

MA2001

MULTI-VARIABLE CALCULUS AND LINEAR ALGEBRA

Chapter 3

Multiple Integrals

Dr. Emily Chan

Chapter 3: Multiple Integrals

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- 2 Change of Variables in Double Integrals**
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1 Double Integrals

Consider a function $f(x)$ of one variable defined over $a \leq x \leq b$.

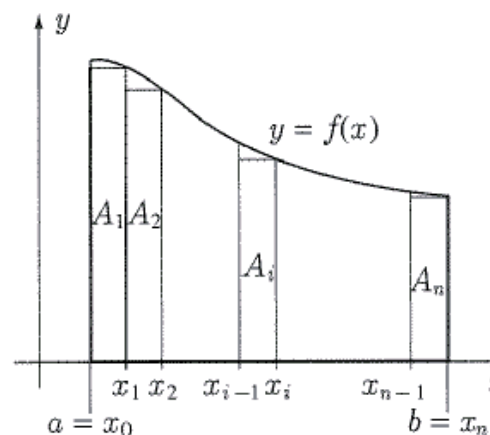
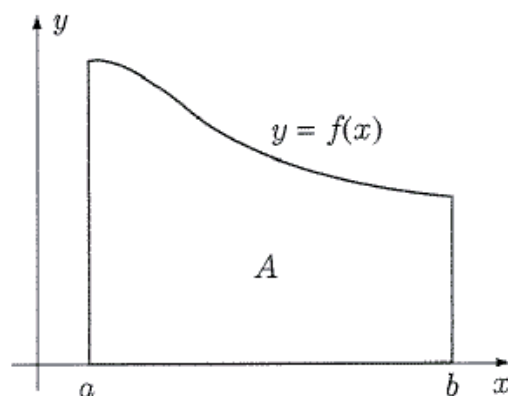
Let $x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b$ and $\delta x_i = x_i - x_{i-1}$.

Let $P = \{\delta x_i : 1 \leq i \leq n\}$ and $|P| = \max_{1 \leq i \leq n} \{\delta x_i : 1 \leq i \leq n\}$.

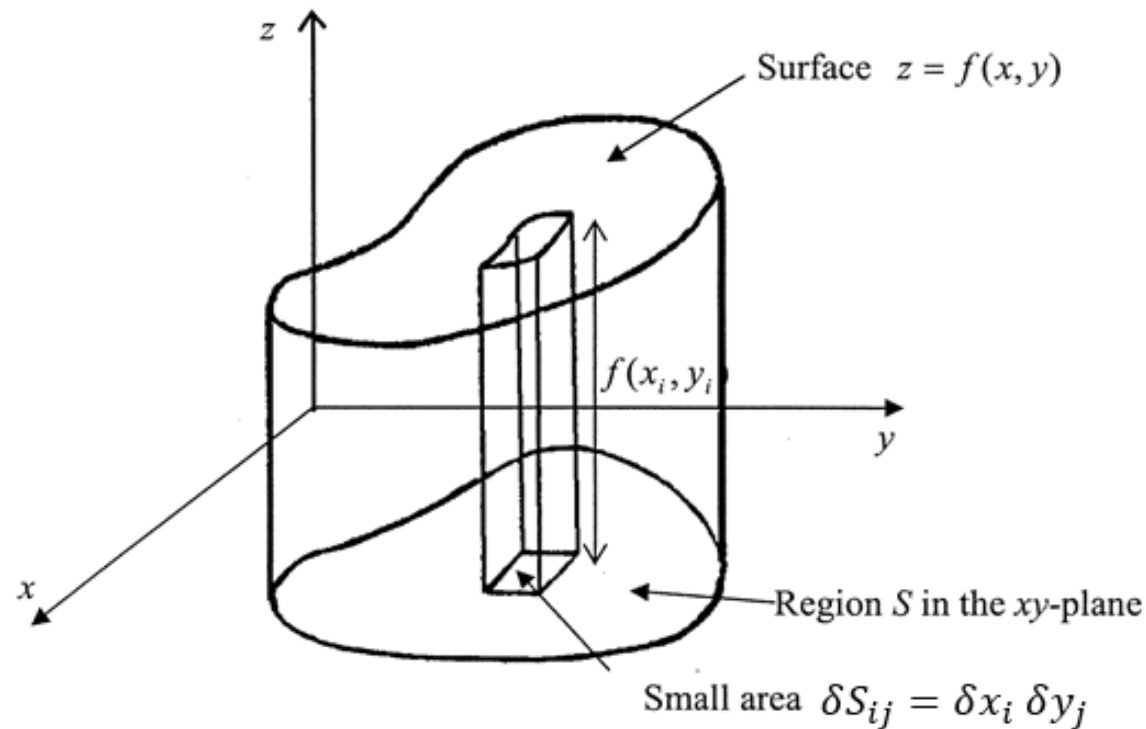
We define the definite integral of f from a to b as

$$\int_a^b f(x) dx = \lim_{|P| \rightarrow 0} \sum_{i=1}^n f(x_i) \delta x_i$$

which corresponds geometrically, if $f(x)$ is positive, to the area between the graph, the x -axis and the lines $x = a$ and $x = b$.



A function $f(x, y)$ of two variables defines a surface and, if $f(x, y)$ is positive, we may analogously determine the volume enclosed by this surface and a cylinder erected on a region S in the xy -plane.



Divide S into n small area elements such that the $(i, j)^{\text{th}}$ element has area $\delta S_{ij} = \delta x_i \delta y_j$ and contains the point (x_i, y_j) . Then $f(x_i, y_j) \delta x_i \delta y_j$ is the volume of the column with base area $\delta S_{ij} = \delta x_i \delta y_j$ and height $f(x_i, y_j)$.

Let $P = \{\delta S_{ij} = \delta x_i \delta y_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

We define the **double integral** of $f(x, y)$ over the region S to be

$$\iint_S f(x, y) dx dy = \lim_{\text{all } \delta x_i, \delta y_j \rightarrow 0} \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) \delta x_i \delta y_j .$$

The direct evaluation of the limit in the definition of double integral is impractical. We will calculate the double integral by means of the **iterated integrals**.

Rectangular regions

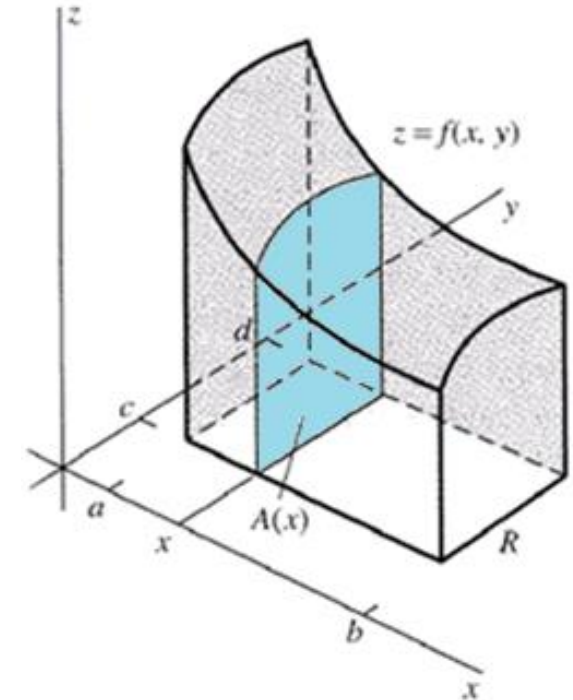
Let $f(x, y)$ be continuous on the region $S: a \leq x \leq b, c \leq y \leq d$.

Let $A(x) = \int_c^d f(x, y) dy$, which is the area of the cross section at x , then

$$\iint_S f(x, y) dx dy = \int_a^b A(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx.$$

Similarly, we also have

$$\iint_S f(x, y) dx dy = \int_c^d \left[\int_a^b f(x, y) dx \right] dy.$$



Example 1

Evaluate $\iint_S (2xy + y^2) \, dx \, dy$, where S is the rectangle $1 \leq x \leq 2$, $0 \leq y \leq 1$.

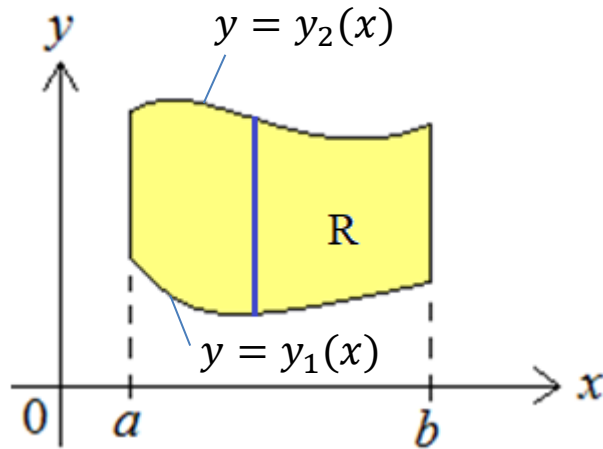
Solution

$$\begin{aligned} \iint_S (2xy + y^2) \, dx \, dy &= \int_0^1 \left[\int_1^2 (2xy + y^2) \, dx \right] dy = \int_0^1 [x^2 y + xy^2]_1^2 \, dy \\ &= \int_0^1 [(4y + 2y^2) - (y + y^2)] \, dy = \int_0^1 (3y + y^2) \, dy = \left[\frac{3y^2}{2} + \frac{y^3}{3} \right]_0^1 = \left(\frac{3}{2} + \frac{1}{3} \right) - 0 = \frac{11}{6} \end{aligned}$$

$$\begin{aligned} \text{or } \iint_S (2xy + y^2) \, dx \, dy &= \int_1^2 \left[\int_0^1 (2xy + y^2) \, dy \right] dx = \int_1^2 \left[xy^2 + \frac{y^3}{3} \right]_0^1 \, dx \\ &= \int_1^2 \left(x + \frac{1}{3} \right) dx = \left[\frac{x^2}{2} + \frac{x}{3} \right]_1^2 = \left(\frac{2^2}{2} + \frac{2}{3} \right) - \left(\frac{1^2}{2} + \frac{1}{3} \right) = \frac{11}{6} \end{aligned}$$

