L08: Introduction to Transmission Lines

EE3109 Applied Electromagnetics

Lecture Notes

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Outline

- Transmission Lines – Bridging electromagnetics with circuit theory
- The TL Model
- Basic TL Relationships

What is a Transmission Line?

https://i.stack.imgur.com/H5vVZ.jpg

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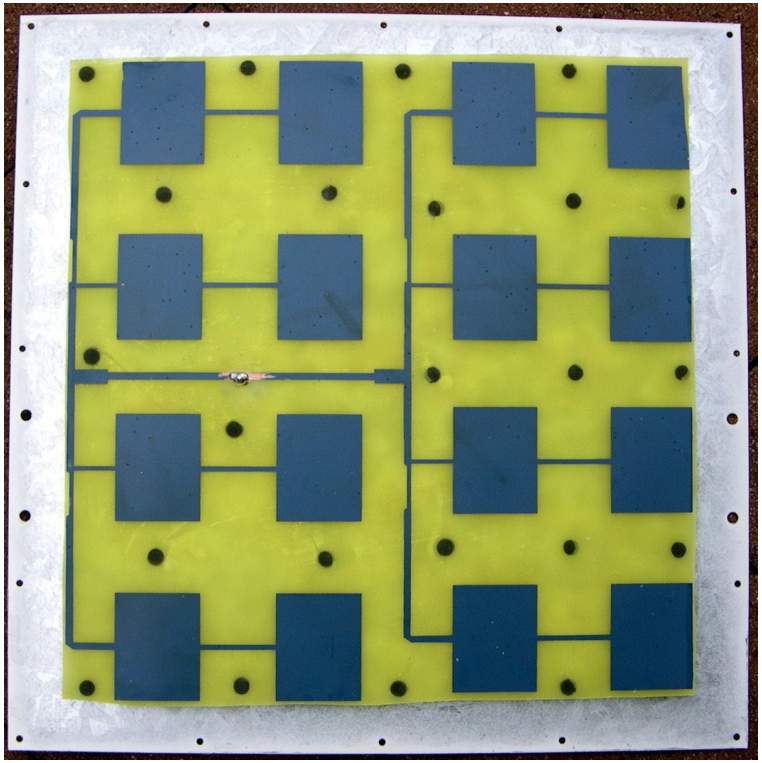

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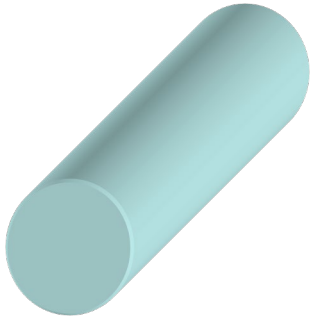
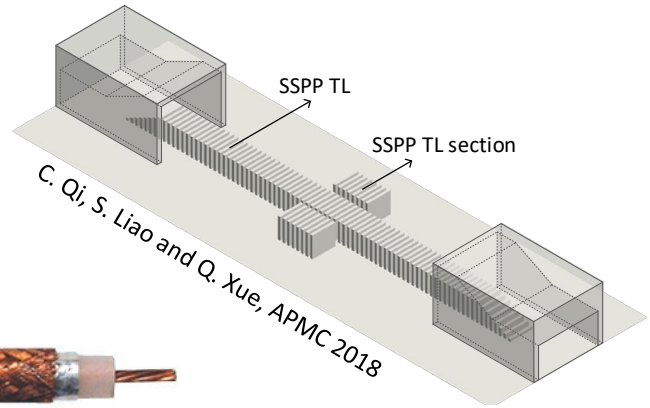
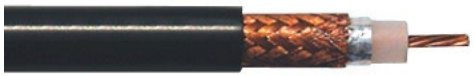
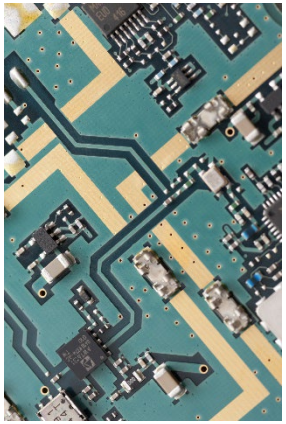
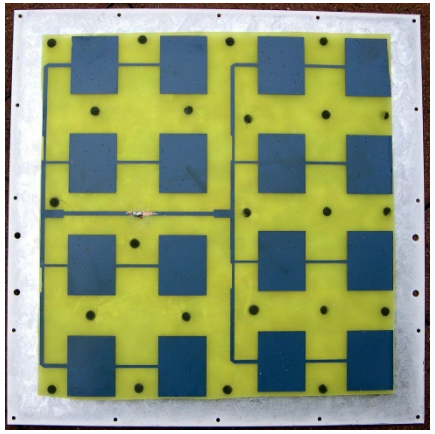


