

MA2001

MULTI-VARIABLE CALCULUS AND LINEAR ALGEBRA

Chapter 0 (Preliminary)

Matrices, Determinants, Vectors and Single Variable Calculus

Chapter 0 (Preliminary): Matrices, Determinants, Vectors and Single Variable Calculus

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1 Matrices and Determinants

1.1 Matrices

A **matrix** is a rectangular array of elements. The elements in a matrix are called the **entries** of the matrix. We say that matrix A is an **$m \times n$ matrix** (read as m -by- n matrix) if it has m rows and n columns. That is,

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{n \text{ columns}} \left. \vphantom{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} \right\} m \text{ rows}$$

where a_{ij} is the entry on the i -th row and j -th column. The entry a_{ij} is called the **$(i, j)^{\text{th}}$ entry** of A .

- **Square matrix**: A matrix whose number of rows and number of columns are equal.

For example, $\begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix}$ is a 2×2 square matrix.

- **Row vector**: A matrix which has only one row.

For example, $(1 \ -2 \ 3 \ 7)$ is a row vector.

- **Column vector**: A matrix which has only one column.

For example, $\begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}$ is a column vector.

- **Zero matrix**: A matrix whose entries are all zeros.

For example, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a 3×2 zero matrix.

Operations of matrices

➤ Addition/ Subtraction

If A and B are both $m \times n$ matrices (i.e. they have the **same size**), then the **sum** $A + B$ and the **difference** $A - B$ are defined as the $m \times n$ matrices such that

$$A \pm B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$$

Example 1

$$(a) \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 + (-2) & 2 + 4 \\ 3 + 1 & 4 + 3 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ 4 & 7 \end{pmatrix}$$

$$(b) \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} -2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 - (-2) & 2 - 4 \\ 3 - 1 & 4 - 3 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}$$

$$(c) \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix} \text{ is } \underline{\text{undefined}}, \text{ since the two matrices do not have the same size.}$$

Properties of matrix addition:

Suppose that A , B and C are matrices with the same size. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$

► Multiplication

If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then the **product** AB is defined as the $m \times n$ matrix such that

$$AB = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mp} \end{pmatrix}}_{m \times p \text{ matrix}} \underbrace{\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pn} \end{pmatrix}}_{p \times n \text{ matrix}} = \underbrace{\begin{pmatrix} \sum_{k=1}^p a_{1k}b_{k1} & \cdots & \sum_{k=1}^p a_{1k}b_{kn} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^p a_{mk}b_{k1} & \cdots & \sum_{k=1}^p a_{mk}b_{kn} \end{pmatrix}}_{m \times n \text{ matrix}}$$

where the $(i, j)^{\text{th}}$ entry of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^p a_{ik}b_{kj}.$$

Remarks

1. AB is only defined if the number of columns of the first matrix (i.e. A) is equal to the number of rows of the second matrix (i.e. B).
2. In general, $AB \neq BA$.

Example 2

Given the matrices $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 4 & -5 \end{pmatrix}$ and $C = \begin{pmatrix} -9 & 8 & 7 \\ 6 & 5 & -4 \\ 3 & -2 & 1 \end{pmatrix}$.

Find (a) AB , (b) BA and (c) CA .

Solution

$$(a) \quad AB = \underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \end{pmatrix}}_{2 \times \textcolor{red}{3} \text{ matrix}} \underbrace{\begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 4 & -5 \end{pmatrix}}_{\textcolor{red}{3} \times 2 \text{ matrix}} = \underbrace{\begin{pmatrix} 0 - 3 + 20 & -2 + 9 - 25 \\ 0 + 4 + 24 & -4 - 12 - 30 \end{pmatrix}}_{2 \times 2 \text{ matrix}} = \begin{pmatrix} 17 & -18 \\ 28 & -46 \end{pmatrix}$$

$$(b) \quad BA = \underbrace{\begin{pmatrix} 0 & -2 \\ -1 & 3 \\ 4 & -5 \end{pmatrix}}_{3 \times \textcolor{red}{2} \text{ matrix}} \underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \end{pmatrix}}_{\textcolor{red}{2} \times 3 \text{ matrix}} = \begin{pmatrix} 0 - 4 & 0 + 8 & 0 - 12 \\ -1 + 6 & -3 - 12 & -5 + 18 \\ 4 - 10 & 12 + 20 & 20 - 30 \end{pmatrix}$$
$$= \begin{pmatrix} -4 & 8 & -12 \\ 5 & -15 & 13 \\ -6 & 32 & -10 \end{pmatrix}$$

Note: Clearly, $AB \neq BA$.

(c) $CA = \underbrace{\begin{pmatrix} -9 & 8 & 7 \\ 6 & 5 & -4 \\ 3 & -2 & 1 \end{pmatrix}}_{3 \times \mathbf{3} \text{ matrix}} \underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \end{pmatrix}}_{\mathbf{2} \times 3 \text{ matrix}}$ is **not defined**, because the number of columns of

C is not the same as the number of rows of A .