Non-rectangular regions

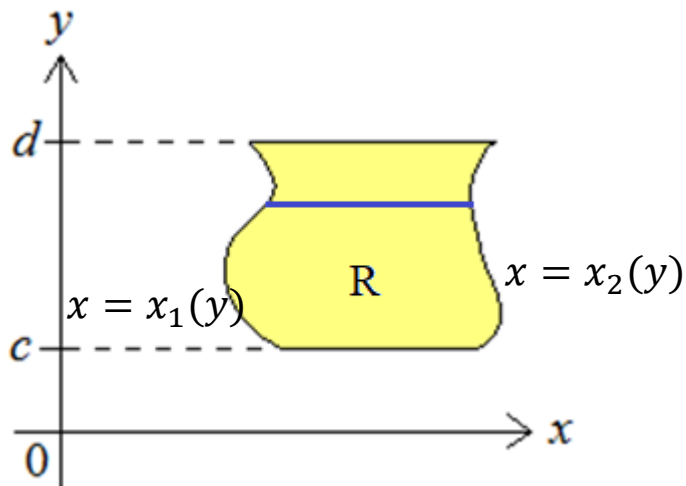
➤ Vertically simple region



$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, y_1(x) \leq y \leq y_2(x)\}$$

$$\iint_R f(x, y) dx dy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y) dy \right] dx$$

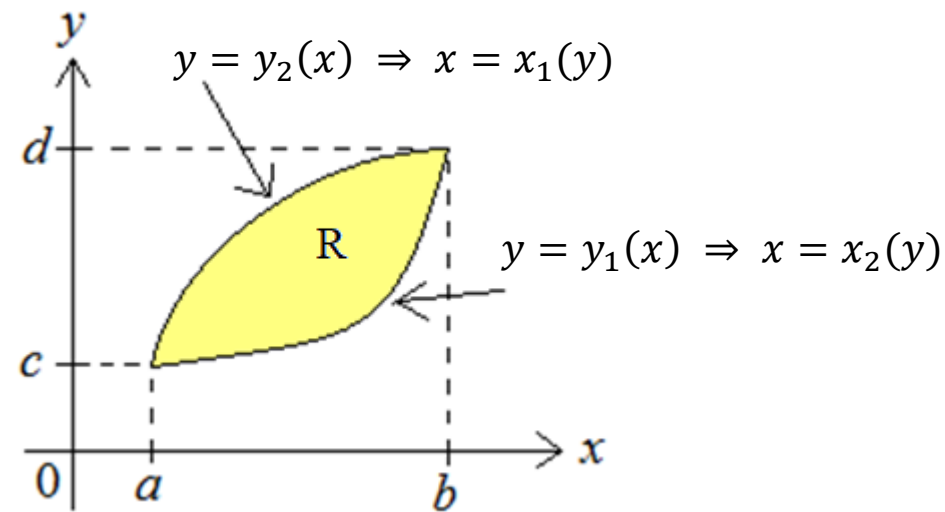
➤ Horizontally simple region



$$R = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, x_1(y) \leq x \leq x_2(y)\}$$

$$\iint_R f(x, y) dx dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y) dx \right] dy$$

➤ **Both vertically and horizontally simple region**



$$\iint_R f(x, y) \, dx \, dy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y) \, dy \right] dx$$

or

$$\iint_R f(x, y) \, dx \, dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y) \, dx \right] dy$$

Remark:

$$\iint_S 1 \, dx \, dy = \text{area of the region } S \text{ if } f(x, y) \equiv 1.$$

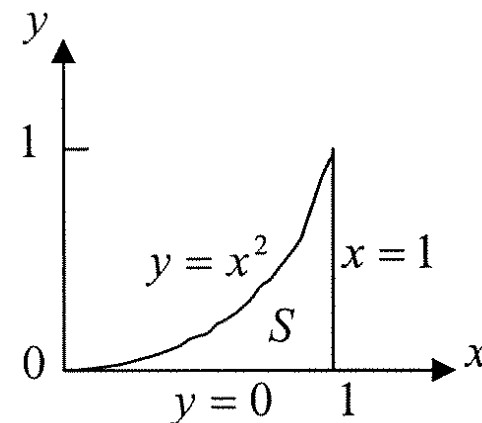
Example 2

Evaluate $\iint_S xy^2 \, dx \, dy$, where S is the region bounded by $y = 0$, $y = x^2$ and $x = 1$.

Solution

A sketch of the region S is important, but it is not necessary to be able to visualize the surface $f(x, y)$.

A sketch of the region S is shown on the right.



$$\iint_S xy^2 \, dx \, dy = \int_0^1 \left[\int_0^{x^2} xy^2 \, dy \right] dx = \int_0^1 \left[\frac{xy^3}{3} \right]_0^{x^2} dx = \int_0^1 \frac{x(x^2)^3}{3} dx = \int_0^1 \frac{x^7}{3} dx = \left[\frac{x^8}{24} \right]_0^1 = \frac{1}{24}$$

or

$$\iint_S xy^2 \, dx \, dy = \int_0^1 \left[\int_{\sqrt{y}}^1 xy^2 \, dx \right] dy = \int_0^1 \left[\frac{x^2 y^2}{2} \right]_{\sqrt{y}}^1 dy = \int_0^1 \left(\frac{y^2}{2} - \frac{y^3}{2} \right) dy = \left[\frac{y^3}{6} - \frac{y^4}{8} \right]_0^1 = \frac{1}{24}$$

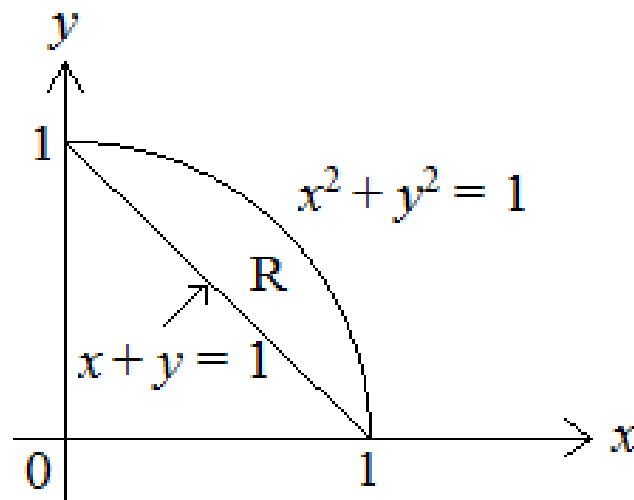
Example 3

Find the limits of integration in

$$\iint_R f(x, y) \, dx \, dy$$

where R is the region in the first quadrant bounded by $x^2 + y^2 = 1$ and $x + y = 1$.

Solution

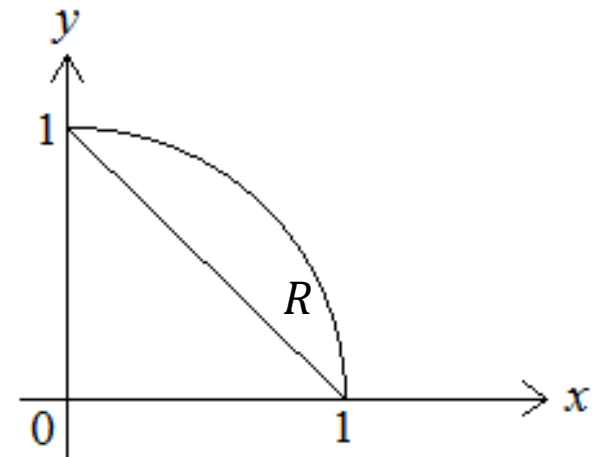


Observe that R is both vertically simple and horizontally simple.

Method 1: (Treat R as a vertically simple region)

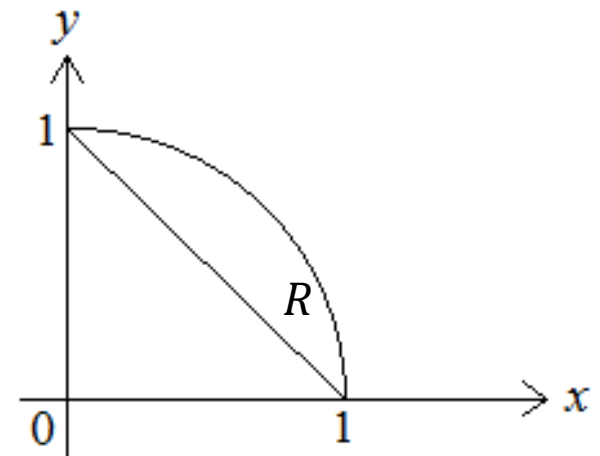
Fix x first, then find the limits for y .

$$\iint_R f(x, y) \, dx \, dy = \int_0^1 \left(\int_{1-x}^{\sqrt{1-x^2}} f(x, y) \, dy \right) dx$$

**Method 2: (Treat R as a horizontally simple region)**

Fix y first, then find the limits for x .

$$\iint_R f(x, y) \, dx \, dy = \int_0^1 \left(\int_{1-y}^{\sqrt{1-y^2}} f(x, y) \, dx \right) dy$$



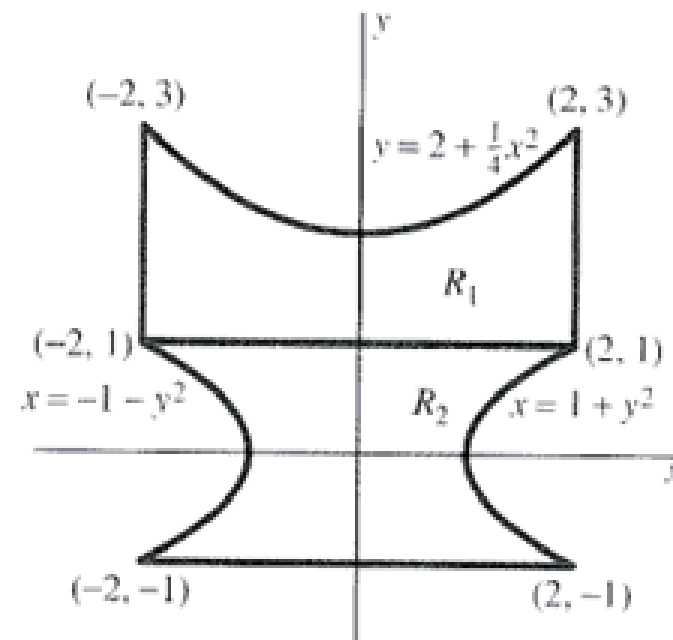
Example 4

The non-simple region R is the union of the non-overlapping simple regions R_1 and R_2 .

$$\iint_R f(x, y) \, dx \, dy$$

$$= \iint_{R_1} f(x, y) \, dx \, dy + \iint_{R_2} f(x, y) \, dx \, dy$$

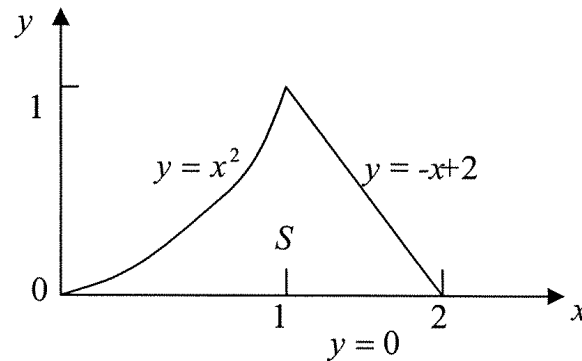
$$= \int_{-2}^2 \left[\int_1^{2+\frac{x^2}{4}} f(x, y) \, dy \right] dx + \int_{-1}^1 \left[\int_{-1-y^2}^{1+y^2} f(x, y) \, dx \right] dy$$



Note: In some cases, one order of integration may be easier than the other.

