

Test 2 Solution

6:00pm~6:40pm (YEUNG LT-15 or 17)

- As a closed-book test, you are not allowed to use any written materials online/offline.
- Write your answers on the question papers directly.
- Non-graphical calculators are allowed. Laptops/phones must be switched off.
- Unless otherwise specified, SI units are assumed in the questions and 3 significant figures should be used in numerical answers.
- The total score is 100. There are a total of **7 pages.**

The following formulas may or may not be useful.

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \mathbf{E}(x, y, z) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{a}}_R & d\mathbf{B} &= \frac{\mu_0}{4\pi} \frac{Id\mathbf{L} \times \hat{\mathbf{a}}_R}{R^2} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} & \oint_S \mathbf{D} \cdot d\mathbf{S} &= Q & \oint_L \mathbf{H} \cdot d\mathbf{L} &= I \\ \nabla \cdot \mathbf{D} &= \rho_v & \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon_r \mathbf{E} & \mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 \mu_r \mathbf{H} \\ \nabla \cdot \mathbf{B} &= 0 & \epsilon_0 &= 8.854 \times 10^{-12} \text{ (F/m)} & \mu_0 &= 4\pi \times 10^{-7} \text{ (H/m)}\end{aligned}$$

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \hat{\mathbf{a}}_\rho \quad \mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{\mathbf{a}}_r + \sin\theta \hat{\mathbf{a}}_\theta) \quad \mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\mathbf{a}}_\phi \quad \mathbf{B} = \mu_0 n I \hat{\mathbf{a}}_z$$

$$\begin{aligned}V_{AB} &= -\int_B^A \mathbf{E} \cdot d\mathbf{L} & \nabla^2 V &= -\frac{\rho_v}{\epsilon} & V &= -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \\ \nabla^2 &= \nabla \cdot \nabla\end{aligned}$$

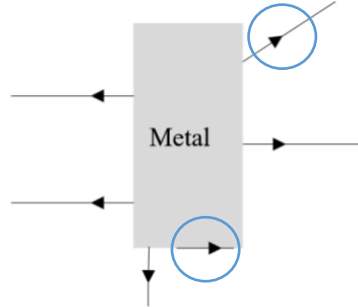
$$\begin{aligned}\nabla &= \frac{\partial}{\partial x} \hat{\mathbf{a}}_x + \frac{\partial}{\partial y} \hat{\mathbf{a}}_y + \frac{\partial}{\partial z} \hat{\mathbf{a}}_z & \nabla \times \mathbf{F} &= \frac{1}{\rho} \begin{vmatrix} \hat{\mathbf{a}}_\rho & \rho \hat{\mathbf{a}}_\phi & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} \\ \nabla f &= \frac{\partial f}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial f}{\partial z} \hat{\mathbf{a}}_z & \nabla \times \mathbf{F} &= \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin \theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin \theta F_\phi \end{vmatrix} \\ \nabla f &= \frac{\partial f}{\partial r} \hat{\mathbf{a}}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\mathbf{a}}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi & \nabla \times \mathbf{F} &= \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin \theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin \theta F_\phi \end{vmatrix}\end{aligned}$$

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{\partial}{\partial \phi} F_\phi + \frac{\partial}{\partial z} (\rho F_z) \right] \\ \nabla \cdot \mathbf{F} &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 \sin \theta F_r) + \frac{\partial}{\partial \theta} (r \sin \theta F_\theta) + \frac{\partial}{\partial \phi} (r F_\phi) \right]\end{aligned}$$

Question 1 (14 points)

Give short answers to the following.

- (a) The electric field inside a dielectric is always non-zero. True/False
- (b) The electric field inside a perfect conductor is always zero. True/False
- (c) The electric field in a resistor is inversely proportional to the current. True/False
- (d) Ohm's law states that charge is proportional to capacitance. True/False
- (e) A capacitor with a capacitance of $5\ \mu\text{F}$ is connected to a voltage source of $5\ \text{V}$. How much charge is stored?
- (f) For a rectangular metal in air (below), six electric field lines are drawn, but two of the field lines are incorrect. Please circle the two incorrect field lines.



Solution:

- (a) False
- (b) True
- (c) False
- (d) False
- (e) $Q = CV = 5 \times 10^{-6} \times 5 = 2.5 \times 10^{-5} \text{ (C)}$
- (f) See the blue circles in the above figure.

