

# Unit 1

## Sets

# The Barber Paradox

- The barber is a man in town who shaves those and only those men who do not shave themselves.
- *Q:* Who shaves the barber?
- (1-min video)  
[https://www.youtube.com/watch?v=qQs2ZHV\\_WBk](https://www.youtube.com/watch?v=qQs2ZHV_WBk)



# Outline of Unit 1

- 1.1 Basic Concepts
- 1.2 Set Identities
- 1.3 Proving Set Equalities
- 1.4 Russell's Paradox

# Unit 1.1

## Basic Concepts

# Set

- ❑ A set is a **collection of objects**.
- ❑ Suppose the set  $A$  contains an object called  $x$ .
  - $x$  is an **element** (or **member**) of  $A$ .
  - $x$  **belongs** to  $A$  (or  $x$  is in  $A$ ), denoted by  $x \in A$ .
- ❑ The **roster notation** of a set simply lists all members of the set inside braces  $\{ \}$ .
- ❑ Example:

$C = \{10\text{¢}, 20\text{¢}, 50\text{¢}, \$1, \$2, \$5, \$10\}$



- The order is not important.
- The same element needs not appear more than once.  
(Duplicate elements are redundant and can be removed.)

# Set-Builder Notation

- We can “build” a set by describing what properties its members have.

- **Set-builder notation:**

$$T = \{ x \in S \mid P(x) \}.$$

such that

- **Example:**

- Let  $C = \{10\text{¢}, 20\text{¢}, 50\text{¢}, \$1, \$2, \$5, \$10\}$ .
- $D$  is the set whose elements are all elements  $x \in C$  such that  $x$  is completely bronze in color.
- $D = \{x \in C \mid x \text{ is completely bronze in color.}\}$   
 $= \{10\text{¢}, 20\text{¢}, 50\text{¢}\}$



# Cardinality

- ❑ The **cardinality** of a set  $A$  is defined as the number of elements in the set.
- ❑ It is denoted by  $|A|$ .
  - If  $|A|$  is finite,  $A$  is called a **finite** set.
  - Otherwise,  $A$  is called an **infinite** set.
- ❑ Example:

$$C = \{10\text{¢}, 20\text{¢}, 50\text{¢}, \$1, \$2, \$5, \$10\}$$

$$|C| = 7.$$



# Subset

- ❑  $A$  is a **subset** of  $B$ , written as  $A \subseteq B$ , if every member of  $A$  is also a member of  $B$ .
- ❑  $B$  is then said to be a **superset** of  $A$ .
- ❑ A subset  $A$  of  $B$  is called a **proper subset** of  $B$  if  $B$  contains some elements that are not in  $A$ .
  - i.e.,  $A$  is not the same as  $B$ .
- ❑ Example:
  - The set of all women is a proper subset of the set of all human beings.

# Proof of Subset Relationship

□ To prove a subset relationship, the **element argument** is usually used.

□ To prove  $A \subseteq B$ ,

- 1) Suppose  $x$  is an arbitrarily chosen element of  $A$ .
- 2) Show that  $x$  is also an element of  $B$ .
- 3) Therefore (by Universal Generalization),  $A \subseteq B$ .

o Universal Generalization (UG) has been discussed in EE1001. We will ignore this concept in this course.

# Example

□ Prove that  $A \subseteq B$ , where

$$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$$

$$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}.$$

□ Proof:

- Let  $x$  be an element of  $A$ , so there is an integer  $r$  such that

$$x = 6r + 12 = 3(2r + 4).$$

- Since  $2r + 4$  is an integer, by definition of  $B$ ,  $x$  is an element of  $B$ .
- Therefore,  $A \subseteq B$ . *Q.E.D.*

# Set Equality

- Two sets are the **same** (or **equal**) if and only if
  - they contain the same elements, or equivalently,
  - each is a subset of the other.

