

Limits

(a) $\lim_{x \rightarrow \infty} \frac{x^2 + 1}{2x^3 - x}$ (b) $\lim_{x \rightarrow \infty} \frac{x + \sqrt{x^4 - x^2 + 1}}{2x^2 + 1 + \sqrt{x^4 + 1}}$ (c) $\lim_{x \rightarrow 0} \frac{m \sin(mx) - n \sin(nx)}{\tan(mx) + \tan(nx)}$

(a)

$$\lim_{x \rightarrow \infty} \frac{x^2 + 1}{2x^3 - x} = 0$$

(b)

$$\lim_{x \rightarrow \infty} \frac{x + \sqrt{x^4 - x^2 + 1}}{2x^2 + 1 + \sqrt{x^4 + 1}} = \frac{1}{3}$$

(c)

2.

a) $\frac{1}{3}$	*b) Does Not Exist (Left Hand Limit $\neq$ Right Hand Limit)	c) $\frac{a}{b}$
d) 0	e) 9	f) 2
g) -6		

Solution:

*4. Evaluate $\lim_{n \rightarrow \infty} \cos \frac{\theta}{2} \cos \frac{\theta}{4} \cos \frac{\theta}{8} \cdots \cos \frac{\theta}{2^n}$, where $\theta \neq 0$.

Solution:

Note that

$$\begin{aligned} \cos \frac{\theta}{2} \cos \frac{\theta}{4} \cos \frac{\theta}{8} \cdots \left(\sin \frac{\theta}{2^n} \cos \frac{\theta}{2^n} \right) &= \cos \frac{\theta}{2} \cos \frac{\theta}{4} \cos \frac{\theta}{8} \cdots \cos \frac{\theta}{2^{n-1}} \left(\frac{1}{2} \sin \frac{\theta}{2^{n-1}} \right) \\ &= \frac{1}{2} \cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \cos \frac{\theta}{2^3} \cdots \cos \frac{\theta}{2^{n-2}} \left(\frac{1}{2} \sin \frac{\theta}{2^{n-2}} \right) = \frac{1}{2^2} \cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \cos \frac{\theta}{2^3} \cdots \cos \frac{\theta}{2^{n-3}} \left(\frac{1}{2} \sin \frac{\theta}{2^{n-3}} \right) \\ &= \cdots = \frac{1}{2^{n-2}} \cos \frac{\theta}{2} \left(\frac{1}{2} \sin \frac{\theta}{2} \right) = \frac{1}{2^{n-1}} \left(\frac{1}{2} \sin \theta \right) = \frac{1}{2^n} \sin \theta. \\ \Rightarrow \cos \frac{\theta}{2} \cos \frac{\theta}{4} \cos \frac{\theta}{8} \cdots \cos \frac{\theta}{2^n} &= \frac{\sin \theta}{2^n \sin \frac{\theta}{2^n}} = \frac{\sin \theta}{\frac{\theta \sin \frac{\theta}{2^n}}{\frac{\theta}{2^n}}} = \frac{\sin \theta}{\theta} \frac{1}{\frac{\sin \frac{\theta}{2^n}}{\frac{\theta}{2^n}}}. \end{aligned}$$

Continuity

5. Discuss the continuity of the following functions at $x = 0$:

$$(a) \quad f(x) = \frac{x^2}{x} \quad (b) \quad h(x) = \begin{cases} |x| & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \quad (c) \quad f(x) = \begin{cases} x^2 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Solution:

(a)

$$f(x) = \frac{x^2}{x} \text{ is not continuous at } x = 0.$$

(b)

$$\lim_{x \rightarrow 0^+} h(x) = \lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0 = \lim_{x \rightarrow 0^-} h(x) = \lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) \Rightarrow \lim_{x \rightarrow 0} h(x) = 0$$

But $\lim_{x \rightarrow 0} h(x) = 0 \neq h(0) = 1$. We conclude that $h(x) = \begin{cases} |x| & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$ is not continuous at $x = 0$.

(c)

$f(x)$ is continuous at $x = 0$.

6. Define $f(0)$ for the following functions such that they are continuous at $x = 0$.

$$(a) \quad f(x) = \sin x \sin \frac{1}{x} \quad (b) \quad f(x) = \frac{\tan(2x)}{x}.$$

Solution

(a)

$$0 \leq \left| \sin x \sin \frac{1}{x} \right| = \left| \sin x \right| \left| \sin \frac{1}{x} \right| \leq \left| \sin x \right|. \text{ As } \lim_{x \rightarrow 0} |\sin x| = 0, \text{ we have } \lim_{x \rightarrow 0} \left| \sin x \sin \frac{1}{x} \right| = 0$$

Define $f(0) = 0$, we then have $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \sin x \sin \frac{1}{x} = 0 = f(0)$. That means $f(x)$ is continuous at $x = 0$.

