

EF2452: Mathematics for Economics and Finance

Lecture 9

Part I of II

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Main Topics

- Determine the rank of a consistent system
- Find the inverse of a matrix using elementary row operations

Rank of a consistent system

- In general, the rows of the echelon form of a consistent system are of two types.
- The first type, which we call a “zero” row, consists entirely of zeros: $(0 \ 0 \ \dots \ 0|0)$.
- The other type is a row like $(* \ * \ \dots \ * |b)$, where $b \neq 0$, and at least one of the $*$ s is not zero. We call this a “non-zero” row.
- The number of non-zero rows in the echelon form is known as the rank of the system.

Rank of a consistent system: Example

- Suppose we are given a system for which the augmented matrix reduces to the echelon form

$$A = \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

- Here, the rank (number of non-zero rows) is $r = 3$, which is less than the number of unknowns, $n = 6$.
- The system does not provide enough information to specify the values of $x_1, x_2, \dots, x_n$ uniquely. There are infinitely many solutions.

Summary of solution to a general linear system

- If the echelon form has a row $(0 \ 0 \ \dots \ 0|a)$, with $a \neq 0$, the original system is inconsistent; it has no solutions.
- If the echelon form has no rows of the above type, it is consistent, and the general solution involves the following
 1. $r = n$: unique solution
 2. $r < n$: infinitely many solutions

where r is the rank of the system and n is the total number of unknowns.

Inverse of a square matrix

- We say that the square matrix A has an inverse Z if

$$AZ = ZA = I$$

- If a matrix has an inverse, it has only one.
- A matrix may have no inverse.
- When A has an inverse, we denote it by A^{-1} and we say that A is invertible.
- A square matrix A is invertible if and only if, for any $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. In other words, the rank of the system must be n .

Calculation of the inverse of a square matrix

- Suppose we have the following square matrix

$$A = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

- Write the given matrix in the following form with the identity matrix I (of the same size as A) on the right

$$(A|I) = \left(\begin{array}{ccc|ccc} * & * & * & 1 & 0 & 0 \\ * & * & * & 0 & 1 & 0 \\ * & * & * & 0 & 0 & 1 \end{array} \right)$$

Calculation of the inverse of a square matrix

- Use elementary row operations to write the above in the form

$$(I|B) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \end{array} \right)$$

- Then B is the inverse of A .
- If, at any stage, we create a row of the form

$$(0 \quad 0 \quad \dots \quad 0 \mid * \quad * \quad \dots \quad *),$$

then A is not invertible.

References

Following chapters of the textbook:

17 Linear Equations II

18 Inverse Matrices