Example 5

Consider an integrable function $f(x, y)$, which is defined on a region of xy -plane as shown below:



We observe that

$$\iint_S f(x, y) \, dx dy = \int_0^1 \left[\int_{\sqrt{y}}^{2-y} f(x, y) \, dx \right] dy$$

looks like easier than

$$\iint_S f(x, y) \, dx dy = \int_0^1 \left[\int_0^{x^2} f(x, y) \, dy \right] dx + \int_1^2 \left[\int_0^{2-x} f(x, y) \, dy \right] dx .$$

Example 6

Change the order of the integration in $\int_0^1 \left[\int_x^{\sqrt{x}} f(x, y) dy \right] dx$.

Solution

The line $y = x$ and the parabola $y = \sqrt{x}$ cut at $(0, 0)$ and $(1, 1)$.

The domain of integration is the region bounded by $y = x$ and $y = \sqrt{x}$ (the same as $x = y^2$).

So

$$\int_0^1 \left[\int_x^{\sqrt{x}} f(x, y) dy \right] dx = \int_0^1 \left[\int_{y^2}^y f(x, y) dx \right] dy.$$

Example 7

Change the order of the integration in $\int_0^1 \left[\int_{-y}^y f(x, y) dx \right] dy$.

Solution

Let R be the region of integration.

Then R is given by the inequalities $-y \leq x \leq y$ and $0 \leq y \leq 1$.

Therefore, R is bounded by $y = x$ and $y = -x$ between $y = 0, y = 1$.

To find limits for integration in the reverse order, we imagine a vertical line passing from bottom to the top through the region.

From $x = -1$ to $x = 0$, it enters at $y = -x$ and leaves at $y = 1$.

Then from $x = 0$ to $x = 1$, it enters at $y = x$ and leaves at $y = 1$.

$$\therefore \int_0^1 \left[\int_{-y}^y f(x, y) dx \right] dy = \iint_R f(x, y) dx dy = \int_{-1}^0 \left[\int_{-x}^1 f(x, y) dy \right] dx + \int_0^1 \left[\int_x^1 f(x, y) dy \right] dx$$

2 Change of Variables in Double Integrals

For the **definite integral** $\int_a^b f(x) dx$, we know that the change of variable $x = x(u)$ produces

$$\int_a^b f(x) dx = \int_\alpha^\beta f[x(u)] \frac{dx}{du} du,$$

where $x(\alpha) = a$, $x(\beta) = b$.

For the **double integral**

$$\iint_S f(x, y) dx dy,$$

the change of variables $x = x(u, v)$, $y = y(u, v)$ gives

$$\boxed{\iint_S f(x, y) dx dy = \iint_{S^*} f[x(u, v), y(u, v)] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv},$$

where

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

is called **Jacobian** of the transformation.

S^* is the region in the uv -plane corresponding to the region S in the xy -plane in a one-to-one manner. That is, for every point (u, v) in S^* , there is only one corresponding point (x, y) in S . Conversely, for every point (x, y) in S , there is only one corresponding point (u, v) in S^* .

Furthermore, the Jacobian J is either positive throughout S^* or negative throughout S^* .

$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = |J|$ is the absolute value of the Jacobian.

Example: Polar coordinates

Using **polar coordinates**, we have

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Then

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r$$

i.e. $|J| = |r| = r$ and

$$dx \, dy \rightarrow r \, dr \, d\theta$$

where $r \, dr \, d\theta$ corresponds to an element of area in the $r\theta$ -plane.

Then the integral becomes

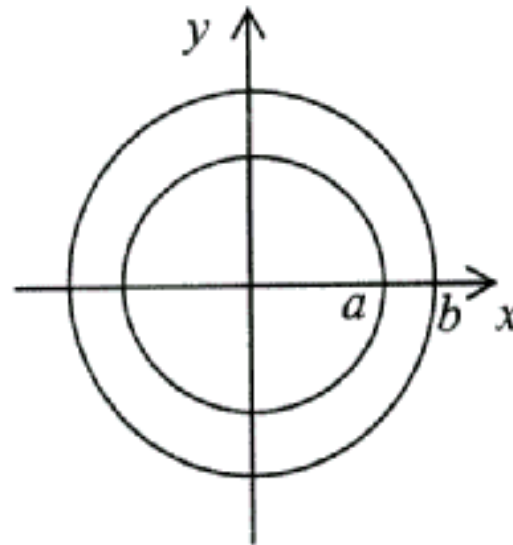
$$\iint_S f(x, y) \, dx \, dy = \iint_{S^*} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$

where S^* is the region in the $r\theta$ -plane corresponding to region S in the xy -plane.

Example 8

Find the moment of inertia of a hollow circular cylinder of inner radius a , outer radius b , height h and constant density ρ about the axis of the cylinder. The formula for the moment of inertia of the hollow circular cylinder is

$$MI = \iint_S \rho h(x^2 + y^2) dx dy.$$



Solution

Using **polar coordinates**:

$$\boxed{x = r \cos \theta}, \quad \boxed{y = r \sin \theta},$$

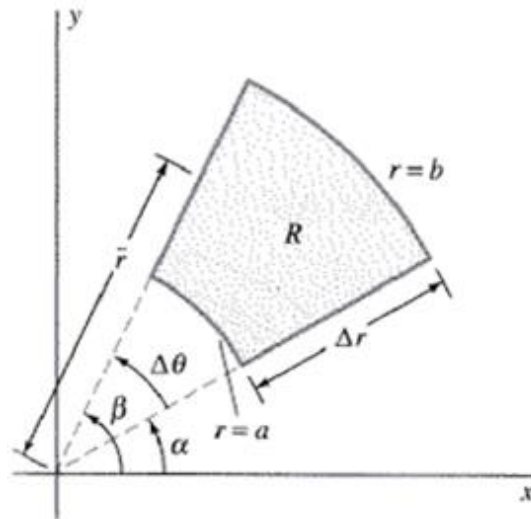
the hollow circular ring S of inner radius a , outer radius b is transformed into a rectangle R in the $r\theta$ -plane, which is bounded by $\theta = 0$, $\theta = 2\pi$, $r = a$, $r = b$.

The region of integration is now a rectangle R in the $r\theta$ -plane,

$$R: \begin{cases} a \leq r \leq b \\ 0 \leq \theta \leq 2\pi \end{cases}$$

For polar coordinates, the area of a small polar rectangle R is approximately $\bar{r} \Delta r \Delta \theta$.

(Note $\delta \equiv \Delta$).



Using polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, we have

$$J = r \Rightarrow |J| = r$$

and $\boxed{dx \, dy \rightarrow r \, dr \, d\theta}$.

$$\begin{aligned} MI &= \iint_S \rho h (x^2 + y^2) \, dx \, dy = \iint_R \rho h r^2 \, r \, dr \, d\theta = \int_0^{2\pi} \left[\int_a^b \rho h r^3 \, dr \right] d\theta \\ &= \rho h \int_0^{2\pi} \left[\int_a^b r^3 \, dr \right] d\theta = \rho h \left(\int_a^b r^3 \, dr \right) \left(\int_0^{2\pi} 1 \, d\theta \right) \\ &= \rho h \left(\left[\frac{r^4}{4} \right]_a^b \right) ([\theta]_0^{2\pi}) = \rho h \left(\frac{b^4}{4} - \frac{a^4}{4} \right) (2\pi - 0) \\ &= \rho h \pi \frac{(b^2 - a^2)(b^2 + a^2)}{2} \\ &= \frac{M}{2} (b^2 + a^2) \end{aligned}$$

where mass $= M = \rho h \pi (b^2 - a^2)$.

Example 9

Evaluate the integral $\iint_Q e^{\frac{y-x}{y+x}} dx dy$, where Q is bounded by $x + y = 2$, $x = 0$, $y = 0$.

(Hint: Use the change of variables $u = y - x$ and $v = y + x$.)

Solution

We use the change of variables

$$u = y - x \text{ and } v = y + x.$$

Solving for x and y , we find that

$$x = \frac{v - u}{2} \text{ and } y = \frac{v + u}{2}.$$

Under the transformations $u = y - x$ and $v = y + x$, the lines $x + y = 2$, $x = 0$, $y = 0$ map onto the lines $v = 2$, $u - v = 0$, $u + v = 0$, respectively.

Points inside Q are carried into points inside the triangular region T bounded by $v = 2$,

$$u - v = 0, u + v = 0. \text{ Thus, } T: \begin{cases} 0 \leq v \leq 2 \\ -v \leq u \leq v \end{cases}.$$

The corresponding Jacobian is

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

or

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = -\frac{1}{2}$$

Therefore,

$$\begin{aligned} \iint_Q e^{\frac{y-x}{y+x}} dx dy &= \iint_T e^{\frac{u}{v}} \left| -\frac{1}{2} \right| du dv = \frac{1}{2} \int_0^2 \left[\int_{-v}^v e^{\frac{u}{v}} du \right] dv = \frac{1}{2} \int_0^2 \left[\frac{e^{\frac{u}{v}}}{\frac{1}{v}} \right]_{-v}^v dv \\ &= \frac{1}{2} \int_0^2 v(e - e^{-1}) dv = \frac{1}{2} \left(e - \frac{1}{e} \right) \left[\frac{v^2}{2} \right]_0^2 = \frac{1}{2} \left(e - \frac{1}{e} \right) (2 - 0) = e - \frac{1}{e} \end{aligned}$$

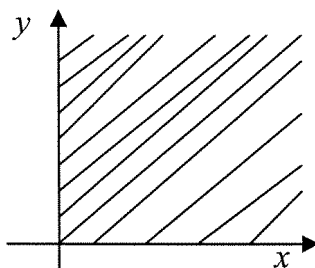
Example 10

By considering $I^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right)$, evaluate $I = \int_0^\infty e^{-x^2} dx$.