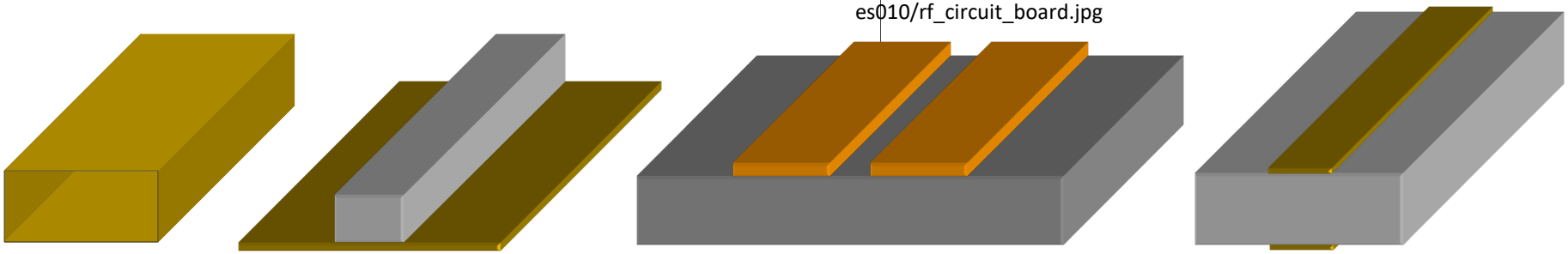








http://www.freeimageslive.co.uk/files/images010/rf_circuit_board.jpg



A **transmission line (TL)** is a structure that guides EM waves in a desired direction. Usually the length of the TL is appreciable compared to the wavelength, such that a change in phase can be observed in the voltage or current wave.

Example: the Optical Fiber

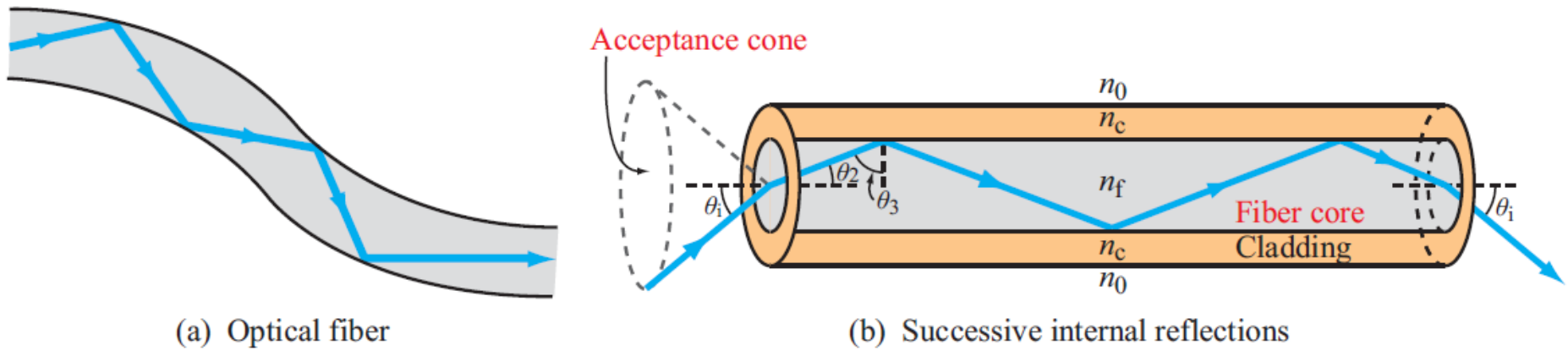


Figure 8-12: Waves can be guided along optical fibers as long as the reflection angles exceed the critical angle for total internal reflection.

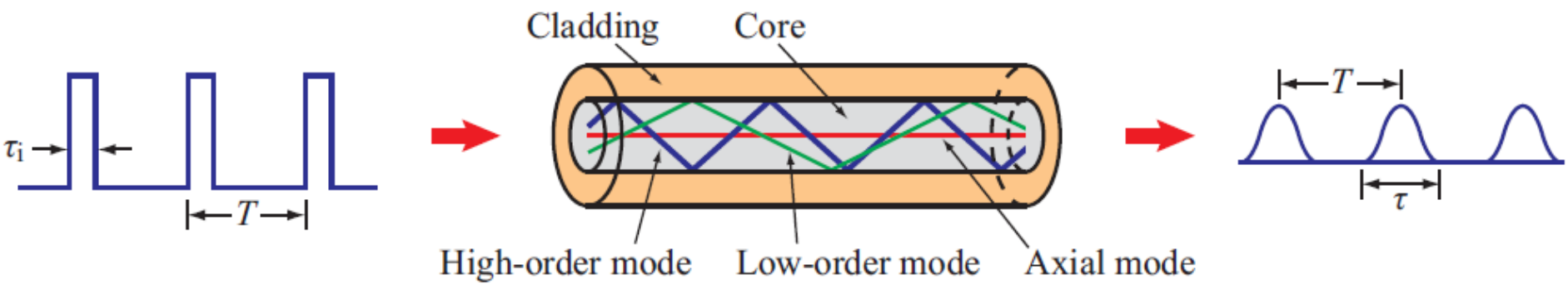


Figure 8-13: Distortion of rectangular pulses caused by modal dispersion in optical fibers.