Properties of matrix multiplication:

Suppose that A , B and C are matrices such that the given matrix multiplication can be performed. Then

1. $kAB = (kA)B = A(kB)$, where k is a scalar.
2. $ABC = (AB)C = A(BC)$
3. $A(B + C) = AB + AC$
4. $(B + C)A = BA + CA$

➤ Scalar multiplication

If k is a scalar and A is an $m \times n$ matrix, we define the scalar multiplication as the $m \times n$ matrix such that

$$kA = \begin{pmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{pmatrix}.$$

For example, if $A = \begin{pmatrix} -1 & 2 & 4 \\ 5 & 7 & -3 \end{pmatrix}$, then

$$2A = 2 \begin{pmatrix} -1 & 2 & 4 \\ 5 & 7 & -3 \end{pmatrix} = \begin{pmatrix} -2 & 4 & 8 \\ 10 & 14 & -6 \end{pmatrix}$$

and

$$-A = - \begin{pmatrix} -1 & 2 & 4 \\ 5 & 7 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -4 \\ -5 & -7 & 3 \end{pmatrix}.$$

Properties of scalar multiplication:

Suppose that both A and B are $m \times n$ matrices, and p, q are constants, then

(a) $pA = Ap$

(b) $(p \pm q)A = pA \pm qA$

(c) $(pq)A = p(qA) = q(pA)$

(d) $q(A \pm B) = qA \pm qB$

► Transpose

If A is an $m \times n$ matrix, then the **transpose** of A (written as A^T) is an $n \times m$ matrix which is obtained by interchanging the rows and columns of A :

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}^T = \underbrace{\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}}_{n \times m \text{ matrix}}$$

That is, the elements in the i^{th} row of A becomes the elements in the i^{th} column of A^T .

For example, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$

For example, $\begin{pmatrix} 1 & -1 & 0 \\ 4 & 2 & 5 \\ -2 & 0 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & -2 \\ -1 & 2 & 0 \\ 0 & 5 & 3 \end{pmatrix}.$

For example, $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}^T = (2 \ 1 \ 3).$ (The transpose of a column vector is a row vector.)

Properties of transpose:

Suppose that the sizes of the matrices A and B are such that the given operations can be performed, then

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(kA)^T = kA^T$ where k is a scalar.

(d) $(AB)^T = B^T A^T$ (not $A^T B^T$)

Square matrix

Given an $n \times n$ square matrix A , i.e.

$$A = \begin{pmatrix} \mathbf{a_{11}} & a_{12} & \cdots & a_{1n} \\ a_{21} & \mathbf{a_{22}} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \mathbf{a_{nn}} \end{pmatrix}.$$

We say that the entries $a_{11}, a_{22}, \dots, a_{nn}$ are on the main diagonal of A . The entries $\mathbf{a_{11}}, \mathbf{a_{22}}, \dots, \mathbf{a_{nn}}$ are called the diagonal entries of A . The entries not on the main diagonal are called the off-diagonal entries.

Here are some special square matrices:

❖ Diagonal matrix

This is a matrix (usually refers to square matrix) for which all the entries not on the main diagonal are zeros.

For example, $\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 2 \end{pmatrix}$ and $\begin{pmatrix} -2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 3 \end{pmatrix}$ are diagonal matrices.

The $n \times n$ diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$$

can also be written as

$$D = \text{diag}(d_1, d_2, d_3, \dots, d_n).$$

A useful property of diagonal matrix:

$$\text{If } D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}, \text{ then}$$

$$D^m = \begin{pmatrix} (d_1)^m & 0 & 0 & \cdots & 0 \\ 0 & (d_2)^m & 0 & \cdots & 0 \\ 0 & 0 & (d_3)^m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (d_n)^m \end{pmatrix}, \text{ where } m \text{ is a positive integer.}$$

❖ Identity matrix (denoted by I_n)

This is an $n \times n$ square matrix for which all the entries on the main diagonal are 1, and all the off-diagonal entries are 0.

For example,

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called a 2×2 identity matrix,

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is called a 3×3 identity matrix,

$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is called a 4×4 identity matrix.

A useful property of identity matrix:

If A is any $m \times n$ matrix, then

$$I_m A = A \quad \text{and} \quad A I_n = A.$$

❖ Symmetric matrix

An $n \times n$ square matrix A is said to be **symmetric** if $A^T = A$.

For example,

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 5 \\ 2 & -3 & 0 \\ 5 & 0 & -1 \end{pmatrix} \quad \text{are both symmetric.}$$

1.2 Determinant

Let A be a square matrix. The determinant of A , written as $\det A$ or $|A|$, is a number associated with it and is defined as follows:

(i) If A is a 1×1 matrix, i.e. $A = (a)$, then $\boxed{\det A = a}$.

(ii) If A is a 2×2 matrix, i.e. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\boxed{\det A = ad - bc}$.

(iii) If A is an $n \times n$ matrix, where $n \geq 3$, i.e. $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$, then

$$\boxed{\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}} , \quad i = 1, 2, \dots, n, \quad (*)$$

where $\boxed{A_{ij} = (-1)^{i+j} M_{ij}}$.