Solution

$$\begin{aligned} I^2 &= \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) = \int_0^\infty \left(\int_0^\infty e^{-x^2} dx\right) e^{-y^2} dy = \int_0^\infty \left(\int_0^\infty e^{-x^2} e^{-y^2} dx\right) dy \\ &= \int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)} dx\right) dy \end{aligned}$$

Note that the integral $\int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)} dx\right) dy$ is equivalent to integrating the function $f(x, y) = e^{-(x^2+y^2)}$ over the first quadrant of the xy -plane.



Under the transformations $x = r \cos \theta$ and $y = r \sin \theta$, the first quadrant R of xy -plane is corresponding to the region $\bar{R}$, which is bounded by $r = 0$, $\theta = 0$, $\theta = \frac{\pi}{2}$ in $r\theta$ -plane.

We observe that under $x = r \cos \theta$ and $y = r \sin \theta$, the quarter circle $x^2 + y^2 = r_0^2$, where $x, y \geq 0$, corresponds to the line segment $\begin{cases} r = r_0 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases}$ in $r\theta$ -plane.

Hence, the region R in the xy -plane is transformed into region $\bar{R}$ in the $r\theta$ -plane, where

$$\bar{R}: \begin{cases} 0 \leq r < \infty \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases}.$$

Also, under polar transformation, $dx dy \rightarrow r dr d\theta$. So

$$\begin{aligned} I^2 &= \int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)} dx \right) dy = \iint_R e^{-(x^2+y^2)} dx dy = \iint_{\bar{R}} e^{-r^2} r dr d\theta \\ &= \int_0^\infty \left(\int_0^{\frac{\pi}{2}} e^{-r^2} r d\theta \right) dr = \int_0^\infty e^{-r^2} r [\theta]_0^{\frac{\pi}{2}} dr = \frac{\pi}{2} \int_0^\infty e^{-r^2} r dr \end{aligned}$$

$$\begin{aligned}
 &= \frac{\pi}{2} \int_0^{\infty} e^{-u} \frac{1}{2} du = \frac{\pi}{4} [-e^{-u}]_0^{\infty} = -\frac{\pi}{4} (e^{-\infty} - 1) = -\frac{\pi}{4} (0 - 1) = \frac{\pi}{4} \\
 &\text{Let } u=r^2 \\
 &\Rightarrow du=2rdr \\
 &r=0 \rightarrow u=0 \\
 &r=\infty \rightarrow u=\infty
 \end{aligned}$$

Hence, we have

$$\begin{aligned}
 I^2 &= \left(\int_0^{\infty} e^{-x^2} dx \right) \left(\int_0^{\infty} e^{-y^2} dy \right) = \frac{\pi}{4} \\
 \Rightarrow I &= \int_0^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}
 \end{aligned}$$

□

3 Triple Integrals

Suppose that a scalar field $f(x, y, z)$ is defined at all points (x_i, y_i, z_i) within a region V of three-dimensional space. Divide V into N sub-regions such that the i -th sub-region has volume $\delta V_{ijk} = \delta x_i \delta y_j \delta z_k$ and contains a point (x_i, y_j, z_k) .

Let $P = \{\delta V_{ijk} = \delta x_i \delta y_j \delta z_k : 1 \leq i \leq N_1, 1 \leq j \leq N_2, 1 \leq k \leq N_3\}$.

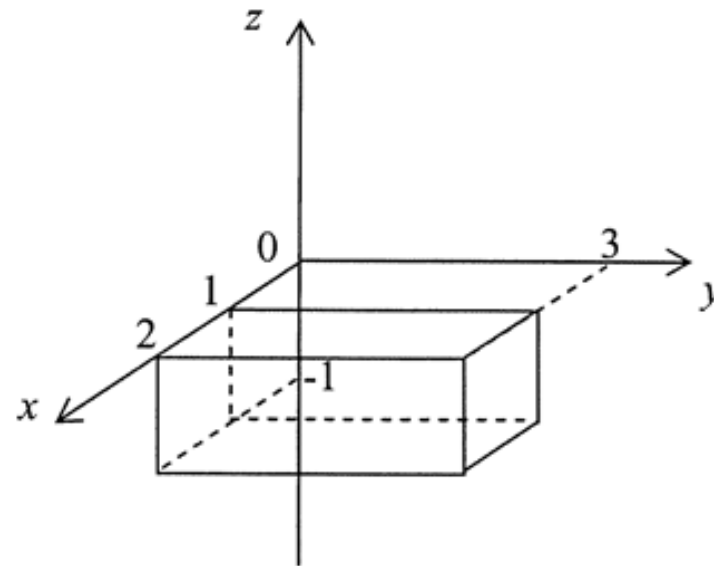
We define the **triple integral** of $f(x, y, z)$ over the region V to be

$$\iiint_V f(x, y, z) dx dy dz = \lim_{\text{all } \delta x_i, \delta y_j, \delta z_k \rightarrow 0} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{k=1}^{N_3} f(x_i, y_j, z_k) \delta x_i \delta y_j \delta z_k .$$

We can evaluate triple integrals by iterated single integration.

Example 11 (Rectangular block)

The density of a rectangular block V bounded by the planes $x = 1$, $x = 2$, $y = 0$, $y = 3$, $z = -1$, $z = 0$ is given by the scalar function $\rho(x, y, z) = x(y + 1) - z$. Find the **mass** of the block.



Solution

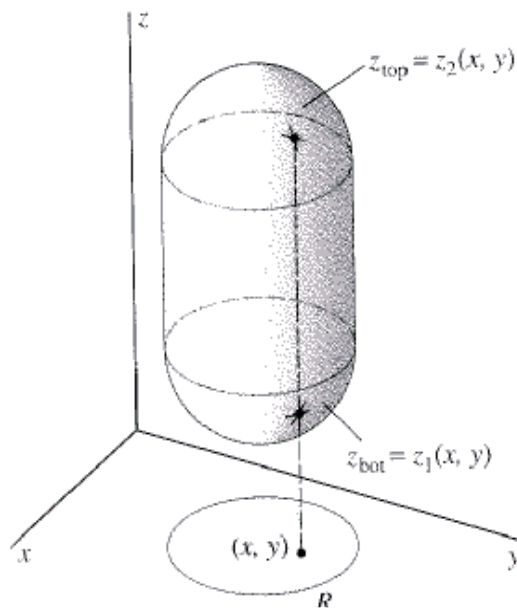
$$\begin{aligned} \text{Mass} &= \iiint_V \rho(x, y, z) \, dx \, dy \, dz = \iiint_V [x(y+1) - z] \, dx \, dy \, dz \\ &= \int_1^2 \left\{ \int_0^3 \left[\int_{-1}^0 [x(y+1) - z] \, dz \right] dy \right\} dx = \int_1^2 \left\{ \int_0^3 \left[x(y+1)z - \frac{z^2}{2} \right]_{-1}^0 dy \right\} dx \\ &= \int_1^2 \left\{ \int_0^3 \left[x(y+1) + \frac{1}{2} \right] dy \right\} dx = \int_1^2 \left[\frac{xy^2}{2} + xy + \frac{1}{2}y \right]_0^3 dx \\ &= \int_1^2 \left(\frac{9}{2}x + 3x + \frac{3}{2} \right) dx = \left[\frac{15}{4}x^2 + \frac{3}{2}x \right]_1^2 \\ &= 18 - \frac{21}{4} \\ &= \frac{51}{4} \end{aligned}$$

Non-rectangular block

Suppose that the solid T with piecewise smooth boundary is **z-simple**: Each line parallel to the z -axis intersects T (if at all) in a single line segment. This means that T can be described by the inequalities $z_1(x, y) \leq z \leq z_2(x, y)$ with $(x, y) \in R$, where R is the vertical projection of T onto the xy -plane. Then

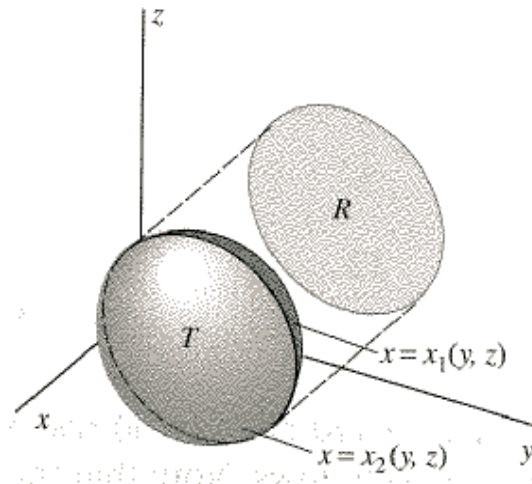
$$\iiint_T f(x, y, z) \, dx \, dy \, dz = \iint_R \left[\int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) \, dz \right] dx \, dy.$$

z-simple



Similarly, we may integrate first with respect to x if the space region T is **x -simple**. That is, T is defined by $x_1(y, z) \leq x \leq x_2(y, z)$ with $(y, z) \in R$, where R is the projection of T onto the yz -plane.