Waveguides & Transmission Lines

- **Waveguide**: general EM structure that guide waves
  - Has **many different "modes"**: orthogonal E-field and H-field distributions along the cross-section, that don't change in the longitudinal direction
- **Transmission Lines (TL)**: abstracted single mode waveguides
  - When a waveguide has **one mode**, one can simplify its analysis defining currents, voltages and resistances on the line, thereby **simplifying the EM problem to a circuit problem**

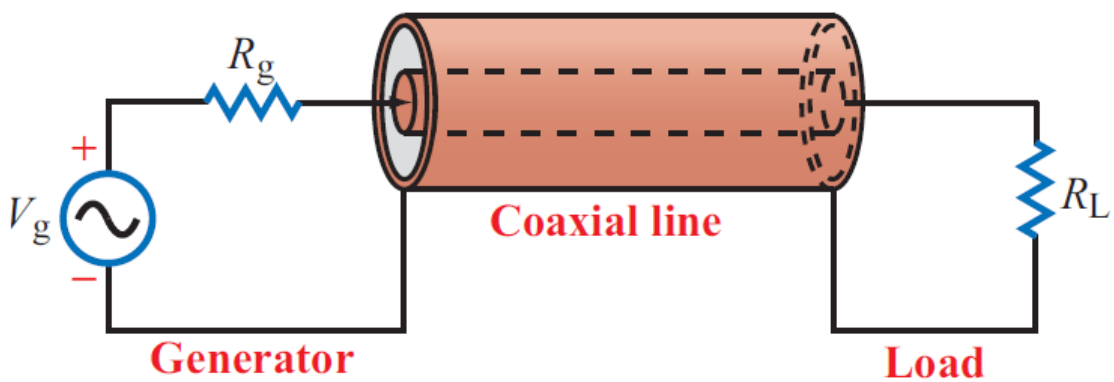
Transmission Lines vs. Waveguides

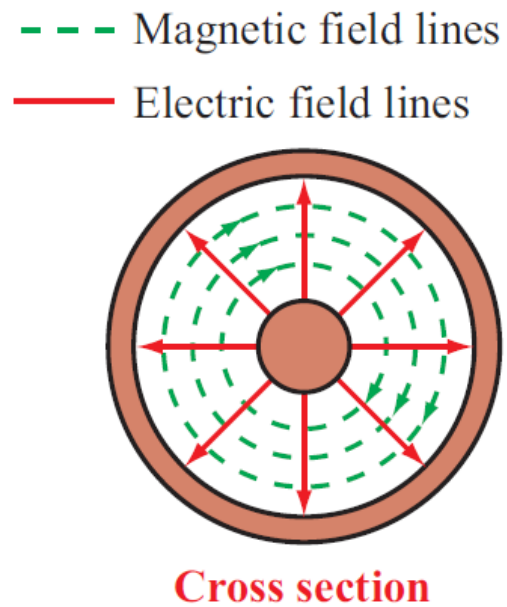
Transmission line theory

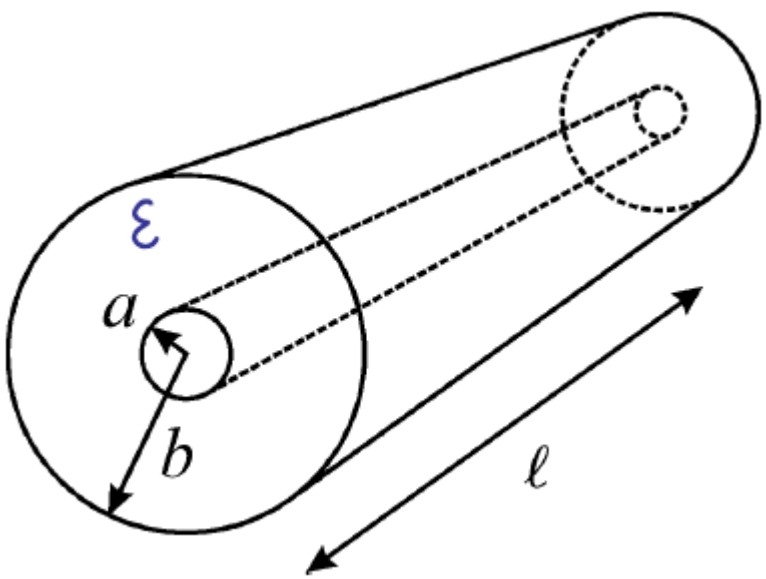
<u>Waveguide</u>
EM field theory
Maxwell's equations

