

MA2001 (17/18A) Quiz I

Question. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}.$$

- Find the determinant of A.
- Explain why you can deduce from part i) that 0 is an eigenvalue of A.
- Find the characteristic equation of the matrix A and hence determine all the eigenvalues of A. Find corresponding eigenvectors. Use the matrix A to check that your eigenvectors are correct.
- Can the matrix A be diagonalizable? Explain carefully, starting what it means to diagonalize a matrix, why this can or cannot be done for the matrix A.

Solution. i) $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2 + (-2) = 0$

ii) Since $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$ with $\lambda_1, \lambda_2, \lambda_3$ the eigenvalues of A, we have (by substituting $\lambda=0$),

$$\det A = \lambda_1 \lambda_2 \lambda_3.$$

Hence, when $\det A = 0$, 0 must be one of its eigenvalues

iii) $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 0 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1-\lambda & -1 \end{vmatrix}$
 $= (1-\lambda)[(1-\lambda)(2-\lambda)] + [-1 - (1-\lambda)]$
 $= (1-\lambda)^2(2-\lambda) - (2-\lambda)$
 $= (2-\lambda)[(1-\lambda)^2 - 1] = (2-\lambda)(1 - 2\lambda + \lambda^2 - 1) = (2-\lambda)(\lambda^2 - 2\lambda)$
 $= -\lambda(\lambda-2)^2$

$\Rightarrow$ Eigenvalues are $\lambda = 0, 2$ (multiplicity 2)

a) For $\lambda=0$,

$$\begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 1 & 0 & 2 & | & 0 \end{pmatrix} \xrightarrow{r_3 \leftrightarrow r_1} \begin{pmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

 $\Rightarrow \begin{cases} x_1 + x_2 + x_3 = 0 \\ -x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, t \neq 0. \Rightarrow \vec{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \text{ is one of eig.}$

b) For $\lambda=2$.

$$\begin{pmatrix} -1 & 1 & 1 & | & 0 \\ 0 & -1 & -1 & | & 0 \\ 1 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -1 & -1 & | & 0 \\ -1 & 1 & 1 & | & 0 \end{pmatrix} \xrightarrow{r_3 + r_1} \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -1 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix} \xrightarrow{r_3 + r_2} \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ -(x_2 + x_3) = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad t \neq 0 \Rightarrow \vec{x}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \text{ is one of e.v.}$$

c) check,

Note that

$$A\vec{x}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 0 \cdot \vec{x}_1.$$

$$A\vec{x}_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = 2 \cdot \vec{x}_2.$$

Hence, $\vec{x}_1$ and $\vec{x}_2$ are indeed eigenvectors of A .

(IV)* Diagonalizable matrix: A matrix A is called diagonalizable if there exists an invertible matrix P such that

$$P^{-1}AP = D$$

is diagonal.

* From the definition, we have

$$AP = PD,$$

which implies that the columns of P must be the eigenvectors of A .

* In the case $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$, we have shown that A has only two independent eigenvectors. It is impossible to form a matrix P such that P is invertible and $P^{-1}AP = D$.