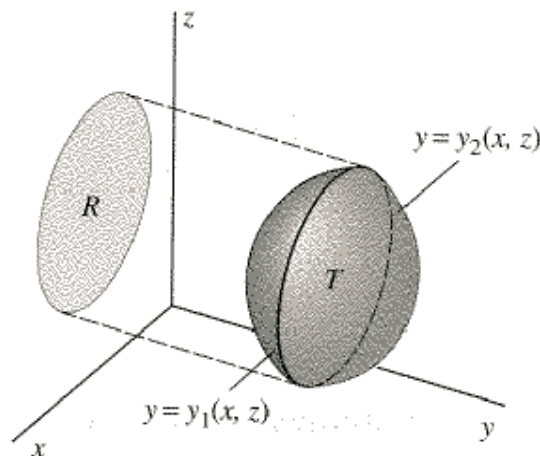
x -simple



$$\iiint_T f(x, y, z) \, dx \, dy \, dz = \iint_R \left[\int_{x_1(y, z)}^{x_2(y, z)} f(x, y, z) \, dx \right] dy \, dz$$

Also, we may integrate first with respect to y if the space region T is **y-simple**. That is, T is defined by $y_1(x, z) \leq y \leq y_2(x, z)$ with $(x, z) \in R$, where R is the projection of T onto the xz -plane.

y-simple



$$\iiint_T f(x, y, z) \, dx \, dy \, dz = \iint_R \left[\int_{y_1(x, z)}^{y_2(x, z)} f(x, y, z) \, dy \right] dx \, dz$$

Remarks

(a) If $f(x, y, z) \equiv 1$, then

$$\iiint_V 1 \, dx dy dz = \text{volume of the region } V.$$

(b) If the scalar function $\rho(x, y, z)$ gives the **density** at a point (x, y, z) of the region V , then

$$\iiint_V \rho(x, y, z) \, dx dy dz = \text{mass of the region } V.$$

(c) If the scalar function $\rho(x, y, z)$ gives the **charge density** at a point (x, y, z) of the region V , then

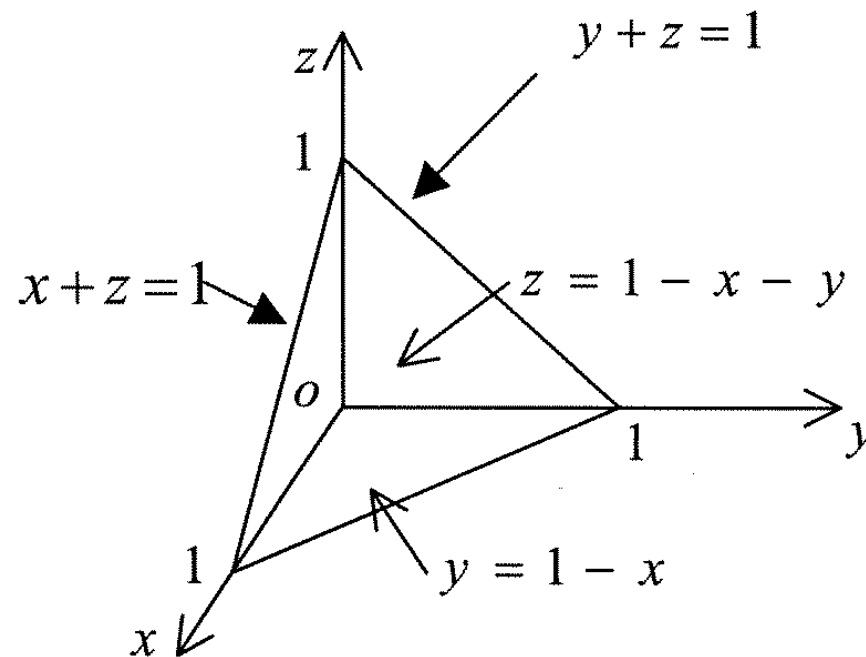
$$\iiint_V \rho(x, y, z) \, dx dy dz = \text{total charge within the region } V.$$

Example 12

Evaluate

$$\iiint_V \frac{1}{(x + y + 2z + 1)^3} dx dy dz$$

where V is the region enclosed by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.



Solution

The projection of V on xy -plane is σ_{xy} which is bounded by $x = 0$, $y = 0$, $y = 1 - x$.

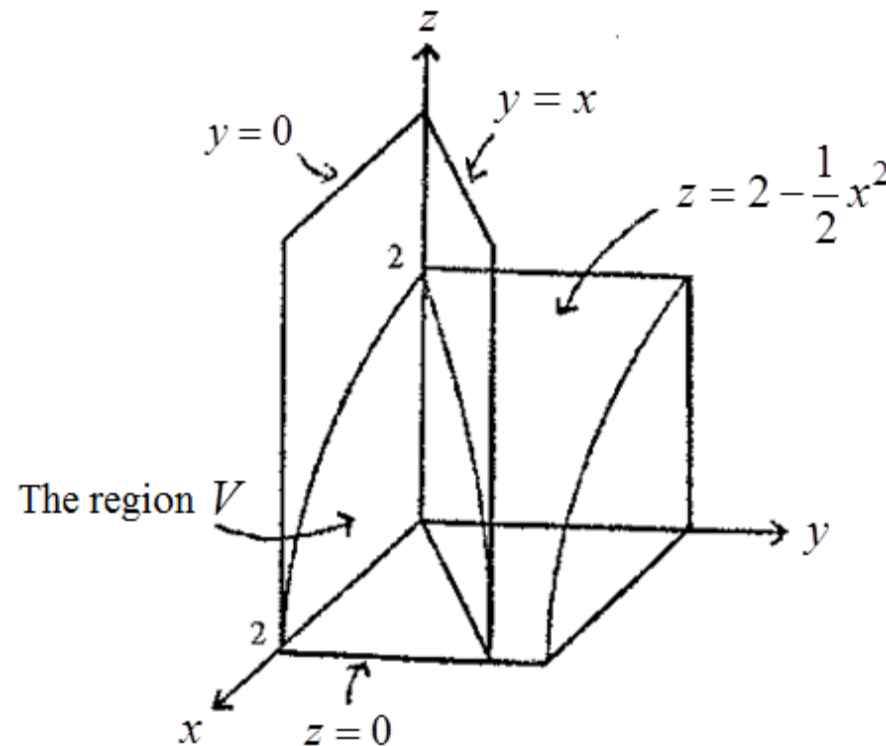
$$\begin{aligned}
 \iiint_V \frac{1}{(x+y+2z+1)^3} dx dy dz &= \iint_{\sigma_{xy}} \left[\int_0^{1-x-y} \frac{1}{(x+y+2z+1)^3} dz \right] dx dy \\
 &= \int_0^1 \left\{ \int_0^{1-x} \left[\int_0^{1-x-y} \frac{1}{(x+y+2z+1)^3} dz \right] dy \right\} dx = \int_0^1 \left\{ \int_0^{1-x} \left[\frac{-1}{4(x+y+2z+1)^2} \right]_0^{1-x-y} dy \right\} dx \\
 &= \int_0^1 \left\{ \int_0^{1-x} \left[\frac{1}{4(x+y+1)^2} - \frac{1}{4(3-x-y)^2} \right] dy \right\} dx = \int_0^1 \left[\frac{-1}{4(x+y+1)} - \frac{1}{4(3-x-y)} \right]_0^{1-x} dx \\
 &= \int_0^1 \left[\frac{-1}{8} - \frac{1}{8} - \left(-\frac{1}{4(x+1)} - \frac{1}{4(3-x)} \right) \right] dx = \left[-\frac{x}{4} + \frac{1}{4} \ln|x+1| - \frac{1}{4} \ln|3-x| \right]_0^1 \\
 &= \left(-\frac{1}{4} + \frac{1}{4} \ln 2 - \frac{1}{4} \ln 2 \right) - \left(0 + \frac{1}{4} \ln 1 - \frac{1}{4} \ln 3 \right) \\
 &= \frac{\ln 3}{4} - \frac{1}{4}
 \end{aligned}$$

Example 13

Evaluate

$$\iiint_V 2xyz \, dx dy dz,$$

where V is the region bounded by the parabolic cylinder $z = 2 - \frac{1}{2}x^2$ and the planes $y = x$, $y = 0$ and $z = 0$ as shown in the following diagram.



Solution

Projecting V onto the xz -plane gives σ_{xz} which is bounded by $z = 2 - \frac{1}{2}x^2$, $x = 0$, $z = 0$.

$$\begin{aligned}
 \iiint_V 2xyz \, dx \, dy \, dz &= \iint_{\sigma_{xz}} \left[\int_0^x 2xyz \, dy \right] dx \, dz = \int_0^2 \left[\int_0^{2-\frac{1}{2}x^2} \left(\int_0^x 2xyz \, dy \right) dz \right] dx \\
 &= \int_0^2 \left(\int_0^{2-\frac{1}{2}x^2} [xy^2z]_0^x dz \right) dx = \int_0^2 \left(\int_0^{2-\frac{1}{2}x^2} x^3z \, dz \right) dx \\
 &= \int_0^2 \left(\left[\frac{x^3z^2}{2} \right]_0^{2-\frac{1}{2}x^2} \right) dx = \int_0^2 \frac{x^3 \left(2 - \frac{1}{2}x^2 \right)^2}{2} dx = \int_0^2 \left(2x^3 - x^5 + \frac{1}{8}x^7 \right) dx \\
 &= \left[\frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{64} \right]_0^2 = \left(\frac{2^4}{2} - \frac{2^6}{6} + \frac{2^8}{64} \right) - 0 \\
 &= \frac{4}{3}
 \end{aligned}$$

Alternatively, projecting V onto the xy -plane gives σ_{xy} which is bounded by $y = 0$, $y = x$, $x = 2$.

$$\begin{aligned}
 \iiint_V 2xyz \, dx \, dy \, dz &= \iint_{\sigma_{xy}} \left[\int_0^{2-\frac{1}{2}x^2} 2xyz \, dz \right] dx \, dy = \int_0^2 \left[\int_0^x \left(\int_0^{2-\frac{1}{2}x^2} 2xyz \, dz \right) dy \right] dx \\
 &= \int_0^2 \left[\int_0^x [xyz^2]_0^{2-\frac{1}{2}x^2} dy \right] dx = \int_0^2 \left[\int_0^x xy \left(2 - \frac{1}{2}x^2 \right)^2 dy \right] dx \\
 &= \int_0^2 \left[\frac{xy^2}{2} \left(2 - \frac{1}{2}x^2 \right)^2 \right]_0^x dx = \int_0^2 \frac{x^3}{2} \left(2 - \frac{1}{2}x^2 \right)^2 dx \\
 &= \int_0^2 \left(2x^3 - x^5 + \frac{1}{8}x^7 \right) dx = \left[\frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{64} \right]_0^2 = \left(\frac{2^4}{2} - \frac{2^6}{6} + \frac{2^8}{64} \right) - 0 \\
 &= \frac{4}{3}
 \end{aligned}$$

Example 14

Evaluate

$$\iiint_V xyz \, dx dy dz,$$

where V is the region enclosed by the surface $x^2 + y^2 + z^2 = 1$ where $x \geq 0$, $y \geq 0$, $z \geq 0$ and the planes $x = 0$, $y = 0$, $z = 0$.