<u>Electronic Circuit</u>
Circuit theory
Ohm's law

Example: Coaxial Cable

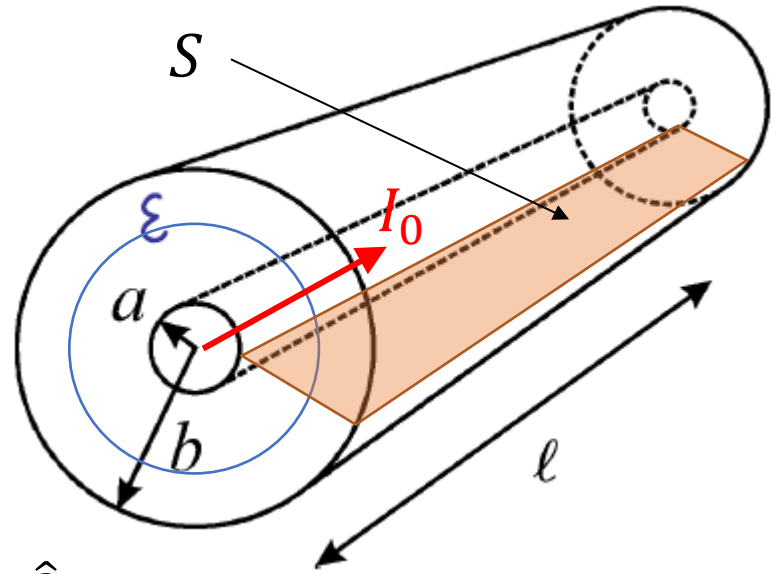






Electric Field E is radial
Magnetic Field H is azimuthal
Propagation is into the page

Inductance



For $a < r < b$,

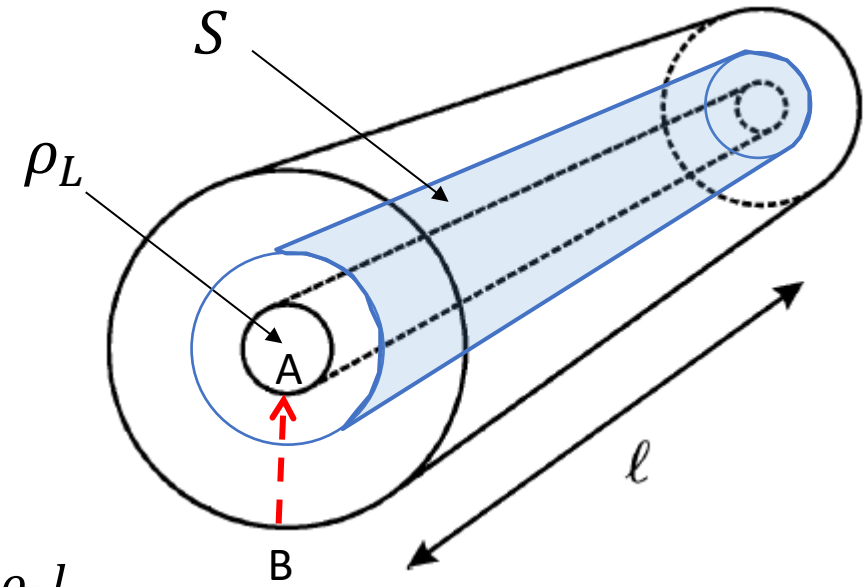
$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_0 = H_\phi 2\pi r = I_0 \Rightarrow \mathbf{H} = \frac{I_0}{2\pi r}\hat{\mathbf{a}}_\phi$$

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I_0}{2\pi r}\hat{\mathbf{a}}_\phi$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^l \int_a^b \frac{\mu_0 I_0}{2\pi r} dr dz = \frac{\mu_0 I_0 l}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$L = \frac{\Phi}{I_0} = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)$$

Capacitance



For $a < r < b$,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_{enc} \Rightarrow D_\rho 2\pi r l = \rho_L l$$

$$D_\rho = \frac{\rho_L}{2\pi r}; E_\rho = \frac{D_\rho}{\varepsilon} = \frac{\rho_L}{2\pi r \varepsilon}$$

$$\mathrm{V} = -\int_B^A \mathbf{E} \cdot d\mathbf{l} = -\int_b^a E_\rho dr = \frac{\rho_L}{2\pi\varepsilon} \ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q_{enc}}{V} = 2\pi\varepsilon l \left[\ln\left(\frac{b}{a}\right)\right]^{-1}$$