Here, M_{ij} is called the minor of a_{ij} which is the determinant of the submatrix of A obtained by deleting the i^{th} row and j^{th} column of A . The number A_{ij} is called the cofactor of element a_{ij} .

In (*), the determinant of A is calculated by expanding $\det A$ along the i^{th} row. The determinant can also be obtained by expanding it along the j^{th} column, i.e.

$$\boxed{\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}} , \quad j = 1, 2, \dots, n. \quad (**)$$

That is, the same value can be obtained by expanding the determinant along any row or any column.

Note that $(-1)^{i+j}$ is either 1 or -1 . The pattern of $(-1)^{i+j}$ is as follows:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Example 3

Compute the determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ 6 & 7 & -2 \end{pmatrix}$ by expanding the determinant along (a) the first row, and (b) the second column.

Solution

$$\begin{aligned} \text{(a)} \quad |A| &= \begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ -1 & 4 & 5 \\ 6 & 7 & -2 \end{vmatrix} = 1 \times \begin{vmatrix} 4 & 5 \\ 7 & -2 \end{vmatrix} - 2 \times \begin{vmatrix} -1 & 5 \\ 6 & -2 \end{vmatrix} + 3 \times \begin{vmatrix} -1 & 4 \\ 6 & 7 \end{vmatrix} \\ &= 1(-8 - 35) - 2(2 - 30) + 3(-7 - 24) \\ &= -80 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad |A| &= \begin{vmatrix} 1 & \mathbf{2} & 3 \\ -1 & \mathbf{4} & 5 \\ 6 & \mathbf{7} & -2 \end{vmatrix} = -2 \times \begin{vmatrix} -1 & 5 \\ 6 & -2 \end{vmatrix} + 4 \times \begin{vmatrix} 1 & 3 \\ 6 & -2 \end{vmatrix} - 7 \times \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} \\ &= -2(2 - 30) + 4(-2 - 18) - 7[5 - (-3)] \\ &= -80 \end{aligned}$$

Properties of determinants

If A and B are both $n \times n$ matrices, m is a positive integer, and k is a scalar, then we have the following results:

1. If all elements in a row (or column) of A are zeros, then $\det A = 0$.
2. The sign of $\det A$ is reversed if any two rows (or two columns) of A are interchanged.
3. $\det(A^T) = \det A$.
4. $\det(AB) = \det A \det B$ (Note: This is only true if both A and B are square matrices).

It follows that $\det(A^m) = \det(AA \cdots A) = (\det A) \cdots (\det A) = (\det A)^m$.

5. If the elements in any row (or column) of A are multiplied by a number, then $\det A$ is multiplied by that number.

For example,
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \textcolor{red}{k}a_{21} & \textcolor{red}{k}a_{22} & \cdots & \textcolor{red}{k}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \textcolor{red}{k} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

It follows that $\det(kA) = k^n \det A$.

6. The value of a determinant is unchanged if a scalar multiple of one row (or column) is added to another row (or column).

For example,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & \cdots & \mathbf{a_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \xrightarrow{R_2 + kR_1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \mathbf{a_{21} + ka_{11}} & \mathbf{a_{22} + ka_{12}} & \cdots & \mathbf{a_{2n} + ka_{1n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \checkmark$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & \mathbf{a_{1n}} \\ a_{21} & a_{22} & \cdots & \mathbf{a_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \mathbf{a_{nn}} \end{vmatrix} \xrightarrow{C_n - kC_1} \begin{vmatrix} a_{11} & a_{12} & \cdots & \mathbf{a_{1n} - ka_{11}} \\ a_{21} & a_{22} & \cdots & \mathbf{a_{2n} - ka_{21}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \mathbf{a_{nn} - ka_{n1}} \end{vmatrix} \quad \checkmark$$

1.3 Inverse Matrix

Given an $n \times n$ square matrix A . The inverse of A , denoted by A^{-1} , is an $n \times n$ matrix such that

$$AA^{-1} = A^{-1}A = I_n$$

where I_n is the $n \times n$ identity matrix.

Note that not every square matrix has an inverse.

- ❖ If A has an inverse (i.e. A^{-1} exists), then A is called an invertible (or non-singular) matrix.
- ❖ If A does not have an inverse (i.e. A^{-1} does not exist), then A is called a non-invertible (or singular) matrix.

Theorem (To check the existence of inverse matrix)

Let A be an $n \times n$ square matrix. Then

A is invertible	if and only if	$\det A \neq 0$.
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□

How to find the inverse of an invertible matrix?

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ be an $n \times n$ square matrix. If $\det A \neq 0$, then A^{-1} exists.

Method 1: $A^{-1} = \frac{1}{\det A} \text{adj } A$. Here, the matrix $\text{adj } A$ (called the adjoint of A) is the

transposed matrix of cofactors $\text{adj } A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T$, where

$A_{ij} = (-1)^{i+j} M_{ij}$ is the cofactor of the entry a_{ij} , and M_{ij} is called the minor of a_{ij} which is the determinant of the submatrix of A obtained by deleting the i^{th} row and j^{th} column of A .