Question 2 (12 points)

The picture below shows the electric field lines from a pair of static charges of $\pm 0.2 \text{ nC}$. The charges are separated by $1 \text{ }\mu\text{m}$. There are no other charges. Points A and B are vertically positioned with respect to the charge pairs. Point C is also shown.

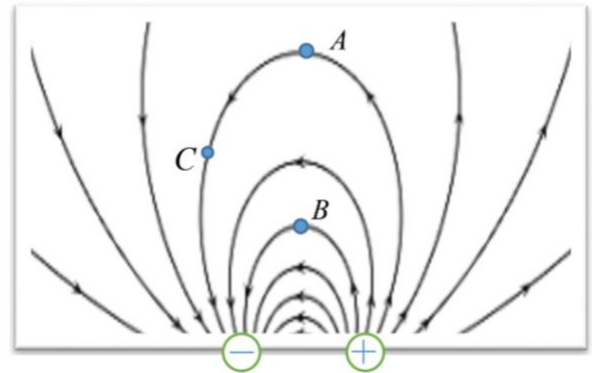
(a) What is the magnitude of the electric dipole moment?

(b) What is $\text{curl } \mathbf{E}$ at point A?

(c) Where can non-zero $\text{div } \mathbf{E}$ be found?

(d) Consider path integral $V_1 = \int_C^A \mathbf{E} \cdot d\mathbf{L}$.
Is it zero, positive, or negative?

(e) Consider path integral $V_2 = \int_B^A \mathbf{E} \cdot d\mathbf{L}$.
Is it zero, positive, or negative?



(f) Find path integral $V_3 = \int_B^C \mathbf{E} \cdot d\mathbf{L}$ in terms of V_1 and V_2 .

Solution:

(a) $p = Qd = 2 \times 10^{-16} \text{ (C} \cdot \text{m)}$

(b) $\text{curl } \mathbf{E}$ at point A is zero.

(c) Non-zero $\text{div } \mathbf{E}$ can be found at the positions of the two static charges.

(d) Negative.

(e) Zero.

(f) $V_3 = V_2 - V_1$ or $V_3 = -V_1$

Question 3 (12 points)

In a region with permittivity of ϵ_0 , the potential field is given in the rectangular coordinates as

$$V(x, y) = V_0 e^{-x^4/a^4}$$

where $V_0 = 5$ V and $a = 0.7$ m. Consider point P at $(+a, +a)$.

- (a) What is the direction of the electric field at P ?
- (b) Find the magnitude of the electric field at P . Show your steps clearly.
- (c) Find the charge density at P . Show your steps clearly.

Solution:

(a) The direction of the electric field at P is $+\hat{x}$.

(b) At point P , we have

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial x} \hat{x} = \frac{4V_0}{a^4} x^3 e^{\frac{-x^4}{a^4}} \hat{x} = \frac{4V_0}{a} e^{-1} \hat{x} = 10.5 \hat{x} \text{ (V/m)}$$

Thus, the magnitude of the electric field at P is 10.5 (V/m).

(c) The charge density at P is

$$\rho = \nabla \cdot \mathbf{D} = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 \nabla \cdot \left(\frac{4V_0}{a^4} x^3 e^{\frac{-x^4}{a^4}} \hat{x} \right) = \frac{4\epsilon_0 V_0}{a^4} e^{\frac{-x^4}{a^4}} \left(3x^2 - \frac{4x^6}{a^4} \right)$$

So we have

$$\rho = \frac{4\epsilon_0 V_0}{a^4} e^{-1} (3a^2 - 4a^2) = -\frac{4\epsilon_0 V_0 e^{-1}}{a^2} = -1.32 \times 10^{-10} \text{ (C/m}^3\text{)}$$

Question 4 (12 points)

Consider a coaxial cable consisting of cylinders of perfect conductors with a length of $L = 12$ m. The inner conductor has a radius of $a = 1$ mm. The outer conductor is thin and has a radius of $b = 5$ mm. The region between the conductors is filled with a dielectric with a relative permittivity of $\epsilon_r = 6$. In other words, the coaxial cable parameters are: $(L, a, b, \epsilon_r) = (12 \text{ m}, 1 \text{ mm}, 5 \text{ mm}, 6)$.

Initially, the cable is connected to a voltage source at 12 V and the energy stored is W_0 (in Joule).