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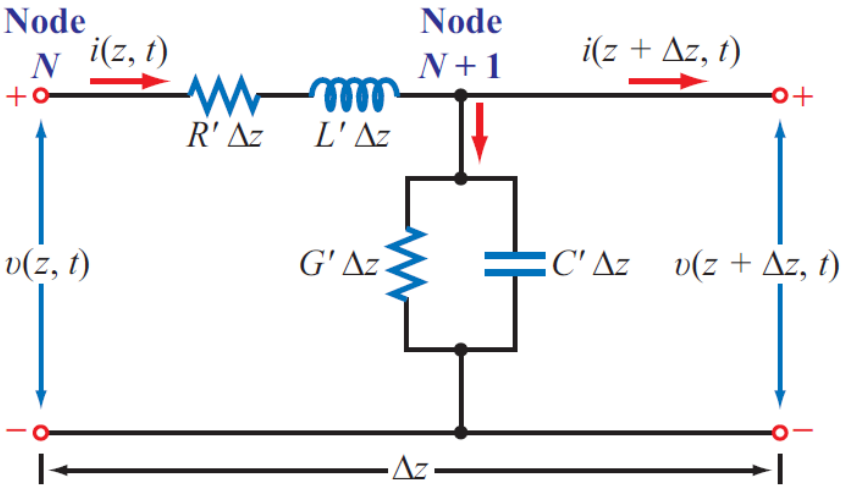
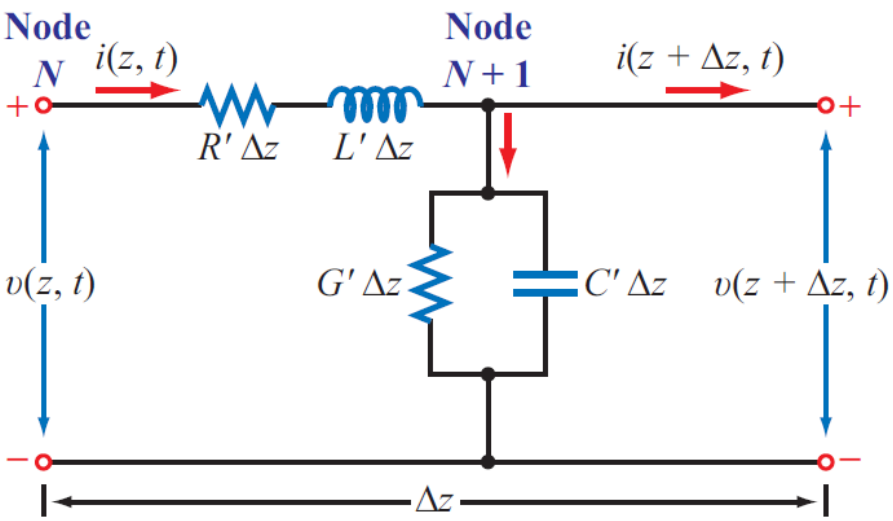
Transmission Line Model

(a) Parallel-wire representation

Transmission-Line Equations



Recap: Transmission Line Model



Solving this electronic circuit gives:

$$-\frac{dV(z)}{dz} = (R' + j\omega L')I(z)\,;$$
$$-\frac{dI(z)}{dz} = (G' + j\omega C')V(z)\,.$$

(Telegrapher's Equations)

(Can we get uncoupled equations… one for $V(z)$ and one for $I(z)$?)

Wave Equations for TL

Telegrapher's Equations

$$-\frac{dV(z)}{dz} = (R' + j\omega L')I(z)\,; \qquad (1)$$

$$-\frac{dI(z)}{dz} = (G' + j\omega C')V(z)\,. \qquad (2)$$

Take d/dz of (1) and sub in (2) ;
Take d/dz of (2) and sub in (1).

Wave Equations

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0\,; \qquad (3)$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0. \qquad (4)$$

where $\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta$

($\gamma \equiv$ propagation constant; $\alpha \equiv$ attenuation constant; $\beta \equiv$ phase constant)

Solutions to Wave Equations

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}; \qquad (5)$$

$$I(z) = I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z}; \qquad (6)$$

Rearrange (6) to get:

$$I(z) = \frac{1}{Z_0}(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}), \text{where}$$

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \equiv \text{Characteristic Impedence}$$

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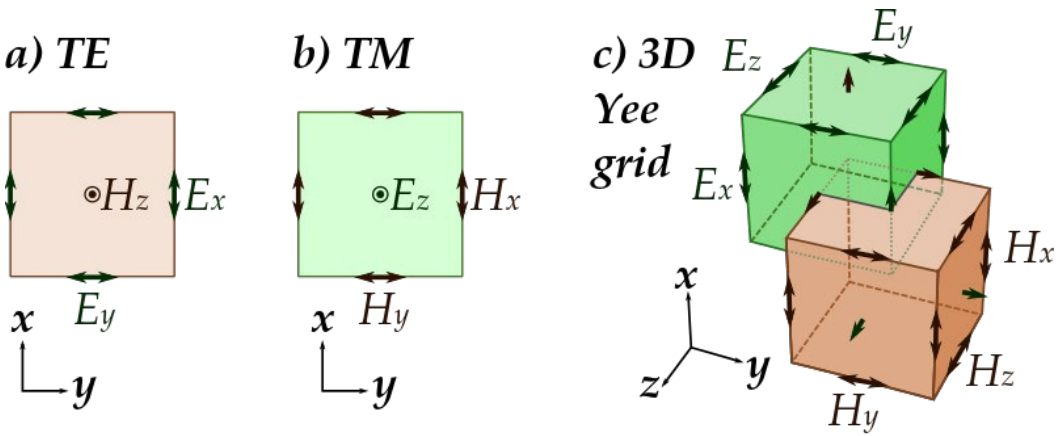
Comparison: EM vs. TL

