

EE3114 Systems and Control

Tutorial - 7

Semester B, 2017-18

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Let $\xi = Px$

where P is non-singular (invertible, full-rank)



$$\begin{cases} \dot{\xi} = \Phi \xi + \Psi u \\ y = \Omega \xi + \Delta u \end{cases}$$

Find: $\Phi = PAP^{-1}$, $\Psi = ?$, $\Omega = ?$, $\Delta = ?$ (5 minutes)

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$$\begin{cases} \dot{\xi} = \Phi \xi + \Psi u \\ y = \Omega \xi + \Delta u \end{cases}$$

Find: $\Phi = PAP^{-1}$, $\Psi = ?$, $\Omega = ?$, $\Delta = ?$

Solution: $\Phi = PAP^{-1}$, $\Psi = PB$, $\Omega = CP^{-1}$, $\Delta = D$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Laplace Transform $\rightarrow$

$$\begin{aligned} sX(s) &= AX(s) + BU(s) & Y(s) &= CX(s) + DU(s) \\ \rightarrow (sI - A)X(s) &= BU(s) \\ \rightarrow X(s) &= (sI - A)^{-1}BU(s) \end{aligned}$$

$$\begin{aligned} \rightarrow Y(s) &= CX(s) + DU(s) \\ &= [C(sI - A)^{-1}B + D]U(s) \\ &:= G(s)U(s) \end{aligned}$$

Similarly,

$$\begin{cases} \dot{\xi} = \Phi \xi + \Psi u \\ y = \Omega \xi + \Delta u \end{cases} \quad \rightarrow \quad \begin{aligned} Y(s) &= [\Omega(sI - \Phi)^{-1}\Psi + \Delta]U(s) \\ &:= \tilde{G}(s)U(s) \end{aligned}$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$Y(s) = [C(sI - A)^{-1}B + D]U(s) := G(s)U(s)$$

$$\begin{cases} \dot{\xi} = \Phi \xi + \Psi u \\ y = \Omega \xi + \Delta u \end{cases}$$

$$Y(s) = [\Omega(sI - \Phi)^{-1}\Psi + \Delta]U(s) := \tilde{G}(s)U(s)$$

Find: Relationship between $G(s)$ and $\tilde{G}(s)$ (10 minutes)

Recall: $\Phi = PAP^{-1}$, $\Psi = PB$, $\Omega = CP^{-1}$, $\Delta = D$

$$Y(s) = [C(sI - A)^{-1}B + D]U(s) := G(s)U(s)$$

$$Y(s) = [\Omega(sI - \Phi)^{-1}\Psi + \Delta]U(s) := \tilde{G}(s)U(s)$$

Recall: $\Phi = PAP^{-1}$, $\Psi = PB$, $\Omega = CP^{-1}$, $\Delta = D$

Solution: $G(s) = \tilde{G}(s)$

Proof:

$$\begin{aligned}\tilde{G}(s) &= \Omega(sI - \Phi)^{-1}\Psi + \Delta \\ &= CP^{-1}(sPP^{-1} - PAP^{-1})^{-1}PB + D \\ &= CP^{-1}[P(sI - A)P^{-1}]^{-1}PB + D \\ &= CP^{-1}P(sI - A)^{-1}P^{-1}PB + D \\ &= C(sI - A)^{-1}B + D \\ &= G(s)\end{aligned}$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$Y(s) = [C(sI - A)^{-1}B + D]U(s) := G(s)U(s)$$

$$\begin{cases} \dot{\xi} = \Phi \xi + \Psi u \\ y = \Omega \xi + \Delta u \end{cases}$$

$$Y(s) = [\Omega(sI - \Phi)^{-1}\Psi + \Delta]U(s) := \tilde{G}(s)U(s)$$

$$\rightarrow G(s) = \tilde{G}(s)$$

Theorem:

Non-singular state transformation $\xi = Px$ does **not** change the Transfer Matrix (Function) of an LTI system

Next, consider:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + \mathbf{D}u \end{cases}$$

Laplace Transform (with initial condition) →

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$\rightarrow (sI - A)X(s) = X(0) + BU(s)$$

$$\rightarrow X(s) = (sI - A)^{-1} X(0) + (sI - A)^{-1} BU(s)$$

Inverse Laplace Transform →

$$\begin{aligned} \mathbf{x}(t) &= L^{-1}\{(sI - A)^{-1}\}X(0) + L^{-1}\{(sI - A)^{-1}BU(s)\} \\ &= \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t - \tau)B\mathbf{u}(\tau)d\tau \end{aligned}$$