Method 2: **Gauss-Jordan elimination method:**

$(A|I_n) \rightarrow \cdots \rightarrow (\text{by performing some row operations}) \rightarrow \cdots \rightarrow (I_n|A^{-1})$

Example 4

Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ by using

- (a) the cofactor method,
- (b) Gauss-Jordan elimination method.

Solution

(a) $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = 4 - 6 = -2 \neq 0$, so A^{-1} exists.

The **adjoint of A** is

$$\text{adj } A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

Hence, the **inverse of A** is

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

(b) By the **Gauss-Jordan elimination method**,

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array} \right) \xrightarrow{R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array} \right)$$

$$\xrightarrow{-\frac{1}{2} \times R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array} \right)$$

$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_A \quad \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I_2} \quad \underbrace{\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}}_{A^{-1}}$

Hence, $A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$

Remark of (a): For any **invertible** 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 5

Find the inverse of the matrix $A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & -3 & 0 \\ -2 & 5 & 3 \end{pmatrix}$ by using

- (a) the cofactor method,
- (b) Gauss-Jordan elimination method.

Solution

$$\begin{aligned} \text{(a)} \quad |A| &= \begin{vmatrix} 0 & 1 & -2 \\ 1 & -3 & 0 \\ -2 & 5 & 3 \end{vmatrix} = -1 \times \begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix} + (-2) \times \begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix} \\ &= -(3 - 0) - 2 \times (5 - 6) = -1 \neq 0 \\ \therefore A^{-1} &\text{ exists.} \end{aligned}$$

Compute A^{-1} by using $A^{-1} = \frac{1}{|A|} \text{adj } A$, where

$$\begin{aligned}
 \text{adj } A &= \begin{pmatrix} \begin{vmatrix} -3 & 0 \\ 5 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ -2 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ -2 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & -2 \\ -3 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & -3 \end{vmatrix} \end{pmatrix}^T \\
 &= \begin{pmatrix} -9 - 0 & -(3 - 0) & 5 - 6 \\ -(3 + 10) & 0 - 4 & -(0 + 2) \\ 0 - 6 & -(0 + 2) & 0 - 1 \end{pmatrix}^T \\
 &= \begin{pmatrix} -9 & -3 & -1 \\ -13 & -4 & -2 \\ -6 & -2 & -1 \end{pmatrix}^T \\
 &= \begin{pmatrix} -9 & -13 & -6 \\ -3 & -4 & -2 \\ -1 & -2 & -1 \end{pmatrix}
 \end{aligned}$$

Hence,

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1} \begin{pmatrix} -9 & -13 & -6 \\ -3 & -4 & -2 \\ -1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 13 & 6 \\ 3 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

(b) By the **Gauss-Jordan elimination method**,

$$\begin{aligned}
 & \left(\begin{array}{ccc|ccc} 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 & 1 & 0 \\ -2 & 5 & 3 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ -2 & 5 & 3 & 0 & 0 & 1 \end{array} \right) \\
 & \xrightarrow{R_3 + 2R_1} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 2 & 1 \end{array} \right) \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array} \right) \\
 & \xrightarrow{R_2 + 2R_3} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 4 & 2 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array} \right) \xrightarrow{R_1 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 13 & 6 \\ 0 & 1 & 0 & 3 & 4 & 2 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array} \right)
 \end{aligned}$$

Hence, $A^{-1} = \begin{pmatrix} 9 & 13 & 6 \\ 3 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$

2 Solving system of linear equations

Consider the following system of m linear equations in n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}.$$

The numbers a_{ij} are the coefficients of the system and they are given. The constants $b_1, b_2, \dots, b_m$ on the right-hand side of the equations are also known. If all the b_i 's are zeros, then the system is called a homogeneous system. If at least one of the b_i 's is non-zero, then the system is called an inhomogeneous (or non-homogeneous) system.

Our objective is to determine the values of those n unknowns $x_1, x_2, \dots, x_n$ which satisfy the m equations simultaneously.

In **matrix form**, the system can be written as

$$\boxed{A\vec{x} = \vec{b}},$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is called the **coefficient matrix** for the linear system;

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a column vector containing the n unknowns $x_1, x_2, \dots, x_n$;

and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is a column vector containing the m numbers $b_1, b_2, \dots, b_m$.

Note: We may also write the system as $A\underline{x} = \underline{b}$ instead of $A\vec{x} = \vec{b}$.

For every system of m linear equations in n unknowns, there are three possibilities for the solutions:

- **Unique solution**
- **No solution**
- **Infinitely many solutions**

We say that the system is **consistent** if it has at least one solution (i.e. either unique solution or infinitely many solutions). The system is **inconsistent** if it has no solution.

2.1 Gaussian elimination

This is a powerful technique for solving systems of equations.

Given a system with m linear equations and n unknowns,

$$A\vec{x} = \vec{b},$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

We construct the **augmented matrix** $(A|\vec{b})$, i.e.

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right).$$

Gaussian elimination is a systematic procedure that uses elementary row operations to attempt to reduce the augmented matrix to the row echelon form.