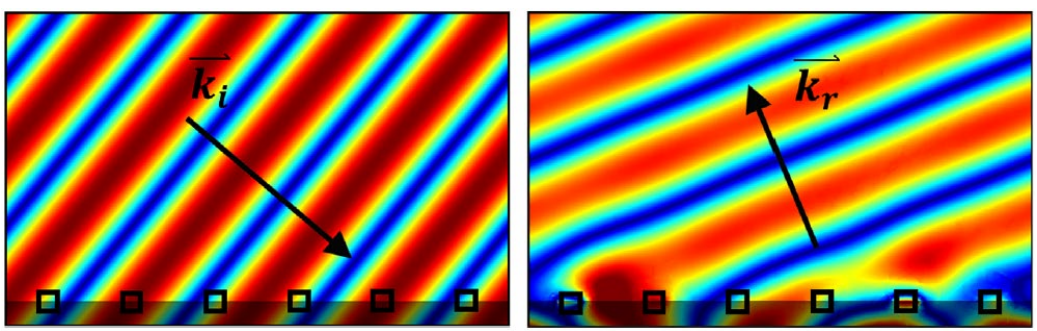
Electromagnetics	Transmission Lines
$\mathbf{E}\,;\ \mathbf{H}$	$V = \int_a^b \mathbf{E}\cdot d\mathbf{l}\ ;\ \ I = \oint_C \mathbf{H}\cdot d\mathbf{l}$
Maxwell's Equations: $\nabla\times\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t}; \nabla\times\mathbf{H} = \varepsilon\frac{\partial\mathbf{E}}{\partial t}$	Telegrapher's Equations: $-\frac{dV}{dz} = (R' + j\omega L')I\ ;\ -\frac{dI}{dz} = (G' + j\omega C')V$
Wave Equations: $\frac{d^2\mathbf{E}}{dz^2} - \gamma^2\mathbf{E} = 0\ ;\ \ \frac{d^2\mathbf{H}}{dz^2} - \gamma^2\mathbf{H} = 0$	Wave Equations: $\frac{d^2V}{dz^2} - \gamma^2 V = 0\ ;\ \ \frac{d^2I}{dz^2} - \gamma^2 I = 0$
Solution to Wave Equations: $\mathbf{E}(\mathbf{r}) = \left(E_0^+ e^{-j\mathbf{k}\cdot\mathbf{r}} + E_0^- e^{+j\mathbf{k}\cdot\mathbf{r}}\right)\hat{\mathbf{e}}$ $\mathbf{H}(\mathbf{r}) = \left(H_0^+ e^{-j\mathbf{k}\cdot\mathbf{r}} - H_0^- e^{+j\mathbf{k}\cdot\mathbf{r}}\right)\hat{\mathbf{h}}$	Solution to Wave Equations: $V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z};$ $I(z) = I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z}$
Intrinsic Impedance: $\eta = \frac{E_0^+}{H_0^+} = \sqrt{\frac{\mu}{\varepsilon_c}}$	Characteristic Impedance: $Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$
Propagation Constant: $jk = \gamma = j\omega\sqrt{\mu\varepsilon_c} = \alpha + j\beta$	Propagation Constant: $\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta$

Computational Electromagnetics:

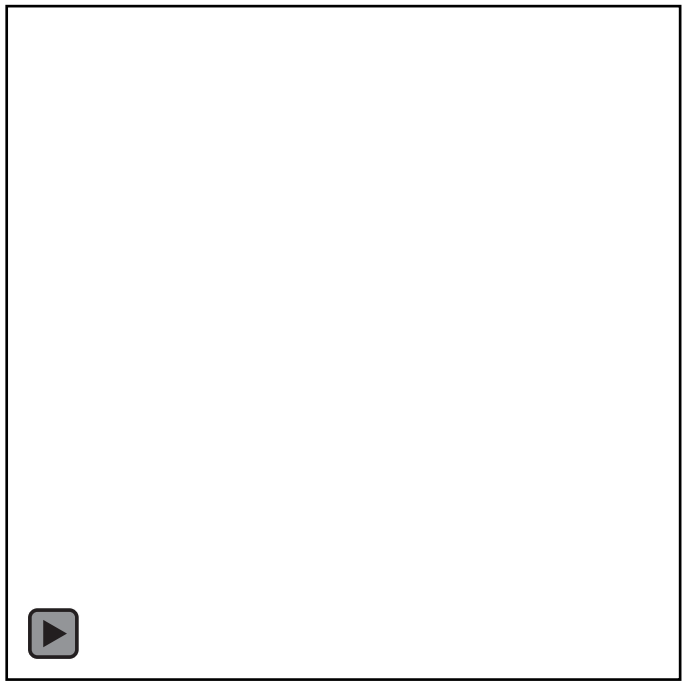
Finite Difference Time Domain (FDTD) and Finite Element Method (FEM)



https://commons.wikimedia.org/wiki/File:FDTD_Yee_grid_2d-3d.svg



AMH Wong and GV Eleftheriades, Phys. Rev. X., 2018.