(b)

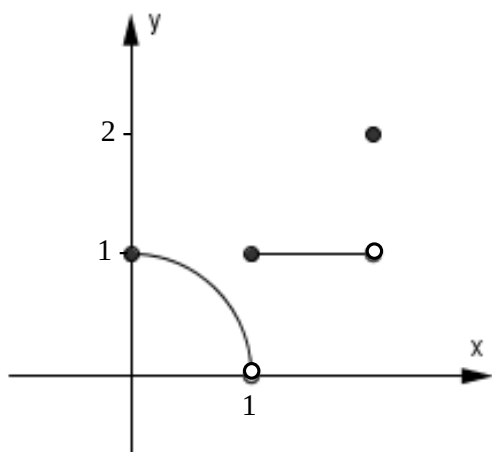
As $f(x) = \frac{\tan(2x)}{x} = \frac{\sin(2x)}{2x} \frac{2}{\cos(2x)}$, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\tan(2x)}{x} = \lim_{x \rightarrow 0} \frac{\sin(2x)}{2x} \frac{2}{\cos(2x)} = 2$.

Define $f(0) = 2$, we then have $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\tan(2x)}{x} = 2 = f(0)$. That means $f(x)$ is continuous at $x = 0$.

7. Sketch the graph of the following function on $[0, 2]$

$$f(x) = \begin{cases} \sqrt{1-x^2} & 0 \leq x < 1 \\ 1 & \text{for } 1 \leq x < 2 \\ 2 & x = 2 \end{cases}$$

- (a) For what values of c in the domain does $\lim_{x \rightarrow c} f(x)$ exist?
 (b) At what points does only the left-hand limit exist?
 (c) At what points does only the right-hand limit exist?



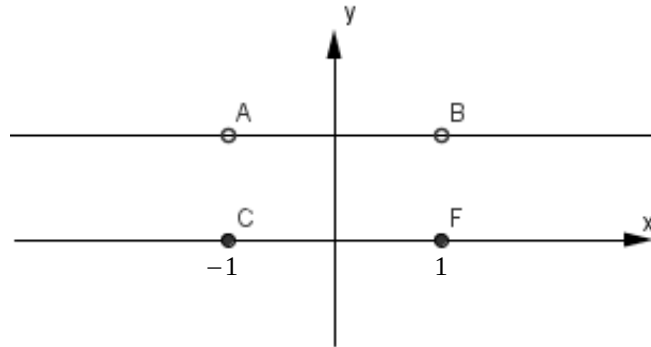
- a) $(0, 1) \cup (1, 2)$ b) $(0, 2]$ c) $[0, 2)$

8. Given the function $y = f(x)$ defined as follows:

$$f(x) = \begin{cases} 0, & x^2 = 1 \\ 1, & \text{otherwise} \end{cases}$$

Sketch the function. At what points is the function discontinuous? Explain.

The function looks like this:



It is discontinuous at $x = -1, 1$, because $\lim_{x \rightarrow -1, 1} f(x) = 1 \neq 0 = f(1) = f(-1)$

Differentiability

9. Given $f(x) = \begin{cases} \frac{1}{2}(x^2 - 4) & 0 < x < 2, \\ 0 & x = 2, \\ \frac{2}{x^2}(x^2 - 4) & x > 2. \end{cases}$ Show that f is differentiable at $x = 2$.

Solution

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0^+} \frac{\frac{2}{(2+h)^2}((2+h)^2 - 4) - 0}{h} = \lim_{h \rightarrow 0^+} \frac{2(2+h)^2 - 8}{h(2+h)^2} = \lim_{h \rightarrow 0^+} \frac{8h + 2h^2}{h(2+h)^2} \\ &= \lim_{h \rightarrow 0^+} \frac{8 + 2h}{(2+h)^2} = 2 \end{aligned}$$

$$\lim_{h \rightarrow 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0^-} \frac{\frac{1}{2}((2+h)^2 - 4) - 0}{h} = \lim_{h \rightarrow 0^-} \frac{4h + h^2}{2h} = \lim_{h \rightarrow 0^-} \frac{4+h}{2} = 2$$

$\therefore f$ is differentiable at $x = 2$.

-End-