Elementary row operations

The three **elementary row operations** that can be used in Gaussian elimination are

- (1) **Interchange two rows.** (For example, $R_1 \leftrightarrow R_2$.)
- (2) **Add a multiple of one row to another row.** (For example, $R_2 - 3R_1$.)
- (3) **Multiply (or divide) a row by a non-zero constant.** (For example, $(-1) \times R_1$.)

The solution of a system of linear equations is not changed by performing the row operations.

Note: In the Gaussian elimination method, only row operations are allowed.

Column operations are not allowed in Gaussian elimination.

Definition (**Row echelon form**)

A matrix is in **row echelon form** if

- all non-zero rows (rows with at least one non-zero entry) are above any zero rows (rows whose entries are all zeros); and
- the **pivot** (which is the first non-zero entry from the left) of a non-zero row is always strictly to the right of the pivot of the row above it.

In other words, if a matrix is in row echelon form, all entries to the left and below each pivot must be zeros, and all the zero rows must appear at the bottom of the matrix.

For example, $\begin{pmatrix} \mathbf{1} & 4 & -3 \\ 0 & \mathbf{2} & 1 \\ 0 & 0 & -\mathbf{1} \end{pmatrix}$, $\begin{pmatrix} \mathbf{2} & 1 & 3 & 0 \\ 0 & 0 & \mathbf{4} & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} \mathbf{3} & 1 & -1 & 2 \\ 0 & \mathbf{2} & 0 & 5 \end{pmatrix}$ are in **row echelon forms**.

(The entries highlighted in red are the **pivots** of their corresponding rows.)

For matrices in the row echelon form, each row has at most one pivot.

Example 6 (Unique solution)

Solve the following system of equations using Gaussian elimination:

$$\begin{cases} x - 5y + 2z = -9 \\ 2x + 4y - 3z = 3 \\ -3x + 9y + 2z = 8 \end{cases}.$$

Solution

$$\left(\begin{array}{ccc|c} 1 & -5 & 2 & -9 \\ 2 & 4 & -3 & 3 \\ -3 & 9 & 2 & 8 \end{array} \right) \xrightarrow[R_3+3R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -5 & 2 & -9 \\ 0 & 14 & -7 & 21 \\ 0 & -6 & 8 & -19 \end{array} \right) \xrightarrow{R_3 - \left(\frac{-6}{14}\right)R_2} \left(\begin{array}{ccc|c} 1 & -5 & 2 & -9 \\ 0 & 14 & -7 & 21 \\ 0 & 0 & 5 & -10 \end{array} \right)$$

This is equivalent to the system

$$\begin{cases} x - 5y + 2z = -9 \\ 14y - 7z = 21 \\ 5z = -10 \end{cases}.$$

Solving backward, we obtain the solution

$$5z = -10 \Rightarrow \mathbf{z = -2}$$

$$14y - 7z = 21 \Rightarrow \mathbf{y = \frac{1}{14}(21 + 7z) = \frac{1}{2}}$$

$$x - 5y + 2z = -9 \Rightarrow \mathbf{x = -9 + 5y - 2z = -\frac{5}{2}}$$

Example 7 (No solution)

Solve the following system of equations using Gaussian elimination:

$$\begin{cases} x + 2y - z = 3 \\ 3x + y = 4 \\ 2x - y + z = 2 \end{cases}$$

Solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & 1 & 0 & 4 \\ 2 & -1 & 1 & 2 \end{array} \right) \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -5 & 3 & -5 \\ 0 & -5 & 3 & -4 \end{array} \right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -5 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

The last row $(0 \ 0 \ 0 \mid 1)$ corresponds to the equation

$$0x + 0y + 0z = 1, \quad \text{i.e. } \mathbf{0 = 1} \text{ (Contradiction!!)}.$$

$\therefore$ The system has no solution.

Remark: After reducing the augmented matrix into row echelon form, if there is a row which has the form $(\mathbf{0 \ 0 \ 0 \ \cdots \ 0 \mid b})$ with $\mathbf{b \neq 0}$, then the system has no solution.

Example 8 (Infinitely many solutions with 1 free variable)

Solve the following system of equations using Gaussian elimination:

$$\begin{cases} x + y - 6z = 6 \\ 2x - y + 3z = 18 \\ x - 2y + 9z = 12 \end{cases}$$

Solution

$$\begin{pmatrix} 1 & 1 & -6 & | & 6 \\ 2 & -1 & 3 & | & 18 \\ 1 & -2 & 9 & | & 12 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -6 & | & 6 \\ 0 & -3 & 15 & | & 6 \\ 0 & -3 & 15 & | & 6 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 1 & -6 & | & 6 \\ 0 & -3 & 15 & | & 6 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \\ \xrightarrow{R_2 / (-3)} \begin{pmatrix} 1 & 1 & -6 & | & 6 \\ 0 & 1 & -5 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The last matrix is equivalent to the system $\begin{cases} x + y - 6z = 6 \\ y - 5z = -2 \end{cases}$

There is no row of the form $(0 \ 0 \ 0 \ | \ b)$, where $b \neq 0$, so the system must be **consistent**.

Note that **column 3 has no pivots**, so the system has **infinitely many solutions** and we may take the corresponding variable (i.e. z) as a **free variable**.

