

Short Notes on Complex Numbers

The imaginary number

An imaginary number carries the symbol of j . In engineering, we typically use j to represent an imaginary number (instead of i , so as not confuse with the symbol for current).

By definition an imaginary number is defined by:

$$j = \sqrt{-1}$$

So now if you want to find the square root of 3, we can use the imaginary number to do so:

$$\sqrt{-3} = j\sqrt{3}$$

By convention, we position the symbol j in front of the number rather than behind.

If we add a real number to an imaginary number, we get a complex number. As an example:

$$z = 3 + j4$$

Here, z is a complex number. Being a complex number, we see that z contains 2 parts: a real part and an imaginary part. If we want to refer to the real part of z , we use Re . If we want to refer to the imaginary part, we use Im . For example:

$$\text{Re}\{z\} = 3$$

$$\text{Im}\{z\} = 4$$

Graphically, we can express a complex number on an Argand diagram like the following:

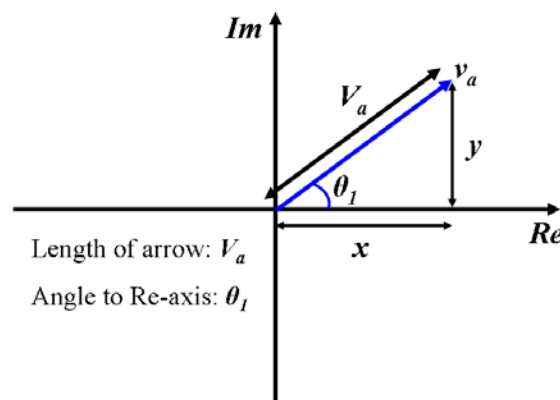


Figure 1: Argand diagram and representation of a complex number

The real part of the complex number has a value of x , while the imaginary part has a value of y . Just to check that you have got the concept, write down the complex number in terms of x and y , followed by identifying the real and imaginary parts:

$$z = x + jy$$

$$\text{Re}\{z\} = x$$

$$\text{Im}\{z\} = y$$

Note that the real part of the complex number is denoted by the x -coordinate of the point on Figure 1 while the imaginary part of the complex number is denoted by the y -coordinate of that point.

Conjugates

The conjugate of z as defined above is denoted as z^* . This can be described as the reflection of z about the x-axis. Hence,

$$\text{If } z = a + jb, \text{ then } z^* = a - jb$$

If we multiply z by z^* , we get:

$$z z^* = (a + jb)(a - jb) = a^2 + b^2$$

This is a useful trick to use to change from a complex to a real form.

Magnitude and phase

Besides referring to z using the rectangular form of: $z = x + jy$, we can refer to it in terms of the magnitude and the phase. This form is known as a **phasor**.

Magnitude

The magnitude refers to the length between the point and the origin (0, 0). With reference to the complex number in Figure 1, this is equal to the value V_a . We use the modulus operator $|z|$ to represent the magnitude of the complex number z :

$$|z| = |x + jy| = V_a$$

By Pythagoras' theorem:

$$|x + jy| = \sqrt{x^2 + y^2}$$

Phase

The phase refers to the angle between of z (from the origin) and the Re-axis. The phase of the complex number z is denoted by the symbol θ_1 . Note that the phase of z is also known as the argument of z , and is referred to using the short form $\arg$:

$$\arg z = \theta_1 = \tan^{-1} \frac{y}{x}$$

Phasor

A phasor is an alternative representation form of a complex number in terms of just its phase and magnitude. For the example in Fig 1, the complex number expressed as a phasor is:

$$z = |z| \angle \theta_1$$

Euler's Identity

By applying basic trigonometry to the x and y components of z in Fig 1, we obtain:

$$\begin{aligned} x &= |z| \cos \theta_1 \\ y &= |z| \sin \theta_1 \end{aligned}$$

Hence we can re-write the rectangular form of z as:

$$z = |z|(\cos \theta_1 + j \sin \theta_1)$$

In fact, we can actually write the complex number as an exponential:

$$e^{j\theta_1} = \cos \theta_1 + j \sin \theta_1$$

This is known as Euler's identity. The proof for the identity is omitted from this short note and beyond the scope expected of EE2301 which focuses on circuits.

For a complex number like the one in Fig 1 that has a magnitude of $|z|$, the Euler form of the complex number is then:

$$z = |z|e^{j\theta_1}$$

We will use the Euler identity in deriving the impedance expressions for a capacitor and also an inductor in the main lecture notes of EE2301.

The focus of this course is not on the mathematics, but using the mathematics as a tool. So do not get caught up with understanding mathematical proofs.

Performing mathematical operations with complex numbers

Addition and subtraction

Let us say there are two complex numbers:

$$z_a = Z_a e^{j\theta_a}$$

$$z_b = Z_b e^{j\theta_b}$$

If we add,

$$z_a + z_b = Z_a e^{j\theta_a} + Z_b e^{j\theta_b}$$

$$z_a + z_b = Z_a \cos \theta_a + Z_b \cos \theta_b + j(Z_a \sin \theta_a + Z_b \sin \theta_b)$$

If we subtract,

$$z_a - z_b = Z_a e^{j\theta_a} - Z_b e^{j\theta_b}$$

$$z_a - z_b = Z_a \cos \theta_a - Z_b \cos \theta_b + j(Z_a \sin \theta_a - Z_b \sin \theta_b)$$

Multiplication

If we multiply, (1) the magnitudes multiply and (2) the phases add up:

$$z_a z_b = Z_a e^{j\theta_a} \times Z_b e^{j\theta_b} = Z_a Z_b e^{j(\theta_a + \theta_b)}$$

$$z_a z_b = Z_a Z_b \angle(\theta_a + \theta_b)$$

Division

Similarly, if we divide, (1) the magnitudes now divide and (2) the phases subtract:

$$\frac{z_a}{z_b} = \frac{Z_a e^{j\theta_a}}{Z_b e^{j\theta_b}} = \frac{Z_a}{Z_b} e^{j(\theta_a - \theta_b)}$$

$$\frac{z_a}{z_b} = \frac{Z_a}{Z_b} \angle(\theta_a - \theta_b)$$