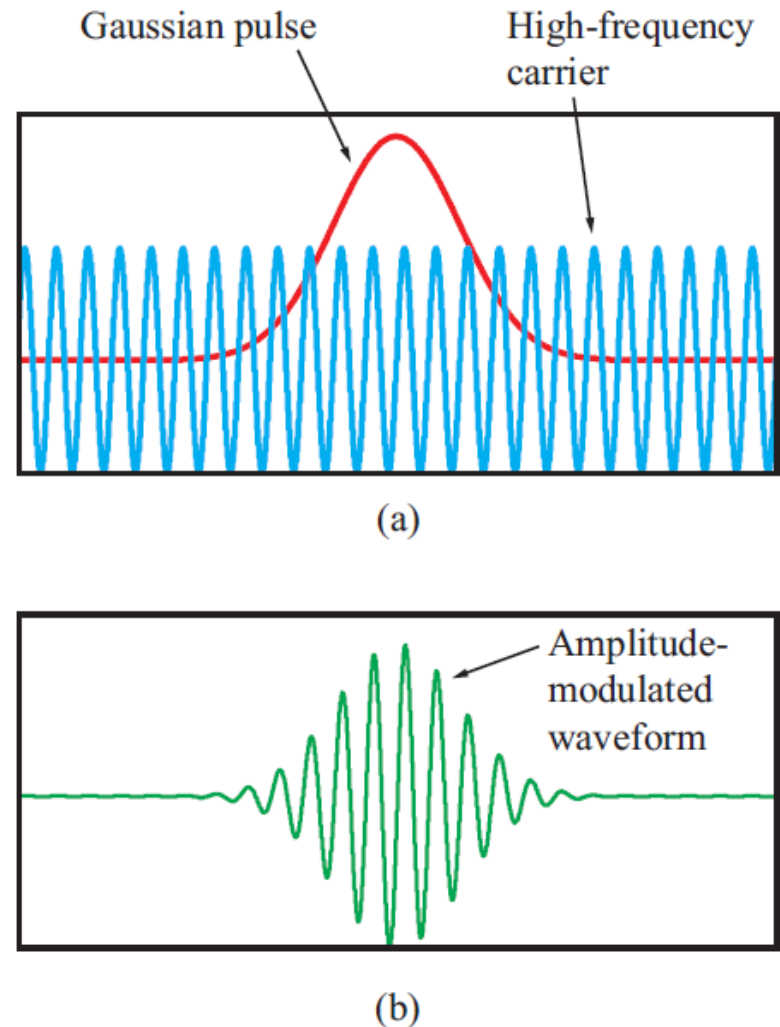


**Figure 8-25:** The amplitude-modulated high-frequency waveform in (b) is the product of the Gaussian-shaped pulse with the sinusoidal high-frequency carrier in (a).

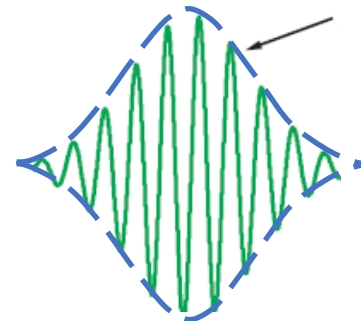
# Why is the group velocity $v_g = \left(\frac{dk}{d\omega}\right)^{-1}$ ?

Consider 2 travelling waves, where  $\Delta\omega \ll \omega_0$ :

$$\begin{aligned} E(z, t) &= E_0 \cos[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z] \\ &\quad + E_0 \cos[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z] \\ &= 2E_0 \cos(t\Delta\omega - z\Delta\beta) \cos(\omega_0 t - \beta_0 z) \end{aligned}$$

The wave inside the envelope propagates with phase velocity :

$$\begin{aligned} \omega_0 t - \beta_0 z &= \text{Constant} \\ v_p &= \frac{dz}{dt} = \frac{\omega_0}{\beta_0} \end{aligned}$$