$$A = B \iff A \subseteq B \text{ and } B \subseteq A.$$

## □ Example

- Suppose

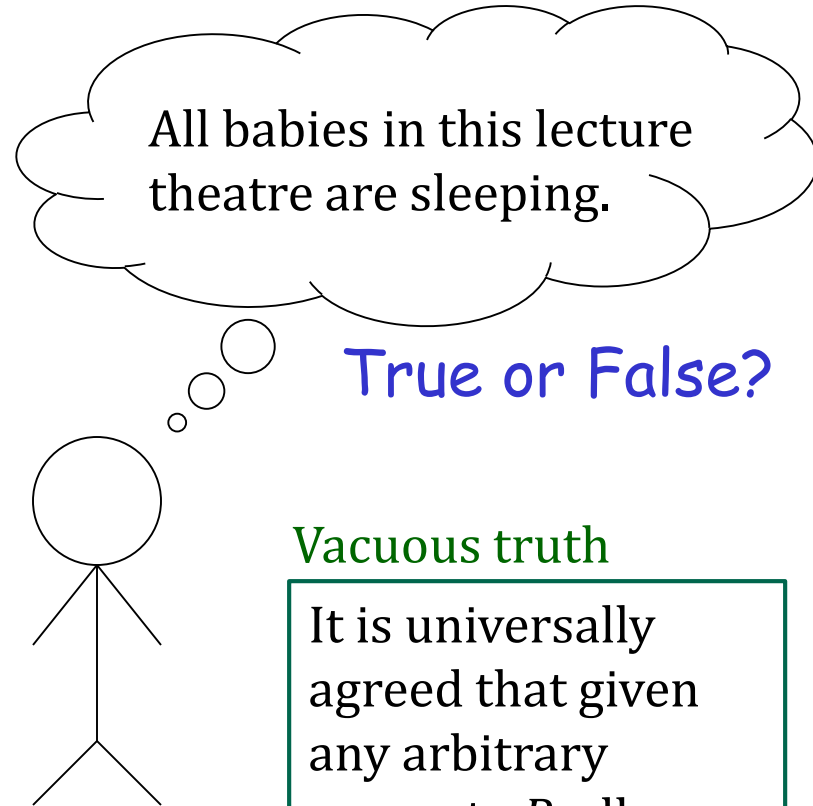
$$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$$

$$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

- Are they equal?

# The Empty Set

- ❑ A set is **empty** if it contains no elements at all.
- ❑ There is only one empty set.
  - If two sets are empty, each set is a subset of the other one, so they are the same set.
- ❑ We denote it by  $\emptyset$ .
- ❑ Remark:
  - The empty set  $\emptyset$  is different from the set containing  $\emptyset$ .
    - i.e.,  $\emptyset \neq \{\emptyset\}$ .

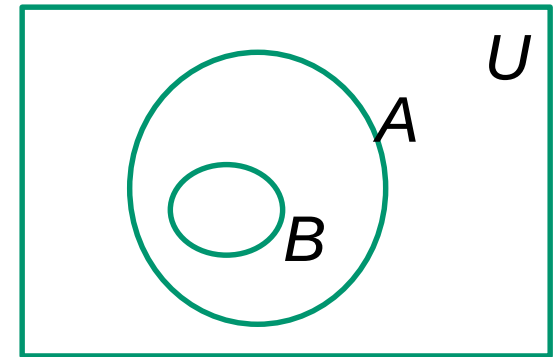


## Vacuous truth

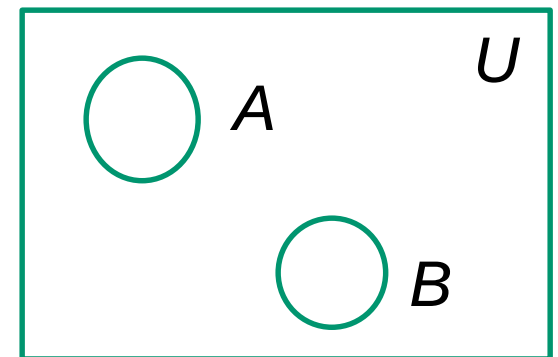
It is universally agreed that given any arbitrary property  $P$ , all elements of  $\emptyset$  have property  $P$ .