Let $z = s$, where $s \in \mathbb{R}$.

Then by solving backward, we have

$$y = -2 + 5z = -2 + 5s$$

and
$$x = 6 - y + 6z = 6 - (-2 + 5s) + 6s = 8 + s.$$

Hence, the system has infinitely many solutions:

$$x = 8 + s, \quad y = -2 + 5s, \quad z = s, \quad \text{where } s \in \mathbb{R}.$$

Remark:

In the above calculation, the last matrix $\left(\begin{array}{ccc|c} \mathbf{1} & 1 & -6 & 6 \\ 0 & \mathbf{1} & -5 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right)$ is in row echelon form. If we just consider the entries on the left-hand side of the vertical line of this matrix, only column 3 has no pivots, so we only have one free variable in our solution.

Example 9 (Infinitely many solutions with 2 free variables)

Solve the following system of equations using Gaussian elimination:

$$\begin{cases} x + 2y + 2z - w = 3 \\ 2x + 5y + 3z + w = 4. \\ 3x + 8y + 4z + 3w = 5 \end{cases}$$

Solution:

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 3 \\ 2 & 5 & 3 & 1 & 4 \\ 3 & 8 & 4 & 3 & 5 \end{array} \right) \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 3 \\ 0 & 1 & -1 & 3 & -2 \\ 0 & 2 & -2 & 6 & -4 \end{array} \right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{cccc|c} \mathbf{1} & 2 & 2 & -1 & 3 \\ 0 & \mathbf{1} & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

There is no row of the form $(0 \ 0 \ 0 \ 0 \mid b)$, where $b \neq 0$, so the system must be **consistent**.

Columns 3 and 4 have no pivots, so we take the corresponding variables (i.e. z and w) as free variables. Then the system has infinitely many solutions.

The last matrix is equivalent to the system

$$\begin{cases} x + 2y + 2z - w = 3 \\ y - z + 3w = -2 \end{cases}.$$

Let $z = s$ and $w = t$, where $s, t \in \mathbb{R}$.

$$\text{Then } y = -2 + z - 3w = -2 + s - 3t$$

$$\text{and } x = 3 - 2y - 2z + w = 3 - 2(-2 + s - 3t) - 2s + t = 7 - 4s + 7t.$$

Hence, the system has infinitely many solutions

$$x = 7 - 4s + 7t, \quad y = -2 + s - 3t, \quad z = s, \quad w = t,$$

where $s, t \in \mathbb{R}$

Remark:

In this example, there are two columns (on the left-hand side of the matrix in row echelon form) with no pivots, so there are two free variables in the solution.

Linear dependency or independency of vectors

The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are said to be **linearly dependent** if at least one of them, say $\vec{v}_l$ can be written as a linear combination of the other $\vec{v}_j$'s (excluding $\vec{v}_l$).

If none of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ can be written as a linear combination of the others, we say that the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are **linearly independent**.

How to check if $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly dependent or independent?

We solve the system of linear equations

$$\boxed{x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_n \vec{v}_n = \vec{0}}.$$

If the trivial solution (i.e. $x_1 = x_2 = \dots = x_n = 0$) is the only solution to this system, then the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are **linearly independent**.

If there exists solutions other than the trivial solution, then the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are **linearly dependent**.

We can solve the above system $(\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n \mid \vec{0})$ by using **Gaussian elimination**.

Remark: (An alternative method for 3 vectors in $\mathbb{R}^3$)

If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are non-zero vectors in $\mathbb{R}^3$, we may also check the linear independence of these vectors by evaluating the scalar triple product $\vec{v}_1 \cdot \vec{v}_2 \times \vec{v}_3$:

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are **linearly independent** if and only if $\vec{v}_1 \cdot \vec{v}_2 \times \vec{v}_3 \neq 0$.

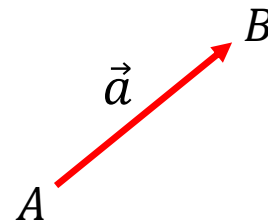
3 Vectors

A vector is a quantity that involves both magnitude (size or length) and direction.

Examples of vectors: force, displacement, magnetic field, etc.

Notations

The vector $\overrightarrow{AB}$ is a directed line segment from the initial point A to the terminal point B . We may also rename this vector, for example, let $\vec{a} = \overrightarrow{AB}$.



The magnitude of the vector $\vec{a}$ is denoted by $|\vec{a}|$.

Note: We may also denote the vector $\vec{a}$ by $\underline{a}$.

➤ **Zero vector $\vec{0}$**

A **zero vector**, denoted by $\vec{0}$, is a vector with magnitude 0. It does not have any direction.

➤ **Unit vector**

A **unit vector** is a vector with magnitude 1.

