

MA1301 Mid-term Test, Section CB1, 19 March, 2018

Name:

Student ID:

1. Compute the following limits:

$$(a) [12 \text{ marks}] \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sqrt{1 + \frac{2}{n}} + \sqrt{1 + \frac{4}{n}} + \sqrt{1 + \frac{6}{n}} + \cdots + \sqrt{3} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sqrt{1 + 2 \frac{k}{n}}$$

Note that $\frac{1}{n} \sum_{k=1}^n \sqrt{1 + 2 \frac{k}{n}}$ is the Riemann Sum of the function

$\sqrt{1+2x}$ on the interval $[0, 1]$. Moreover, as $\sqrt{1+2x}$ is continuous on $[0, 1]$, the Riemann Sum has a limit as $n \rightarrow \infty$.

$$\text{So, } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sqrt{1 + 2 \frac{k}{n}} = \int_0^1 \sqrt{1+2x} \, dx$$

$$= \frac{1}{3} (1+2x)^{\frac{3}{2}} \Big|_0^1$$

$$= \sqrt{3} - \frac{1}{3}$$

$$(b) [10 \text{ marks}] \lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x \frac{t^2}{at^4 + 1} dt, \quad a > 0.$$

Because $\lim_{x \rightarrow 0} \int_0^x \frac{t^2}{at^4 + 1} dt = 0$, this is a limit of $\frac{0}{0}$ type.

By L'Hospital's Rule, we get

$$\lim_{x \rightarrow 0} \frac{\int_0^x \frac{t^2}{at^4 + 1} dt}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \left(\int_0^x \frac{t^2}{at^4 + 1} dt \right)}{(x^3)'} = \lim_{x \rightarrow 0} \frac{\frac{x^2}{ax^4 + 1}}{3x^2}$$

$$= \lim_{x \rightarrow 0} \frac{1}{3} \frac{1}{ax^4 + 1} = \frac{1}{3}$$

2. (a) [5 marks] If $f(x)$ is continuous on $(-\infty, \infty)$ and periodic with period T , show that

$$\int_a^b f(x) dx = \int_{a+kT}^{b+kT} f(x) dx, \quad \text{for } k = 1, 2, 3, \dots$$

(b) [10 marks] Use the above result to compute $\int_0^{100\pi} \sqrt{1 - \cos(2x)} dx$.

(a) As $f(x)$ is periodic with period T , we have

$$f(x) = f(x+T) = f(x+kT), \quad k = 1, 2, 3, \dots$$

Then

$$\begin{aligned} \int_{a+kT}^{b+kT} f(x) dx & \stackrel{x=t+kT}{=} \int_a^b f(t+kT) dt \\ & = \int_a^b f(t) dt \\ & = \int_a^b f(x) dx \end{aligned}$$

(b) Let $f(x) = \sqrt{1 - \cos(2x)}$.

As $f(x+\pi) = \sqrt{1 - \cos(2x+2\pi)} = f(x)$, this is a periodic function with period π . Then

$$\begin{aligned} \int_0^{100\pi} f(x) dx &= \sum_{k=1}^{100} \int_{(k-1)\pi}^{k\pi} f(x) dx \\ &= \sum_{k=1}^{100} \int_0^{\pi} f(x) dx = 100 \int_0^{\pi} \sqrt{1 - \cos(2x)} dx \\ &= 100 \int_0^{\pi} \sqrt{2} \sin x dx \\ &= \boxed{200\sqrt{2}} \end{aligned}$$

3. [10 marks] Find the values of a and b such that the following integral is convergent

$$\int_0^1 \frac{dx}{x^a(1-x)^b}, \quad a, b \in \mathbb{R}.$$

$$\int_0^1 \frac{dx}{x^a(1-x)^b} = \int_0^{\frac{1}{2}} \frac{dx}{x^a(1-x)^b} + \int_{\frac{1}{2}}^1 \frac{dx}{x^a(1-x)^b}$$

Consider the integral

$$\int_{\frac{1}{2}}^1 \frac{dx}{x^a(1-x)^b}.$$

$$\frac{1}{2} \leq x \leq 1$$

$$\Rightarrow 1 \leq \frac{1}{x} \leq 2$$

$$\text{when } a \geq 0, \quad \frac{1}{x^a} \leq 2^a$$

$$\text{when } a < 0, \quad \frac{1}{x^a} \leq 1$$

$$\text{so } \frac{1}{x^a} \leq 2^a + 1 \text{ for all } a$$

$$\text{when } x \in [\frac{1}{2}, 1]$$

$$\text{Then } 0 < \int_{\frac{1}{2}}^1 \frac{dx}{x^a(1-x)^b} < (2^{a+1}) \int_{\frac{1}{2}}^1 \frac{dx}{(1-x)^b}$$

$$\int_{\frac{1}{2}}^1 \frac{dx}{(1-x)^b} = \begin{cases} \ln|1-x| \Big|_{\frac{1}{2}}^1, & \text{if } b=1 \\ -\frac{1}{1-b} (1-x)^{1-b} \Big|_{\frac{1}{2}}^1, & \text{if } b \neq 1 \end{cases}$$

It is convergent only when $1-b > 0 \Rightarrow b < 1$.

Similarly, the integral $\int_0^{\frac{1}{2}} \frac{dx}{x^a(1-x)^b}$ is convergent when $a < 1$.