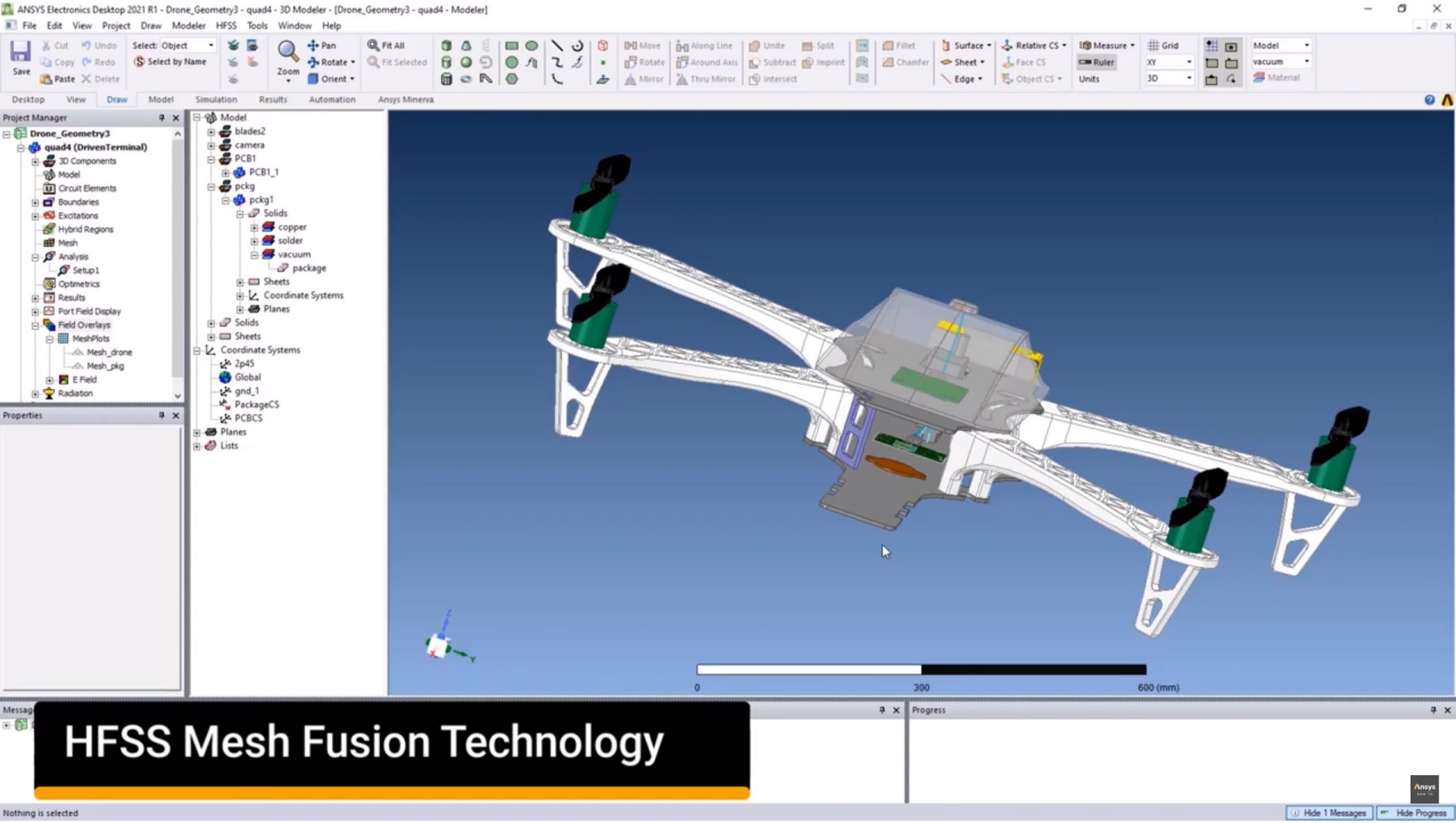


AMH Wong et al., Trans. Antennas Propag., 2021.

- **3D space** is treated as a **3D TL circuit** in a time-domain (FDTD) or frequency domain (FEM) simulation

Computational Electromagnetics

Eg. Ansys HFSS (You will explore this in Lab 1!)



https://www.ansys.com/blog/two-best-features-in-ansys-hfss

Travelling Wave in a TL

Consider a positively travelling wave (in a lowloss TL):

$$V^+(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} \Rightarrow V(z,t) = |V_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \angle V_0^+)$$

$$I^+(z) = \frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j\beta z} \Rightarrow I(z,t) = \frac{|V_0^+|}{Z_0} e^{-\alpha z} \cos(\omega t - \beta z + \angle V_0^+)$$

Voltage amplitudes:

$$\boxed{V_{p-p} = 2|V_0^+|;\ V_{RMS} = \frac{|V_0^+|}{\sqrt{2}}}.$$

Consider the phase of this travelling wave ($\varphi = \omega t - \beta z + \angle V_0^+$):

The wavelength (λ) is the distance through which the wave changes in phase by 2π rad:

$$\lambda \equiv |\Delta z| \ \text{ s.t. } \Delta\varphi = 2\pi \Rightarrow \boxed{\lambda = \frac{2\pi}{\beta}}$$

The phase velocity (v_p) of the wave is the rate at which the constant phase point travels:

Rearrange the phase expression to get $z = \frac{\omega}{\beta} t - \text{Constant} \Rightarrow \boxed{v_p = \frac{dz}{dt} = \frac{\omega}{\beta}}$