Solution

The projection of V onto xy -plane is σ_{xy} which is bounded by the curve $x^2 + y^2 = 1$, where $x \geq 0$, $y \geq 0$ and the lines $x = 0$, $y = 0$.

$$\sigma_{xy}: \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq \sqrt{1 - x^2} \end{cases}$$

Then $0 \leq z \leq \sqrt{1 - x^2 - y^2}$.

$$\begin{aligned}
\iiint_V xyz \, dx dy dz &= \iint_{\sigma_{xy}} \left[\int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \right] dx dy = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \right) dy \right] dx \\
&= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\left[\frac{xyz^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} \right) dy \right] dx = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\frac{xy(1-x^2-y^2)}{2} \right) dy \right] dx \\
&= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\frac{xy - x^3y - xy^3}{2} \right) dy \right] dx = \frac{1}{2} \int_0^1 \left(\left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4} \right]_0^{\sqrt{1-x^2}} \right) dx \\
&= \frac{1}{2} \int_0^1 \left[\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4} \right] dx = \frac{1}{8} \int_0^1 (x - 2x^3 + x^5) dx \\
&= \frac{1}{8} \left[\frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{6} \right]_0^1 \\
&= \frac{1}{48}
\end{aligned}$$

Example 15

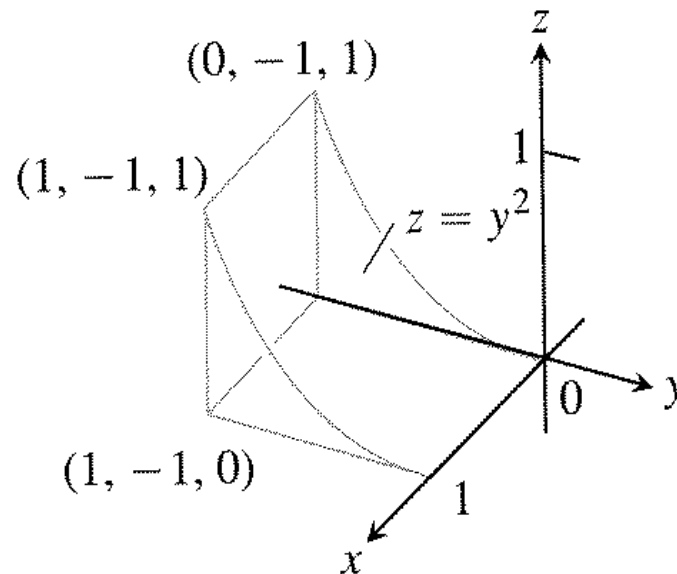
An iterated integral like

$$\int_0^1 \left[\int_{-1}^0 \left(\int_0^{y^2} f(x, y, z) dz \right) dy \right] dx$$

is called an iterated integral with **order $dz dy dx$** .

With the aid of the following figure, change the order of the iterated integral

$\int_0^1 \left[\int_{-1}^0 \left(\int_0^{y^2} f(x, y, z) dz \right) dy \right] dx$ to an equivalent iterated integral with order $dy dz dx$.



Solution

$$\begin{aligned} \int_0^1 \left[\int_{-1}^0 \left(\int_0^{y^2} f(x, y, z) dz \right) dy \right] dx &= \int_0^1 \left(\iint_{\sigma_{yz}} f(x, y, z) dy dz \right) dx \\ &= \int_0^1 \left[\int_0^1 \left(\int_{-1}^{-\sqrt{z}} f(x, y, z) dy \right) dz \right] dx \end{aligned}$$

Example 16

Find the volume V of the region R bounded by the parabolic cylinder $z = 4 - x^2$ in the first octant and the planes $x = 0$, $y = 0$, $y = 6$, $z = 0$.

Solution

$$\begin{aligned} V &= \iiint_R 1 \, dx dy dz = \int_0^2 \left[\int_0^6 \left(\int_0^{4-x^2} 1 \, dz \right) dy \right] dx \\ &= \int_0^2 \left[\int_0^6 [z]_0^{4-x^2} dy \right] dx = \int_0^2 \left[\int_0^6 (4 - x^2) dy \right] dx \\ &= \int_0^2 [(4 - x^2)y]_0^6 dx = \int_0^2 6(4 - x^2) dx \\ &= [24x - 2x^3]_0^2 \\ &= [24(2) - 2(2)^3] - 0 \\ &= 32 \end{aligned}$$

4 Change of Variables in Triple Integrals

Consider the triple integral

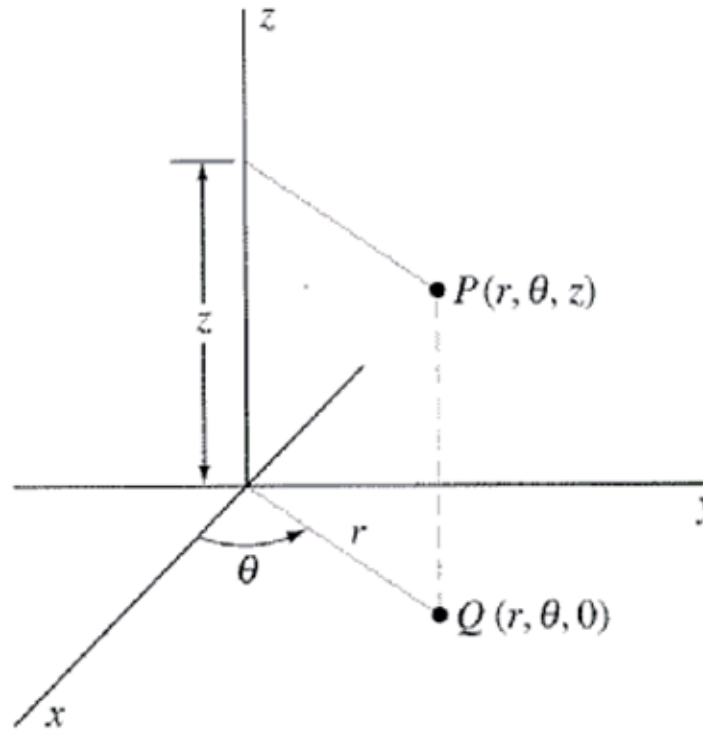
$$I = \iiint_V f(x, y, z) \, dx \, dy \, dz.$$

Using the change of variables $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, we have

$$I = \iiint_{V^*} f[x(u, v, w), y(u, v, w), z(u, v, w)] \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw,$$

where $J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$ is the **Jacobian** of the transformation.

V^* is the region in uvw -space corresponding to the region V in xyz -space in a one-to-one manner. The most popular alternative coordinate systems to Cartesian coordinates are **cylindrical polar coordinates** and **spherical polar coordinates**.

Cylindrical Polar Coordinates:

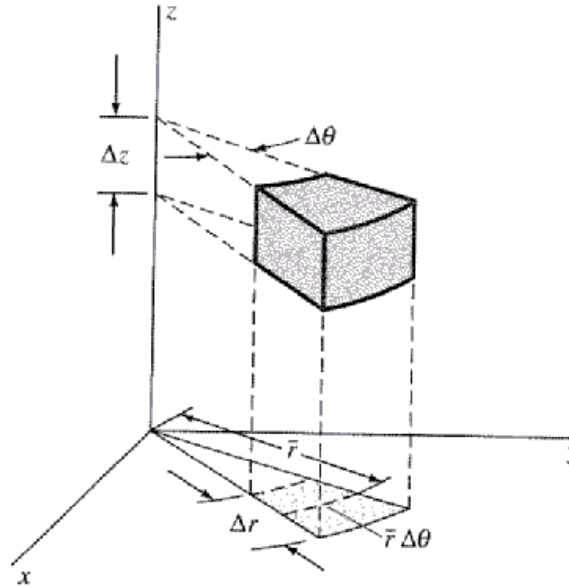
$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

$$r \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad -\infty < z < \infty$$

In particular: $r = \text{constant} \rightarrow \text{cylinder}$

$\theta = \text{constant} \rightarrow \text{plane}$

$z = \text{constant} \rightarrow \text{plane}$



The volume of the cylindrical block is approximately $\bar{r} \Delta r \Delta \theta \Delta z$. (Note $\delta \equiv \Delta$)

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \det \begin{pmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.$$

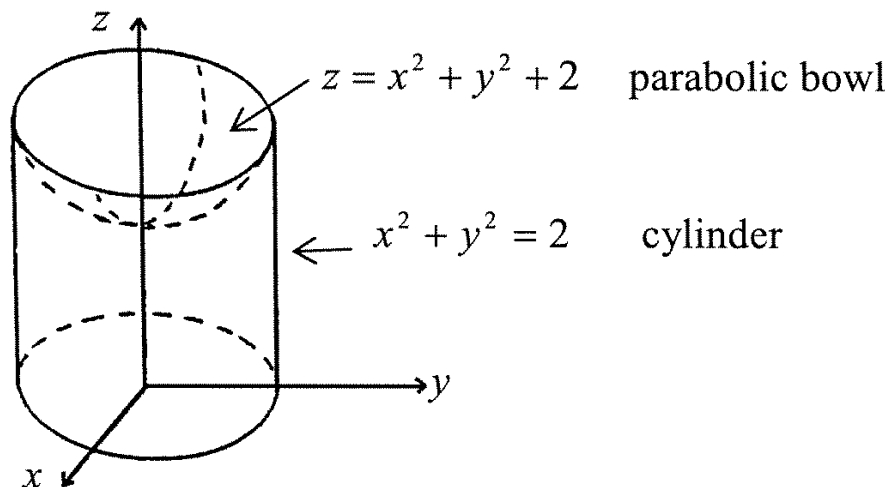
$$\Rightarrow |J| = r$$

$$\therefore \boxed{dx \, dy \, dz \rightarrow r \, dr \, d\theta \, dz},$$

where $r \, dr \, d\theta \, dz$ corresponds to an element of volume in cylindrical polar coordinates.

Example 17

Find the volume of the region V between the surfaces $x^2 + y^2 = 2$, $z = x^2 + y^2 + 2$ and the plane $z = 0$.