# Relationship between Sets

- A **universal** set  $U$  is a set containing everything that we are considering.
- **Venn diagram**
  - $U$  is represented by a rectangular box.
  - Subsets of  $U$  (e. g.  $A$  and  $B$ ) are represented by circles (more precisely, regions inside closed curves).
- $A$  and  $B$  are **disjoint** if they have no elements in common.



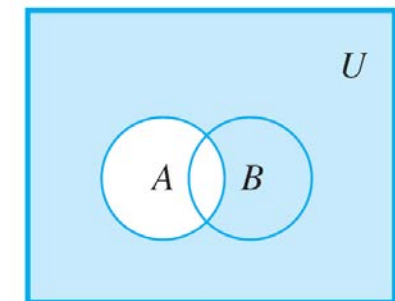
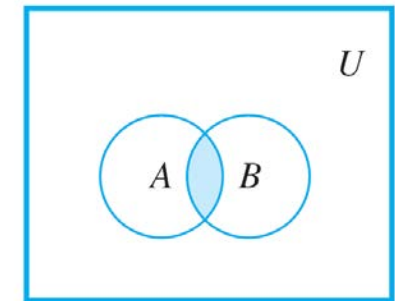
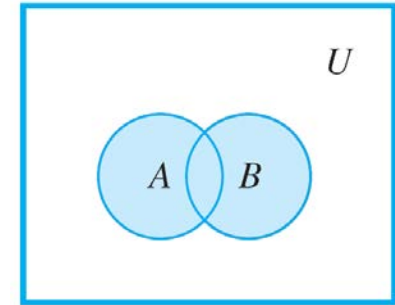
$B$  is a subset of  $A$ .



$A$  and  $B$  are disjoint.

# Three Fundamental Operations

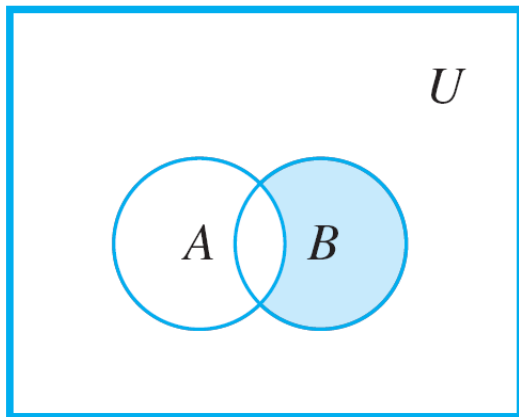
- ❑ The **union** of  $A$  and  $B$ , denoted by  $A \cup B$ , is the set of all elements that belong to **either  $A$  or  $B$ , or in both**.
- ❑ The **intersection** of  $A$  and  $B$ , denoted by  $A \cap B$ , is the set of all elements that are **in both  $A$  and  $B$** .
- ❑ The **complement** of  $A$ , denoted by  $A^c$  or  $\bar{A}$ , is the set of all elements in  $U$  that **do not belong to  $A$** .



# Set Difference

- The difference between B and A refers to the set of all elements that are in B but not in A:

$$B - A \triangleq B \cap A^c$$

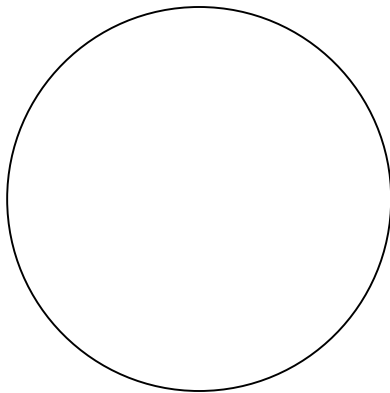


It means **subtract  $A$  from  $B$** .

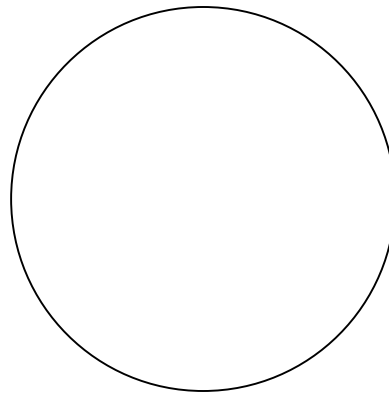
Also denoted as  $B \setminus A$ .