Convolution

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau$$

$$\Phi(t) = e^{At} \quad (A \text{ is } n \times n)$$

$$\Phi(t-\tau) = e^{A(t-\tau)}$$

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau$$

$$e^{\alpha x} = 1 + \alpha x + \frac{\alpha^2}{2!}x^2 + \frac{\alpha^3}{3!}x^3 + \dots\dots$$

$$\rightarrow e^{\alpha A} = I + \alpha A + \frac{\alpha^2}{2!}A^2 + \frac{\alpha^3}{3!}A^3 + \dots\dots \quad (\text{matrix exponential})$$

Cayley-Hamilton Theorem $\rightarrow$

$$e^{\alpha A} = I + \beta_1 A + \frac{\beta_2}{2!}A^2 + \frac{\beta_3}{3!}A^3 + \dots\dots + \frac{\beta_{n-1}}{(n-1)!}A^{n-1}$$

β_i are some functions of α

Cayley-Hamilton Theorem

For $n \times n$ matrix A

A^n can be expressed in terms of $I, A, A^2, A^3, \dots, A^{n-1}$

A^{n+1} can be expressed in terms of $I, A, A^2, A^3, \dots, A^{n-1}$

A^{n+2} can be expressed in terms of $I, A, A^2, A^3, \dots, A^{n-1}$

... ..

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ($n = 2$)

→

$$A^2 = \alpha A + \beta I$$

$$A^3 = a A + b I$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find: α, β, a, b (5 minutes)

Cayley-Hamilton Theorem

For $n \times n$ matrix A

A^{n+1} can be expressed in terms of $I, A, A^2, A^3, \dots, A^{n-1}$

A^{n+2} can be expressed in terms of $I, A, A^2, A^3, \dots, A^{n-1}$

A^{n+3} can be expressed in terms of $I, A, A^2, A^3, \dots, A^{n-1}$

... ..

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad (n = 2)$



Solutions:

$$A^2 = 5A + 2I$$

$$A^3 = 27A + 10I$$

$$A^4 = 145A + 54I$$

... ..

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Recall:

$$e^{\alpha A} = I + \alpha A + \frac{\alpha^2}{2!} A^2 + \frac{\alpha^3}{3!} A^3 + \dots$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad (n = 2)$$



$$\begin{aligned} A^2 &= 5A + 2I \\ A^3 &= 27A + 10I \end{aligned}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise:

$$\begin{aligned} e^{-\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} t} &\approx I + (-t)A + \frac{(-t)^2}{2!} A^2 + \frac{(-t)^3}{3!} A^3 \\ &= ? \end{aligned}$$

(5 minutes)

Recall:

$$e^{\alpha A} = I + \alpha A + \frac{\alpha^2}{2!} A^2 + \frac{\alpha^3}{3!} A^3 + \dots$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad (n = 2)$$



$$\begin{aligned} A^2 &= 5A + 2I \\ A^3 &= 27A + 10I \end{aligned} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} e^{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} t} &\approx I - tA + \frac{t^2}{2!} A^2 - \frac{t^3}{3!} A^3 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - t \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \frac{t^3}{6} \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix} \\ &= \begin{bmatrix} 1 - t + \frac{7}{2}t^2 - \frac{37}{6}t^3 & -2t + 5t^2 - 9t^3 \\ -3t + \frac{15}{2}t^2 - \frac{81}{6}t^3 & 1 - 4t + 11t^2 - \frac{59}{3}t^3 \end{bmatrix} \end{aligned}$$

More about Matrix Exponential (for diagonal matrices)

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix} \Rightarrow e^A = \begin{bmatrix} e^{a_1} & 0 & \cdots & 0 \\ 0 & e^{a_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & e^{a_n} \end{bmatrix}$$

Exercise:

$$\mathbf{x}(t) = \int_0^{t^*} e^{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(t^*-\tau)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}(\tau) d\tau = ? \quad (5 \text{ minutes})$$

(both $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are 2×1 vectors)

More about Matrix Exponential (for diagonal matrices)

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix} \Rightarrow e^A = \begin{bmatrix} e^{a_1} & 0 & \cdots & 0 \\ 0 & e^{a_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & e^{a_n} \end{bmatrix}$$

Solution:

$$\begin{aligned} \mathbf{x}(t) &= \int_0^{t^*} e^{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(t^*-\tau)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}(\tau) d\tau \\ &= \int_0^{t^*} \begin{bmatrix} e^{t^*-\tau} & 0 \\ 0 & e^{t^*-\tau} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}(\tau) d\tau \\ &= \begin{bmatrix} 0 & \int_0^{t^*} e^{t^*-\tau} d\tau \\ \int_0^{t^*} e^{t^*-\tau} d\tau & 0 \end{bmatrix} \mathbf{u}(\tau) d\tau = \begin{bmatrix} (e^{t^*}-1)u_2 \\ (e^{t^*}-1)u_1 \end{bmatrix} \end{aligned}$$

END