**Solution****Cartesian coordinates:**

We observe that the projection of V onto xy -plane is $x^2 + y^2 \leq 2$. So

$$\begin{aligned}
 \text{Volume} &= \iiint_V 1 \, dx \, dy \, dz = \iint_{x^2+y^2 \leq 2} \left(\int_0^{x^2+y^2+2} 1 \, dz \right) dx \, dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left[\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \left(\int_0^{x^2+y^2+2} 1 \, dz \right) dy \right] dx \\
 &= \dots
 \end{aligned}$$

Cylindrical polar coordinates:

Under the cylindrical transformations $x = r \cos \theta$, $y = r \sin \theta$ and $z = z$, the region V is transformed into the region V' in $r\theta z$ -space.

The projection of V' onto $r\theta$ -plane is $0 \leq \theta \leq 2\pi$, $0 \leq r \leq \sqrt{2}$.

Then $0 \leq z \leq x^2 + y^2 + 2$, i.e. $0 \leq z \leq r^2 + 2$.

Thus,

$$\begin{aligned}
 \text{Volume} &= \iiint_V 1 \, dx \, dy \, dz = \iiint_{V'} r \, dr \, d\theta \, dz = \iint_{\substack{0 \leq \theta \leq 2\pi \\ 0 \leq r \leq \sqrt{2}}} \left(\int_0^{r^2+2} r \, dz \right) dr \, d\theta \\
 &= \int_0^{\sqrt{2}} \left[\int_0^{2\pi} \left(\int_0^{r^2+2} r \, dz \right) d\theta \right] dr = \int_0^{\sqrt{2}} \left[\int_0^{2\pi} ([rz]_0^{r^2+2}) d\theta \right] dr = \int_0^{\sqrt{2}} \left[\int_0^{2\pi} r(r^2 + 2) d\theta \right] dr \\
 &= 2\pi \int_0^{\sqrt{2}} (r^3 + 2r) dr = 2\pi \left[\frac{r^4}{4} + r^2 \right]_0^{\sqrt{2}} = 2\pi \left[\left(\frac{(\sqrt{2})^4}{4} + (\sqrt{2})^2 \right) - 0 \right] = 6\pi
 \end{aligned}$$

Example 18

Using cylindrical polar coordinates, evaluate

$$\iiint_V (x^2 + y^2) \, dx \, dy \, dz,$$

where V is the solid enclosed by $x^2 + y^2 = 2z$ and $z = 2$.

Solution

Under the cylindrical transformations $x = r \cos \theta$, $y = r \sin \theta$ and $z = z$, the solid V in xyz -space is transformed into V' in $r\theta z$ -space.

The projection of V' onto $r\theta$ -plane is

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2.$$

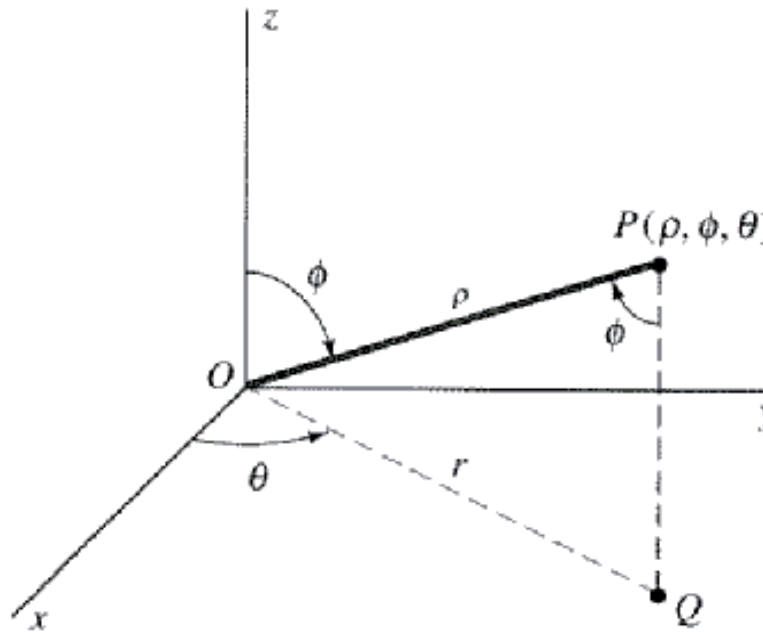
$$\text{Then } \frac{x^2 + y^2}{2} = \frac{r^2}{2} \leq z \leq 2.$$

Thus,

$$V': 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2, \quad \frac{r^2}{2} \leq z \leq 2.$$

Then

$$\begin{aligned}\iiint_V (x^2 + y^2) dx dy dz &= \iiint_{V'} r^2 r dr d\theta dz = \iint_{\substack{0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2}} \left(\int_{\frac{r^2}{2}}^2 r^3 dz \right) dr d\theta \\ &= \int_0^2 \left[\int_0^{2\pi} \left(\int_{\frac{r^2}{2}}^2 r^3 dz \right) d\theta \right] dr = \int_0^2 \left[\int_0^{2\pi} r^3 \left(2 - \frac{r^2}{2} \right) d\theta \right] dr \\ &= \int_0^2 2\pi \left(2r^3 - \frac{r^5}{2} \right) dr = 2\pi \left[\frac{r^4}{2} - \frac{r^6}{12} \right]_0^2 \\ &= 2\pi \left[\left(\frac{2^4}{2} - \frac{2^6}{12} \right) - 0 \right] \\ &= \frac{16\pi}{3}\end{aligned}$$

Spherical Polar Coordinates:

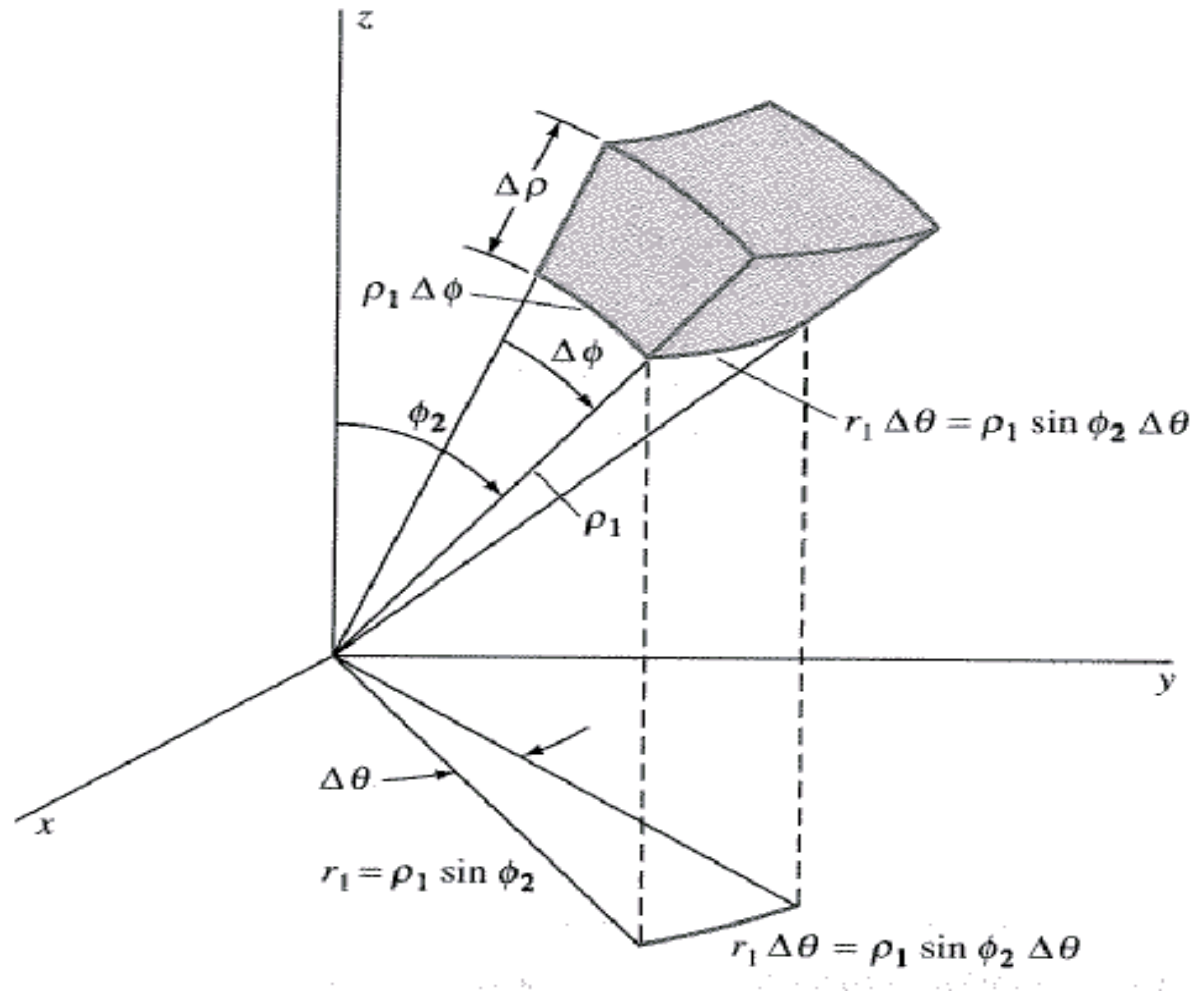
$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

$$\rho \geq 0, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

In particular: $\rho = \text{constant} \rightarrow \text{sphere}$

$\phi = \text{constant} \rightarrow \text{cone}$

$\theta = \text{constant} \rightarrow \text{plane}$



The volume of the spherical block is approximately $\rho^2 \sin \phi \Delta\rho \Delta\phi \Delta\theta$.

The corresponding **Jacobian** is

$$\begin{aligned}
 J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} &= \det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \det \begin{pmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{pmatrix} \\
 &= \cos \phi \begin{vmatrix} \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} - (-\rho \sin \phi) \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} \\
 &= \cos \phi [\rho^2 \cos \phi \sin \phi (\cos^2 \theta + \sin^2 \theta)] + \rho \sin \phi [\rho \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)] \\
 &= \rho^2 (\cos^2 \phi + \sin^2 \phi) \sin \phi \\
 &= \rho^2 \sin \phi
 \end{aligned}$$

$$\Rightarrow |J| = |\rho^2 \sin \phi| = \rho^2 \sin \phi \quad (\because \sin \phi \geq 0 \text{ for } 0 \leq \phi \leq \pi.)$$

$$\therefore \boxed{dx \, dy \, dz \rightarrow \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta},$$

where $\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$ corresponds to an element of volume in spherical polar coordinates.