Given a non-zero vector $\vec{a}$ ($\neq \vec{0}$). The **unit vector in the direction of $\vec{a}$** is given by

$$\frac{\vec{a}}{|\vec{a}|}.$$

Moreover, any non-zero vector $\vec{a}$ can be expressed as the product of a magnitude (a number) and a direction (a unit vector).

$$\vec{a} = \underbrace{|\vec{a}|}_{\text{magnitude}} \times \underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit direction of } \vec{a}}$$

We now recall the results **for vectors in $\mathbb{R}^3$** :

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ be two vectors in $\mathbb{R}^3$, and c is a scalar. Then

➤ **Equality of vectors:**

$$\vec{a} = \vec{b} \quad \text{if and only if} \quad \boxed{a_1 = b_1, \quad a_2 = b_2 \quad \& \quad a_3 = b_3}.$$

➤ **Addition / subtraction** of vectors:

$$\boxed{\vec{a} \pm \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{pmatrix}.$$

➤ **Scalar multiplication** of vectors:

$$\boxed{c\vec{a} = c \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ ca_3 \end{pmatrix}.$$

➤ The **magnitude** of $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

➤ The **zero vector** in $\mathbb{R}^3$ is

$$\vec{0} = 0\vec{i} + 0\vec{j} + 0\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

➤ The **unit vector** in the direction of $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ ($\neq \vec{0}$) is

$$\frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

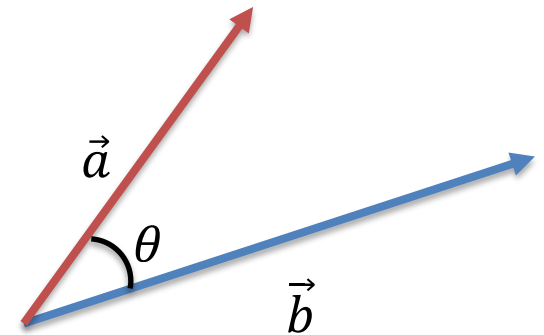
3.1 Scalar / Dot product of vectors

Scalar product (or **dot product**) takes two vectors from the same dimensional space and returns an output which is a scalar (a number).

Let $\vec{a}, \vec{b} \in \mathbb{R}^3$. The scalar / dot product of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta},$$

where θ ($0^\circ \leq \theta \leq 180^\circ$) is the **smaller angle** between $\vec{a}$ and $\vec{b}$ (when $\vec{a}$ and $\vec{b}$ start at the same point).



An alternative formula for computing $\vec{a} \cdot \vec{b}$:

If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then

$$\boxed{\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3}.$$

This formula is often used to compute $\vec{a} \cdot \vec{b}$ if the components of $\vec{a}$ and $\vec{b}$ are known.

Note: For vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$, $\vec{a} \cdot \vec{b}$ can be computed in a similar way.

Properties of scalar product

Let $\vec{a}, \vec{b}$ be vectors and α, β be scalars. Then

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $(\alpha\vec{a} + \beta\vec{b}) \cdot \vec{c} = \alpha(\vec{a} \cdot \vec{c}) + \beta(\vec{b} \cdot \vec{c})$
- $\vec{a} \cdot \vec{0} = \vec{0} \cdot \vec{a} = 0$
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2 \Rightarrow |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$

Result:

Let $\vec{a}, \vec{b} \neq \vec{0}$. Then

$\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are **orthogonal** / **perpendicular** (i.e. $\theta = 90^\circ$).

Therefore, we can use the scalar product to check if two vectors are orthogonal or not.

3.2 Vector / Cross product of vectors

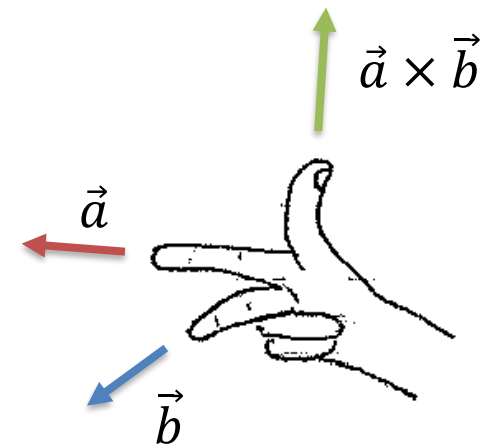
Vector product (or cross product) takes two vectors from $\mathbb{R}^3$ and returns an output which is a **vector**. Note that vector product is defined for vectors in $\mathbb{R}^3$ ONLY.

Definition

Let $\vec{a}$ and $\vec{b}$ be vectors in $\mathbb{R}^3$. The **vector product (or cross product)** of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is given by

$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin \theta) \vec{n},$$

where θ ($0^\circ \leq \theta \leq 180^\circ$) is the smaller angle between $\vec{a}$ and $\vec{b}$ (when $\vec{a}$ and $\vec{b}$ start at the same point), and $\vec{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, following the right-hand rule.



Result:

Let $\vec{a}, \vec{b} \neq \vec{0}$ be two non-zero vectors in $\mathbb{R}^3$. Then

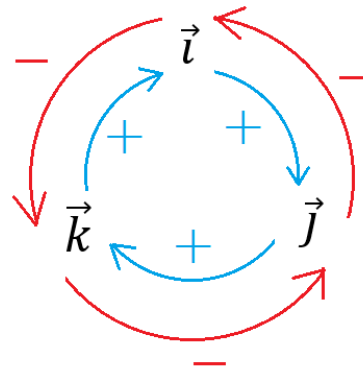
$$\vec{a} \times \vec{b} = \vec{0} \quad \text{if and only if} \quad \vec{a} \text{ and } \vec{b} \text{ are parallel (i.e. } \theta = 0^\circ \text{ or } 180^\circ).$$

Therefore, $\vec{a} \times \vec{a} = \vec{0}$ for any vector $\vec{a}$.