- (a) To double the charge stored, suggest two sets of values for (L, a, b, ϵ_r) .
- (b) Suppose the voltage is decreased to 4 V. Express the stored energy W in terms of the original W_0 .

Solution:

- (a) To double the charge stored, we can set the values of the coaxial cable parameters as (24 m, 1 mm, 5 mm, 6), (12 m, 1 mm, 5 mm, 12), etc..

Other correct combinations include those with the product $L\epsilon_r$ doubled.

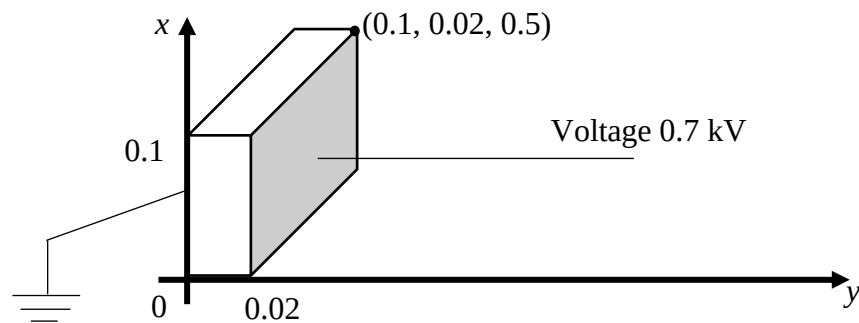
- (b) We know that the energy stored $W_e = \frac{1}{2}CV^2$.

When the voltage is decreased from 12 V to 4 V, we have

$$W = \frac{1}{9}W_0$$

Question 5 (20 points)

The diagram below shows a dielectric of size $0.1 \times 0.02 \times 0.5$ (in meters) along the x -, y -, and z -dimensions, respectively. One corner is at the origin. Another corner is at $(x, y, z) = (0.1, 0.02, 0.5)$.



Parallel to the xz -plane are metal plates that sandwich the dielectric. The metal plate at $y = 0$ is grounded. The metal plate at $y = 0.02$ is connected to a voltage source at 0.7 kV. Let σ (C/m²) be the surface charge density on the metal plate at $y = 0.02$ (m). The dielectric has a relative permittivity that varies with y as:

$$\epsilon_r = \frac{2}{1 + y/0.01}$$

- What is the direction of the electric field in the middle of the dielectric?
- In terms of σ , find the potential $V(y)$ as a function of y between 0 and 0.02 (m).
- Find the total charge stored by the metal plate at $y = 0.02$ (m). Provide a numerical answer.
- State a reason for possible discrepancy in your calculation.

Solution:

(a) The direction of the electric field in the middle of the dielectric is $-\hat{y}$.

(b) We know the electric field between two metal plates is

$$\mathbf{E} = -\frac{\sigma}{\epsilon_0 \epsilon_r} \hat{y} = -\frac{(1 + 100y)\sigma}{2\epsilon_0} \hat{y}$$

The potential between 0 and 0.02 (m) is

$$V = -\int_{y=0}^y \mathbf{E} \cdot d\mathbf{L} = \int_{y=0}^y \frac{(1 + 100y)\sigma}{2\epsilon_0} dy = \frac{\sigma}{2\epsilon_0} (y + 50y^2)$$

(c) The potential difference between the plates is

$$V = -\int_{y=0}^{0.02} \mathbf{E} \cdot d\mathbf{L} = \int_{y=0}^{0.02} \frac{(1 + 100y)\sigma}{2\epsilon_0} dy = 0.02 \frac{\sigma}{\epsilon_0} = 0.7 \text{ kV}$$

So we have $\sigma = 3.10 \times 10^{-7}$ (C/m²)

Then the total charge stored by the metal plate is $\sigma \cdot S = 1.55 \times 10^{-8}$ (C).

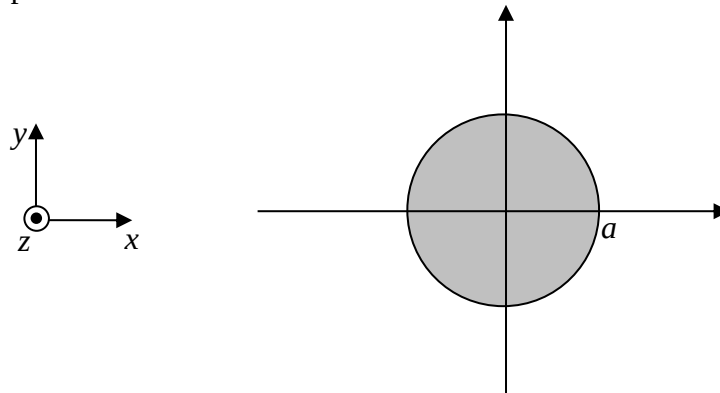
(d) The fringing electric field of the metal plates is ignored in the calculation.