# Disjoint Sets

□  $A$  and  $B$  are **disjoint** iff they have no elements in common, i.e.,  $A \cap B = \emptyset$ .



Human



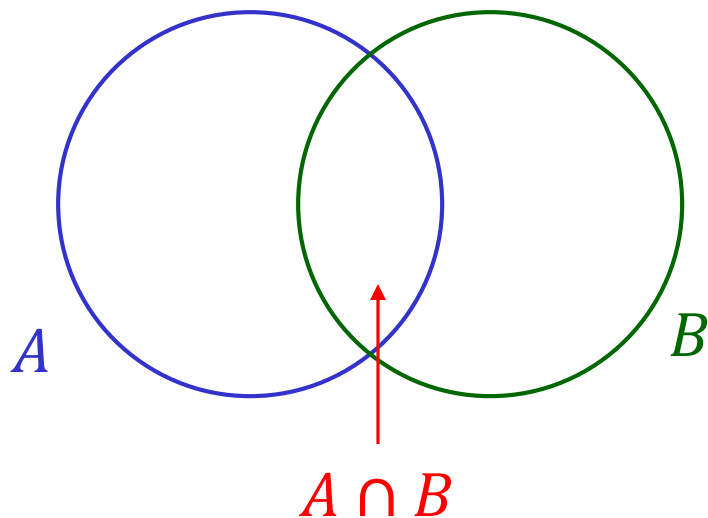
Fish



# Cardinality of Union

□ For finite sets  $A$  and  $B$ ,

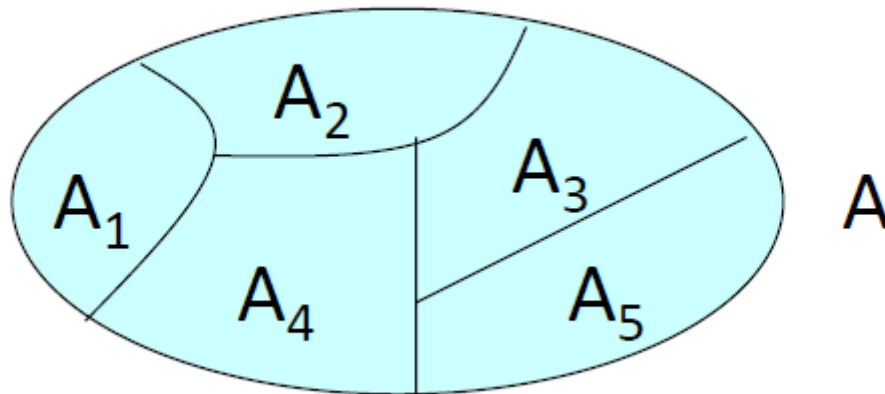
$$|A \cup B| = |A| + |B| - |A \cap B|$$



When adding  $|A|$  and  $|B|$ ,  $A \cap B$  has been **counted twice**. That's why we need to subtract it.

# Partition

- A collection of non-empty sets  $\{A_1, A_2, \dots, A_n\}$  is a **partition** of a set  $A$  iff
- i.  $A = A_1 \cup A_2 \cup \dots \cup A_n$ , and
  - ii.  $A_1, A_2, \dots, A_n$  are **mutually disjoint**, i.e.,  
 $A_i \cap A_j = \emptyset$  for all  $i, j = 1, 2, \dots, n$  and  $i \neq j$ .



# Power Set

□ Given a set  $A$ , the set of all its subsets, denoted by  $\mathcal{P}(A)$ , is called the **power set** of  $A$ .

□ Example:

- Suppose  $A = \{1, 2, 3\}$ .

- List all subsets of  $A$ :

$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$  and  $\{1, 2, 3\}$ .

- Hence,

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

## Cardinality of Power Set

- Suppose  $|A| = n$ .
- What is  $|\mathcal{P}(A)|$ ?

$$A = \{1, 2, 3, \dots, n\}.$$

Subset: {

For each element, there are two possibilities:

- Put it into the subset, or not.