Example 19

Find an equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$ in spherical coordinates.

Solution

$$x^2 + y^2 + (z - 1)^2 = 1$$

$$\begin{array}{l} \Rightarrow \\ \underbrace{x = \rho \sin \phi \cos \theta} \\ \underbrace{y = \rho \sin \phi \sin \theta} \\ \underbrace{z = \rho \cos \phi} \end{array} \quad \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + (\rho \cos \phi - 1)^2 = 1$$

$$\Rightarrow \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + (\rho^2 \cos^2 \phi - 2\rho \cos \phi + 1) = 1$$

$$\Rightarrow \rho^2 (\sin^2 \phi + \cos^2 \phi) - 2\rho \cos \phi = 0$$

$$\Rightarrow \rho^2 - 2\rho \cos \phi = 0$$

$$\Rightarrow \rho(\rho - 2 \cos \phi) = 0$$

$$\Rightarrow \rho = 0 \quad \text{or} \quad \rho = 2 \cos \phi$$

Note that $\rho = 0$ only corresponds to the point $(0, 0, 0)$, it does not represent the given sphere.

$\therefore$ In spherical coordinates, the equation for the given sphere is

$$\rho = 2 \cos \phi.$$

Example 20

Find the volume of the “ice cream cone” D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \frac{\pi}{3}$.

Solution

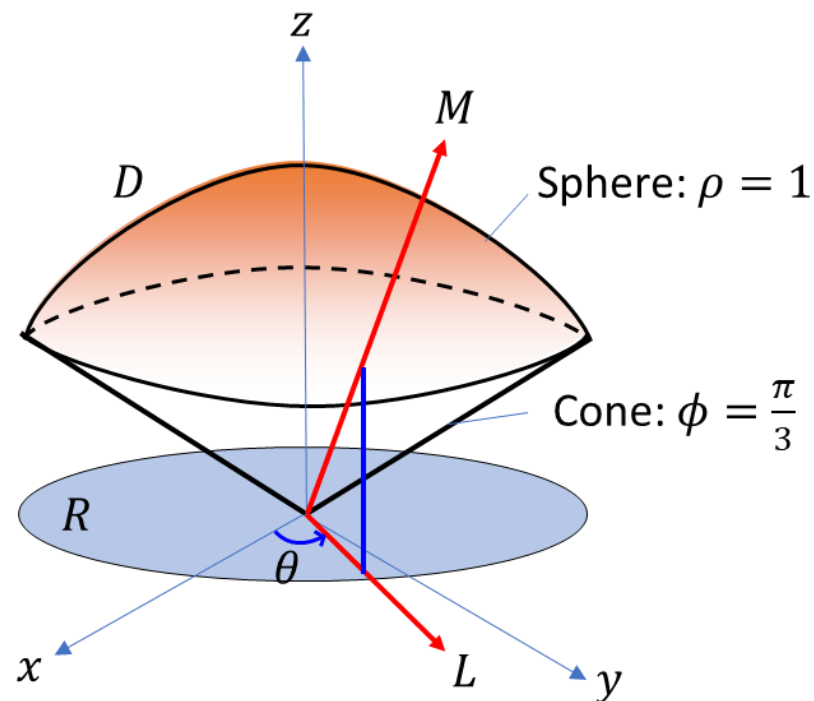
The volume is $V = \iiint_D \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.

To find the limits of integration for evaluating the integral, we take the following steps.

Step 1: Sketch the diagram.

We sketch the solid D and its projection R on the xy -plane.

We draw a ray M from the origin through D making an angle ϕ with the positive z -axis. We also draw L , the projection of M on the xy -plane, along with the angle θ that L makes with the positive x -axis.



Step 2: Find the ρ -limits of integration.

Ray M enters D at $\rho = 0$ and leaves D at $\rho = 1$.

$$\therefore 0 \leq \rho \leq 1.$$

Step 3: Find the ϕ -limits of integration.

The cone $\phi = \frac{\pi}{3}$ makes an angle of $\frac{\pi}{3}$ with the positive z -axis.

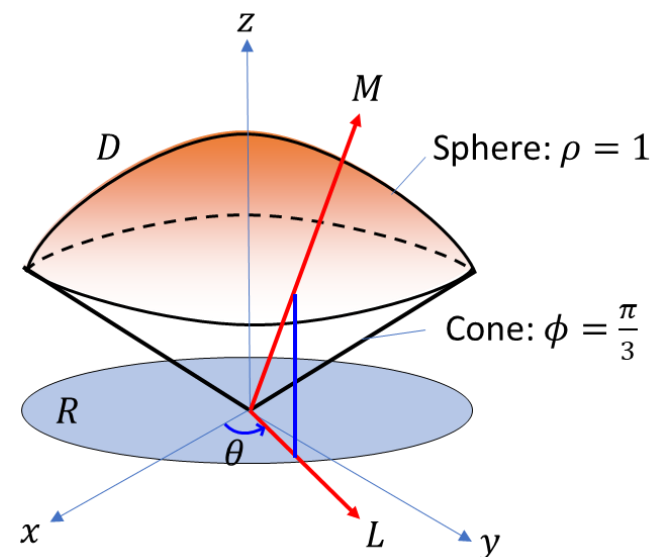
For any given θ , the angle ϕ can run from $\phi = 0$ to $\phi = \frac{\pi}{3}$.

$$\therefore 0 \leq \phi \leq \frac{\pi}{3}.$$

Step 4: Find the θ -limits of integration.

The ray L sweeps over R as θ runs from $\theta = 0$ to $\theta = 2\pi$.

$$\therefore 0 \leq \theta \leq 2\pi.$$



∴ The volume is

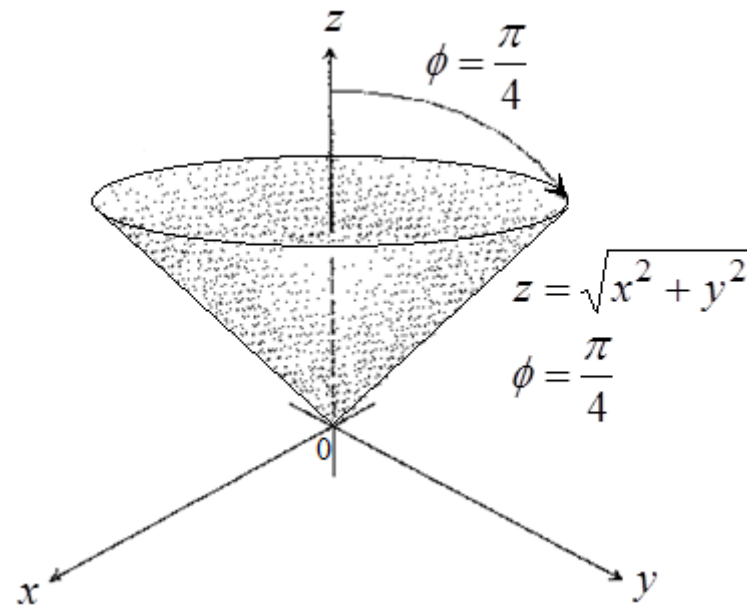
$$\begin{aligned} V &= \iiint_D \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \left[\int_0^{\frac{\pi}{3}} \left(\int_0^1 \rho^2 \sin \phi \, d\rho \right) d\phi \right] d\theta \\ &= \int_0^{2\pi} \left[\int_0^{\frac{\pi}{3}} \left(\left[\frac{\rho^3}{3} \sin \phi \right]_0^1 \right) d\phi \right] d\theta = \int_0^{2\pi} \left[\int_0^{\frac{\pi}{3}} \frac{1}{3} \sin \phi \, d\phi \right] d\theta \\ &= \int_0^{2\pi} \left(\left[-\frac{1}{3} \cos \phi \right]_0^{\frac{\pi}{3}} \right) d\theta = \int_0^{2\pi} -\frac{1}{3} \left(\frac{1}{2} - 1 \right) d\theta \\ &= \frac{1}{6} [\theta]_0^{2\pi} \\ &= \frac{\pi}{3} \end{aligned}$$

Example 21

Convert

$$\int_{-1}^1 \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{\sqrt{x^2+y^2}}^1 1 \, dz \right) dy \right] dx$$

to an integral using spherical coordinates with order $d\theta \, d\rho \, d\phi$ and then evaluate the obtained integral.



Solution

For

$$\int_{-1}^1 \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{\sqrt{x^2+y^2}}^1 1 \, dz \right) dy \right] dx,$$

the integration is done in the solid V which is bounded above by $z = 1$ and bounded below by $z = \sqrt{x^2 + y^2}$.

Under the spherical transformations

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta, \\ z = \rho \cos \phi \end{cases}$$

V is transformed into

$$V^*: \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \frac{\pi}{4} \\ 0 \leq \rho \leq \sec \phi \end{cases}.$$

$$\begin{aligned}
\int_{-1}^1 \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{\sqrt{x^2+y^2}}^1 1 \, dz \right) dy \right] dx &= \int_0^{\frac{\pi}{4}} \left[\int_0^{\sec \phi} \left(\int_0^{2\pi} \rho^2 \sin \phi \, d\theta \right) d\rho \right] d\phi \\
&= \int_0^{\frac{\pi}{4}} \left(\int_0^{\sec \phi} 2\pi \rho^2 \sin \phi \, d\rho \right) d\phi = 2\pi \int_0^{\frac{\pi}{4}} \left(\left[\frac{\rho^3}{3} \sin \phi \right]_0^{\sec \phi} \right) d\phi \\
&= \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \sec^3 \phi \sin \phi \, d\phi = \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \frac{\sin \phi}{\cos^3 \phi} \, d\phi \\
&= -\frac{2\pi}{3} \int_0^{\frac{\pi}{4}} (\cos \phi)^{-3} \, d(\cos \phi) = -\frac{2\pi}{3} \left[\frac{(\cos \phi)^{-2}}{-2} \right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{3} \left(\frac{1}{\cos^2 \frac{\pi}{4}} - \frac{1}{\cos^2 0} \right) = \frac{\pi}{3} \left(\frac{1}{\left(\frac{1}{2}\right)} - \frac{1}{1} \right) \\
&= \frac{\pi}{3}
\end{aligned}$$