Question 6 (30 points)

The figure below shows a wire with a circular cross-section of radius a . The wire carries a current density towards the $+z$ -direction:

$$\mathbf{J} = \frac{I_0}{\pi a \rho} \cos \frac{\pi \rho}{21a} \hat{\mathbf{z}}$$

where I_0 (A) is a given constant and ρ is the radial coordinate. The free-space permeability μ_0 is adopted.



(a) Sketch the current density on the diagram above. Present the variation of the current density by using streamlines.

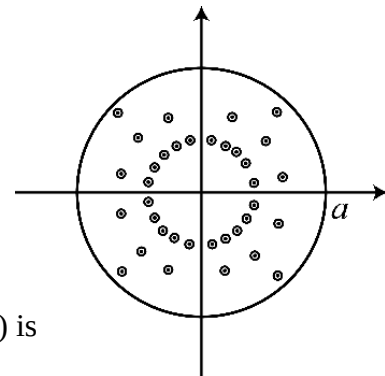
(b) At point $(x, y) = (a, a)$, what is the direction of the magnetic field?

(c) By symmetry arguments, simplify the magnetic flux density $\mathbf{B}(\rho, \phi, z)$. Specifically,

- What is the direction in general?
- Does magnitude of the magnetic field depend on ρ ?
- Does it depend on ϕ ?
- Does it depend on z ?

(d) Find $\mathbf{B}$ as a function of the radial distance ρ for

- Region I: $0 < \rho < a$
- Region II: $a < \rho$.



Solution:

(a) Streamline should be added to the diagram above as:

(b) The direction of the magnetic field at point $(x, y) = (a, a)$ is $(-\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$.

(c) The direction of $\mathbf{B}$ is along $\hat{\phi}$.

The magnitude of the magnetic field is dependent on ρ .

The magnitude of the magnetic field is independent on ϕ .

The magnitude of the magnetic field is independent on z .

(d) We know

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\phi}$$

For region I,

$$I = \int_{\rho=0}^{\rho} |\mathbf{J}| \cdot 2\pi\rho d\rho = \int_{\rho=0}^{\rho} \frac{2I_0}{a} \cos \frac{\pi\rho}{21a} d\rho = \frac{42I_0}{\pi} \sin \frac{\pi\rho}{21a}$$

So we have

$$\mathbf{B} = \frac{21\mu_0 I_0}{\pi^2 \rho} \sin \frac{\pi\rho}{21a} \hat{\phi}$$

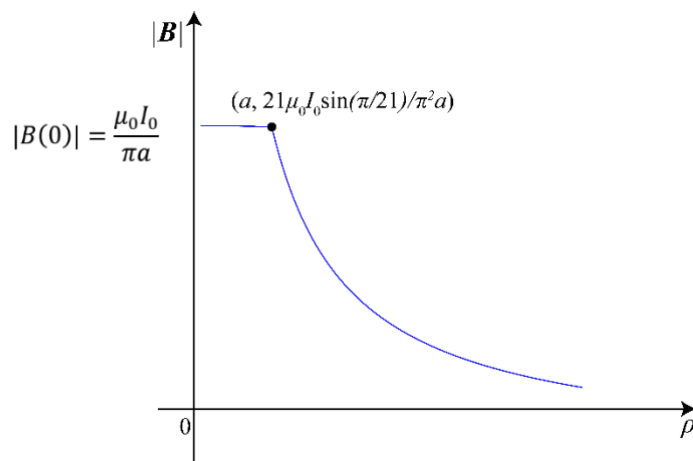
For region II,

$$I = \int_{\rho=0}^a |\mathbf{J}| \cdot 2\pi\rho d\rho = \frac{42I_0}{\pi} \sin \frac{\pi}{21}$$

So we have

$$\mathbf{B} = \frac{21\mu_0 I_0}{\pi^2 \rho} \sin \frac{\pi}{21} \hat{\phi}$$

(e) See the figure below.



~END~