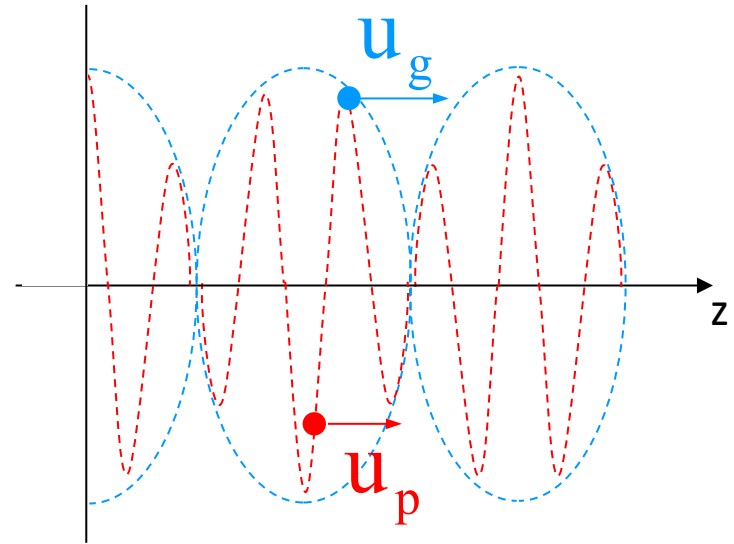
Fast varying carrier;  
Slow varying envelope

velocity of the envelope is determined by:

$$t\Delta\omega - z\Delta\beta = \text{Constant}$$

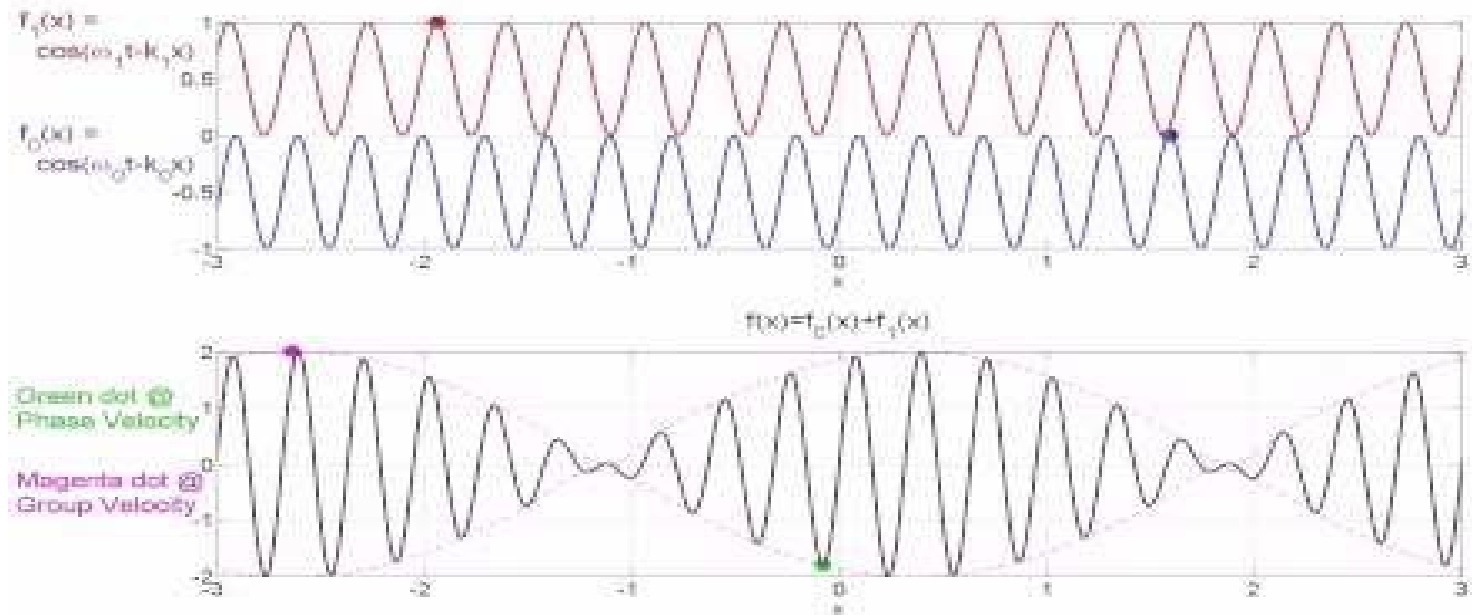
$$v_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta}$$

$$\text{As } \Delta\omega \rightarrow 0, v_g = \frac{dz}{dt} = \left(\frac{d\beta}{d\omega}\right)^{-1}$$



⇒ The group velocity is the velocity of the envelope of the wave packet

# Visualizing Phase and Group Velocities



<https://youtu.be/tlM9vq-bepA>

# Summary

- In this class, we looked at
  - The Poynting vector and how power is transferred by electromagnetic waves
  - The phase and group velocities of an electromagnetic wave