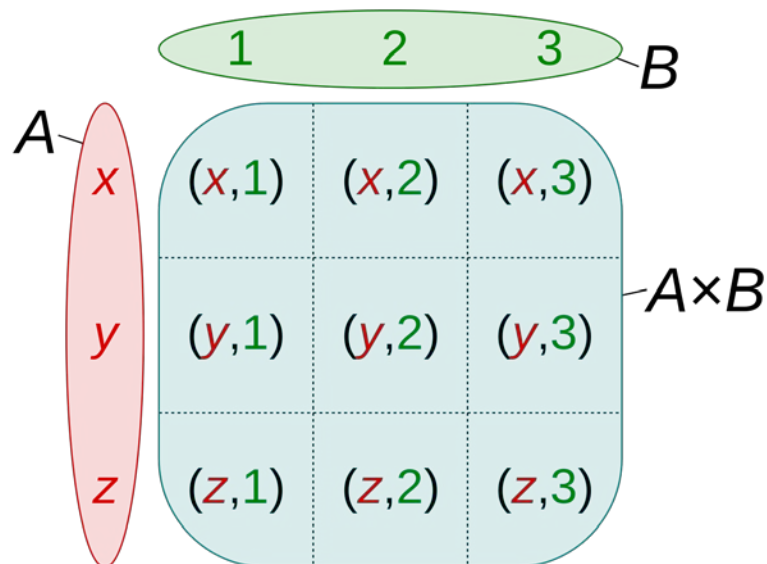
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# Cartesian Product

- The Cartesian product  $A \times B$  of the sets  $A$  and  $B$  is the set of all **ordered pairs**  $(a, b)$ , where  $a \in A$  and  $b \in B$ .

$$A \times B \triangleq \{(a, b) | a \in A \wedge b \in B\}.$$

- Example:



Ordered pair:

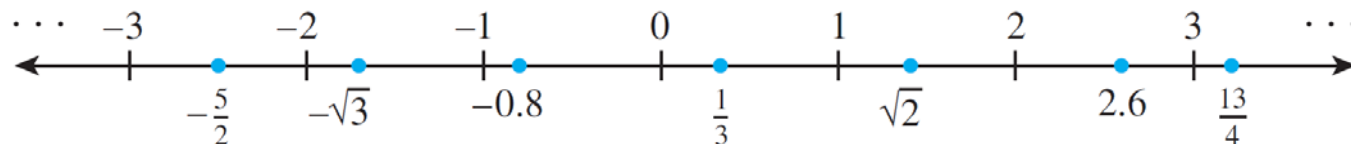
- The order is important:  
 $(a, b) \neq (b, a)$

What is  $|A \times B|$  ?

# Special Sets

- The following sets are commonly used in mathematics:

| Symbol   | Set                         |
|----------|-----------------------------|
| <b>R</b> | set of all real numbers     |
| <b>Z</b> | set of all integers         |
| <b>Q</b> | set of all rational numbers |



- The numbers represented by blue dots all belong to **R** but not to **Z**.
- **Z**<sup>+</sup> and **Z**<sup>-</sup> denote the sets of positive and negative integers, respectively.

# Unit 1.2

## Foundational Rules of Set Theory

# Foundational Rules

□ We have a list of set identities as follows:

Let all sets referred to below be subsets of a universal set  $U$ .

1. *Commutative Laws*: For all sets  $A$  and  $B$ ,

$$(a) A \cup B = B \cup A \quad \text{and} \quad (b) A \cap B = B \cap A.$$

2. *Associative Laws*: For all sets  $A$ ,  $B$ , and  $C$ ,

$$(a) (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$

$$(b) (A \cap B) \cap C = A \cap (B \cap C).$$

# Foundational Rules

3. *Distributive Laws*: For all sets,  $A$ ,  $B$ , and  $C$ ,

$$(a) A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$

$$(b) A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws*: For all sets  $A$ ,

$$(a) A \cup \emptyset = A \quad \text{and} \quad (b) A \cap U = A.$$

5. *Complement Laws*:

$$(a) A \cup A^c = U \quad \text{and} \quad (b) A \cap A^c = \emptyset.$$

6. *Double Complement Law*: For all sets  $A$ ,

$$(A^c)^c = A.$$

7. *Idempotent Laws*: For all sets  $A$ ,

$$(a) A \cup A = A \quad \text{and} \quad (b) A \cap A = A.$$

# Foundational Rules

8. *Universal Bound Laws*: For all sets  $A$ ,

$$(a) A \cup U = U \quad \text{and} \quad (b) A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws*: For all sets  $A$  and  $B$ ,

$$(a) (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (b) (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws*: For all sets  $A$  and  $B$ ,

$$(a) A \cup (A \cap B) = A \quad \text{and} \quad (b) A \cap (A \cup B) = A.$$

11. *Complements of  $U$  and  $\emptyset$* :

$$(a) U^c = \emptyset \quad \text{and} \quad (b) \emptyset^c = U.$$

12. *Set Difference Law*: For all sets  $A$  and  $B$ ,

$$A - B = A \cap B^c.$$

# How to Prove Them?

□ Two methods:

- 1) By definition of set operations and logical equivalence.
  - See the first example...
- 2) By membership table.
  - Similar to truth table in propositional logic.
  - Consider an arbitrary element.
  - For each set in consideration, we use 1 to indicate that the element is in the set and 0 to indicate that it is *not* in the set.
    - See the second example...

# Proof of DeMorgan's Law

Prove that  $\overline{A \cap B} = \bar{A} \cup \bar{B}$ .

Solution (Method 1):

Recall that  $\bar{A}$  is the same as  $A^c$ .

$$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$$

by definition of complement

$$= \{x \mid \neg(x \in (A \cap B))\}$$

by definition of does not belong symbol

$$= \{x \mid \neg(x \in A \wedge x \in B)\}$$

by definition of intersection

$$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$$

by the first De Morgan law for logical equivalences

$$= \{x \mid x \notin A \vee x \notin B\}$$

by definition of does not belong symbol

$$= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$$

by definition of complement

$$= \{x \mid x \in \bar{A} \cup \bar{B}\}$$

by definition of union

$$= \bar{A} \cup \bar{B}$$

by meaning of set builder notation

Q.E.D.

# Proof of Distributive Law

Prove that  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution (Method 2):

| TABLE 2 A Membership Table for the Distributive Property. |   |   |            |                     |            |            |                              |
|-----------------------------------------------------------|---|---|------------|---------------------|------------|------------|------------------------------|
| A                                                         | B | C | $B \cup C$ | $A \cap (B \cup C)$ | $A \cap B$ | $A \cap C$ | $(A \cap B) \cup (A \cap C)$ |
| 1                                                         | 1 | 1 | 1          | 1                   | 1          | 1          | 1                            |
| 1                                                         | 1 | 0 | 1          | 1                   | 1          | 0          | 1                            |
| 1                                                         | 0 | 1 | 1          | 1                   | 0          | 1          | 1                            |
| 1                                                         | 0 | 0 | 0          | 0                   | 0          | 0          | 0                            |
| 0                                                         | 1 | 1 | 1          | 0                   | 0          | 0          | 0                            |
| 0                                                         | 1 | 0 | 1          | 0                   | 0          | 0          | 0                            |
| 0                                                         | 0 | 1 | 1          | 0                   | 0          | 0          | 0                            |
| 0                                                         | 0 | 0 | 0          | 0                   | 0          | 0          | 0                            |



The identity holds because the two columns are identical. Q.E.D.

## Unit 1.3

### Proving Set Equalities

# How to Prove Set Equalities?

Three methods to prove that  $A = B$ .

- 1) To show that each set is a subset of the other, i.e.  $A \subseteq B$  and  $B \subseteq A$ .
  - Each of the two parts can be proved by the **element argument**.
- 2) Use the foundational rules (stated in the previous section) to construct an **algebraic proof**.
- 3) Use **membership table**.