Power in a TL

Consider a positively travelling wave (in a lowloss TL):

$$V^+(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} \Rightarrow V(z,t) = |V_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \angle V_0^+)$$

$$I^+(z) = \frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j\beta z} \Rightarrow I(z,t) = \frac{|V_0^+|}{Z_0} e^{-\alpha z} \cos(\omega t - \beta z + \angle V_0^+)$$

The ***instantaneous power*** carried is:

$$P(z,t) = V(z,t)I(z,t) = \frac{|V_0^+|^2}{Z_0} e^{-2\alpha z} \cos^2(\omega t - \beta z + \angle V_0^+)$$

The ***time-average power*** carried is:

$$P_{av}(z) = \frac{1}{T}\int_0^T P(z,t)\,dt = \frac{|V_0^+|^2}{Z_0} e^{-2\alpha z}\left(\frac{1}{T}\right)\int_0^T \cos^2(\omega t - \beta z + \angle V_0^+)\,dt$$

$$= \frac{|V_0^+|^2}{Z_0 T} e^{-2\alpha z}\int_0^T \frac{1 + \cos\big(2(\omega t - \beta z + \angle V_0^+)\big)}{2}\,dt$$

$$= \frac{|V_0^+|^2}{2Z_0 T} e^{-2\alpha z}\left[t + \frac{1}{2\omega}\cos\big(2(\omega t - \beta z + \angle V_0^+)\big)\right]_{t=0}^{t=T}$$

$T = \frac{2\pi}{\omega}$ is the period of $V(z,t)$ and $I(z,t)$

This term takes on the same value for $t = 0$ and $t = T$.

$$P_{av}(z) = \frac{|V_0^+|^2}{2Z_0} e^{-2\alpha z} = \frac{1}{2}\mathrm{Re}\{V(z)I(z)^*\}$$

More on Impedance

$$\text{Characteristic Impedence: } Z_0 = \frac{V_0^+}{I_0^+} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

- Characteristic Impedance is a property of the frequency and the transmission line
  - In general, the characteristic impedance is a function of frequency
  - However, this will not be the case for lossless TLs.
- Impedance (looking toward the load) is given by:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}}{V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}}$$

Lossless TLs

- Condition: $R' = 0, G' = 0$ (reasonable approximation for many TLs)
- Consequences:

$$\gamma = \sqrt{(j\omega L')(j\omega C')} = j\omega\sqrt{L'C'}$$

$$\Rightarrow \alpha = 0\ ;\ \beta = \omega\sqrt{L'C'}$$

$$Z_0 = \sqrt{\frac{j\omega L'}{j\omega C'}} = \sqrt{\frac{L'}{C'}}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$$

Lowloss TLs

- Condition: $R' \ll \omega L', G' \ll \omega C'$
- Consequences: (you will derive this in a future assignment)

$$\gamma = j\omega\sqrt{L'C'} + \frac{1}{2}\left(\frac{R'}{Z_0} + G'Z_0\right)$$

$$\Rightarrow \alpha = \frac{1}{2}\left(\frac{R'}{Z_0} + G'Z_0\right);\ \beta = \omega\sqrt{L'C'}$$

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$$

Example 1: Wire or TL?

2.2 A transmission line of length l connects a load to a sinusoidal voltage source with an oscillation frequency f. Assuming the velocity of wave propagation on the line is c, for which of the following situations is it reasonable to ignore the presence of the transmission line in the solution of the circuit:

(a) $l = 30$ cm, $f = 20$ kHz,
(b) $l = 50$ km, $f = 60$ Hz,
(c) $l = 30$ cm, $f = 600$ MHz,
(d) $l = 2$ mm, $f = 100$ GHz.

Solution: A transmission line is negligible when $l/\lambda \leq 0.01$.

Example 2: Air Line

An ***air line*** is a transmission line in which air separates the two conductors, which renders $G' = 0$ because $\sigma = 0$. In addition, assume that the conductors are made of a material with high conductivity so that $R' \simeq 0$. For an air line with a characteristic impedance of 50 Ω and a phase constant of 20 rad/m at 700 MHz, find the line inductance L' and the line capacitance C'.

Example 3: Distortionless Line

2.13 In addition to not dissipating power, a lossless line has two important features: (1) it is dispertionless (μ_p is independent of frequency) and (2) its characteristic impedance Z_0 is purely real. Sometimes, it is not possible to design a transmission line such that $R' \ll \omega L'$ and $G' \ll \omega C'$, but it is possible to choose the dimensions of the line and its material properties so as to satisfy the condition

$$R'C' = L'G' \qquad \text{(distortionless line).}$$

Such a line is called a *distortionless* line because despite the fact that it is not lossless, it does nonetheless possess the previously mentioned features of the loss line. Show that for a distortionless line,

$$\alpha = R'\sqrt{\frac{C'}{L'}} = \sqrt{R'G'}, \qquad \beta = \omega\sqrt{L'C'}, \qquad Z_0 = \sqrt{\frac{L'}{C'}}.$$

(Please try for yourselves!)

Summary

- In this class, we
  - Introduced the transmission line model
  - Derived the Telegrapher's Equations and the waves equations
  - Introduced some basic TL Relationships
  - Introduced the lossless TL