Using the definition of vector product, we find that

$$\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{i} = -\vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{j} = -\vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}, \quad \vec{i} \times \vec{k} = -\vec{j}$$

These results are summarized in the following diagram:



$$\text{Also, } \vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}.$$

Properties of vector product

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors in $\mathbb{R}^3$, and α , β be scalars.

$$(i) \quad \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

(Note: $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ have the same magnitude but opposite directions.)

$$(ii) \quad \alpha \vec{a} \times \beta \vec{b} = \alpha \beta (\vec{a} \times \vec{b})$$

$$(iii) \quad \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

and

$$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$

Remark: In general, $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

Alternative method for computing cross product using determinants:

If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are vectors in $\mathbb{R}^3$,

then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \begin{array}{l} \leftarrow \vec{i}, \vec{j}, \vec{k} \\ \leftarrow \text{components of } \vec{a} \\ \leftarrow \text{components of } \vec{b} \end{array}$$

$$= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

Example 10

Let $\vec{a} = 2\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{b} = \vec{i} + 8\vec{j} - 4\vec{k}$. Find $\vec{a} \times \vec{b}$.

Solution:

Method 1:

$$\begin{aligned}\vec{a} \times \vec{b} &= (2\vec{i} - 2\vec{j} + \vec{k}) \times (\vec{i} + 8\vec{j} - 4\vec{k}) \\&= 2 \underbrace{(\vec{i} \times \vec{i})}_{=\vec{0}} + 16 \underbrace{(\vec{i} \times \vec{j})}_{=\vec{k}} - 8 \underbrace{(\vec{i} \times \vec{k})}_{=-\vec{j}} - 2 \underbrace{(\vec{j} \times \vec{i})}_{=-\vec{k}} - 16 \underbrace{(\vec{j} \times \vec{j})}_{=\vec{0}} + 8 \underbrace{(\vec{j} \times \vec{k})}_{=\vec{i}} + \underbrace{\vec{k} \times \vec{i}}_{=\vec{j}} \\&\quad + 8 \underbrace{(\vec{k} \times \vec{j})}_{=-\vec{i}} - 4 \underbrace{(\vec{k} \times \vec{k})}_{=\vec{0}} \\&= 16\vec{k} + 8\vec{j} + 2\vec{k} + 8\vec{i} + \vec{j} - 8\vec{i} \\&= 9\vec{j} + 18\vec{k}\end{aligned}$$

Method 2: (Using determinant method)

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 1 \\ 1 & 8 & -4 \end{vmatrix} \begin{array}{l} \leftarrow \vec{i}, \vec{j}, \vec{k} \\ \leftarrow \text{components of } \vec{a} \\ \leftarrow \text{components of } \vec{b} \end{array} \\ &= \vec{i} \begin{vmatrix} -2 & 1 \\ 8 & -4 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -2 \\ 1 & 8 \end{vmatrix} \\ &= (8 - 8)\vec{i} - (-8 - 1)\vec{j} + [16 - (-2)]\vec{k} \\ &= 9\vec{j} + 18\vec{k}\end{aligned}$$

4 Single Variable Calculus

Recall from MA1200 and MA1201 that if $y = f(x)$ (i.e. y is a function of a single variable x), then we may differentiate $f(x)$ with respect to x to get $\frac{dy}{dx} = f'(x)$, or integrate $f(x)$ with respect to x to get $\int f(x) dx = F(x) + C$, where $\frac{d}{dx} F(x) = f(x)$.

4.1 Differentiation

Let $u = u(x)$ and $v = v(x)$ be two differentiable functions in x , and $f(u)$ be a differentiable function in u .

➤ **Product rule:** $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

➤ **Quotient rule:** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

➤ **Chain rule:** $\frac{d}{dx}[f(u(x))] = \frac{d[f(u)]}{du} \cdot \frac{du}{dx}$

➤ **Implicit differentiation:** Differentiate both sides of $F(x, y) = 0$ w.r.t. x and use the Chain rule.

➤ **Logarithmic differentiation**

➤ **Differentiation of parametric equations:** $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, where $y = y(t)$, $x = x(t)$.

4.2 Integration

➤ $\int f(x) dx = F(x) + C$
 $\Rightarrow \int f(ax + b) dx = \frac{1}{a} F(ax + b) + C$

➤ **Method of substitution:**

$$\int f'(x) g(f(x)) dx \underset{\substack{\equiv \\ \text{let } u=f(x) \\ \Rightarrow \frac{du}{dx}=f'(x)}}{=} \int g(u) du = G(u) + C = G(f(x)) + C$$

➤ **Integration by parts:**

$$\int u dv = uv - \int v du$$

➤ **Integration of rational functions:**

proper/improper rational functions, partial fractions