## Example

Construct an algebraic proof that for all sets  $A$  and  $B$ ,

$$A - (A \cap B) = A - B.$$

Solution:

|                                        |                                   |
|----------------------------------------|-----------------------------------|
| $A - (A \cap B) = A \cap (A \cap B)^c$ | by the set difference law         |
| $= A \cap (A^c \cup B^c)$              | by De Morgan's laws               |
| $= (A \cap A^c) \cup (A \cap B^c)$     | by the distributive law           |
| $= \emptyset \cup (A \cap B^c)$        | by the complement law             |
| $= (A \cap B^c) \cup \emptyset$        | by the commutative law for $\cup$ |
| $= A \cap B^c$                         | by the identity law for $\cup$    |
| $= A - B$                              | by the set difference law.        |

## Exercise

Use membership table to prove that for all sets  $A$  and  $B$ ,

$$A - (A \cap B) = A - B.$$

Solution:

| $A$ | $B$ | $A \cap B$ | $A - A \cap B$ | $A - B$ |
|-----|-----|------------|----------------|---------|
|     |     |            |                |         |
|     |     |            |                |         |
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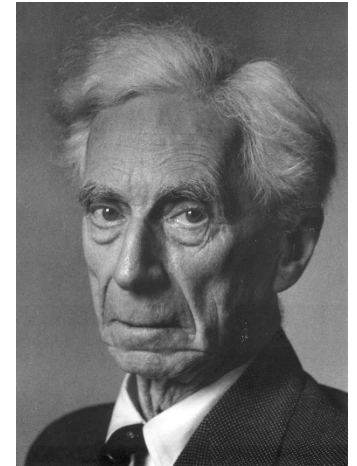
They are equal because the last two columns are identical.

## Unit 1.4

### Russell's Paradox

# Naïve Set Theory

- ❑ In naïve set theory, a set is simply a collection of objects.
- ❑ Given any property, there is a set which contains all objects that have the property.
  - The set is empty if no object has the property.
- ❑ Russell found that a paradox arises!



Bertrand Russell (1872-1970), a British philosopher, logician, and writer.

# Russell's Paradox

- Let  $S$  be the set of all sets that are not members of themselves:

$$S = \{A \mid A \notin A\}.$$

- $Q$ : Is  $S$  an element of itself?

- i.e., is  $S \in S$ ?

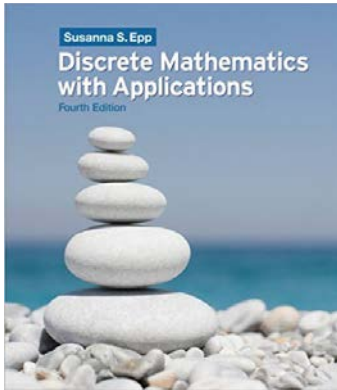
- $A$ : Neither yes nor no, because either way leads to a contradiction:

- Suppose  $S \in S$ . By the defining property of  $S$ ,  $S \notin S$ .
  - Suppose  $S \notin S$ . By the defining property of  $S$ ,  $S \in S$ .

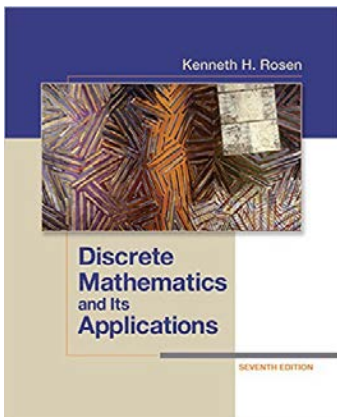
# Remarks

- ❑ The barber's paradox is a popular version of Russell's paradox.
  - Russell uses it to explain the paradox to layman.
- ❑ To resolve Russell's paradox, a set has to be properly defined.
- ❑ Russell's paradox facilitates the development of axiomatic set theory.
  - There are different ways to do it. (Details omitted.)

# Recommended Reading



- Chapter 6, S. S. Epp, *Discrete Mathematics with Applications*, 4<sup>th</sup> ed., Brooks Cole, 2010.



- Sections 2.1 and 2.2, K. H. Rosen, *Discrete Mathematics and its Applications*, 7<sup>th</sup> ed., McGraw-Hill Education, 2011.