Thus, the integral $\int_0^1 \frac{dx}{x^a(1-x)^b}$ is convergent when $\boxed{a < 1 \text{ and } b < 1}$

4. Compute the following integrals:

(a) [13 marks] $\int \frac{2e^{2x} - e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx.$

$$\underline{u = e^x} \int \frac{2u - 1}{\sqrt{3u^2 - 6u - 1}} du$$

$$= \int \frac{2u - 1}{\sqrt{3(u-1)^2 - 4}} du$$

$$= 2 \int \frac{u-1}{\sqrt{3(u-1)^2 - 4}} du + \int \frac{1}{\sqrt{3(u-1)^2 - 4}} du$$

For the first term.

$$I_1 = 2 \int \frac{u-1}{\sqrt{3(u-1)^2 - 4}} du \quad \underline{t = (u-1)^2} \int \frac{dt}{\sqrt{3t-4}}$$

$$= \frac{2}{3} \sqrt{3t-4} + C_1$$

$$= \frac{2}{3} \sqrt{3u^2 - 6u - 1} + C_1$$

(b) [13 marks] $\int \frac{x^5 + x + 1}{x^3 + 1} dx$

$$\frac{x^5 + x + 1}{x^3 + 1} = x^2 - \frac{x^2 - x - 1}{x^3 + 1}$$

$$\frac{x^2 - x - 1}{x^3 + 1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$= \frac{A(x^2 - x + 1) + (Bx + C)(x+1)}{(x+1)(x^2 - x + 1)}$$

$$\Rightarrow A = \frac{1}{3}, \quad B = \frac{2}{3}, \quad C = -\frac{4}{3}$$

Then $\int \frac{x^5 + x + 1}{x^3 + 1} dx$

$$= \int \left(x^2 - \frac{1}{3} \frac{1}{x+1} - \frac{2}{3} \frac{x-2}{x^2-x+1} \right) dx$$

$$= \frac{1}{3} x^3 - \frac{1}{3} \ln|x+1| - \frac{2}{3} \int \frac{x-2}{x^2-x+1} dx$$

For the second term, we use trigonometric substitution.

$$I_2 = \int \frac{1}{\sqrt{3(u-1)^2 - 4}} du$$

$$\underline{u-1 = \frac{2}{\sqrt{3}} \sec \theta} \int \frac{2 \sec \theta \cdot \tan \theta / \sqrt{3}}{2 \tan \theta} d\theta$$

$$= \frac{1}{\sqrt{3}} \int \sec \theta d\theta$$

$$= \frac{\sqrt{3}}{3} \ln |\sec \theta + \tan \theta| + C_2$$

$$= \frac{\sqrt{3}}{3} \ln \left| \frac{\sqrt{3}}{2}(u-1) + \frac{1}{2} \sqrt{3u^2 - 6u - 1} \right| + C_2$$

Recalling $u = e^x$, we have

$$\int \frac{2e^{2x} - e^x}{\sqrt{3e^{2x} - 6e^x - 1}} dx$$

$$= \frac{2}{3} \sqrt{3e^{2x} - 6e^x - 1} + \frac{\sqrt{3}}{3} \ln \left| \sqrt{3}(e^x - 1) + \sqrt{3e^{2x} - 6e^x - 1} \right| + C$$

For $I_1 = \int \frac{x-2}{x^2-x+1} dx = \int \frac{x - \frac{1}{2} - \frac{3}{2}}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx$

$$= \int \frac{x - \frac{1}{2}}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx - \frac{3}{2} \int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx$$

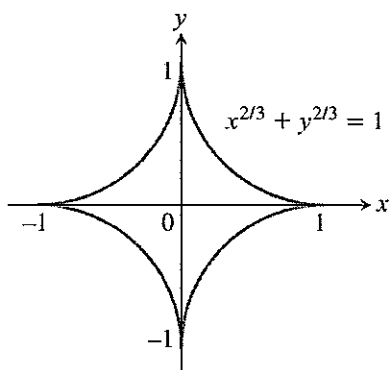
$$= \frac{1}{2} \ln \left| (x-\frac{1}{2})^2 + \frac{3}{4} \right|$$

$$- \sqrt{3} \arctan \left(\frac{2x-1}{\sqrt{3}} \right) + C.$$

So. $\int \frac{x^5 + x + 1}{x^3 + 1} dx$

$$= \frac{1}{3} x^3 - \frac{1}{3} \ln|x+1| - \frac{1}{3} \ln|x^2 - x + 1| + \frac{2\sqrt{3}}{3} \arctan \left(\frac{2x-1}{\sqrt{3}} \right) + C.$$

5. The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called astroids; see the following figure.



(a) [6 marks] Find the length of this astroid.

(b) [6 marks] Find the area of the surface generated by rotating this astroid about the x -axis.

(a) The equation for the astroid in the first quadrant is

$$y = f(x) = (1 - x^{2/3})^{3/2}, \quad x \in [0, 1].$$

As the astroid is symmetric, its length is given by

$$\begin{aligned} L &= 4 \int_0^1 \sqrt{1 + [f'(x)]^2} dx = 4 \int_0^1 \sqrt{1 + (1 - x^{2/3}) x^{-2/3}} dx \\ &= 4 \int_0^1 x^{-1/3} dx = 4 \cdot \frac{3}{2} x^{2/3} \Big|_0^1 = \boxed{6} \end{aligned}$$

(b) The surface area is

$$S = 2 \int_0^1 2\pi y \sqrt{1 + [f'(x)]^2} dx = 4\pi \int_0^1 (1 - x^{2/3})^{3/2} x^{-1/3} dx$$

$$\underline{u = x^{2/3}} \quad 4\pi \int_0^1 (1 - u)^{3/2} \cdot \frac{3}{2} du$$

$$= 6\pi \cdot \left(-\frac{2}{5}\right) (1 - u)^{5/2} \Big|_0^1$$

$$= \boxed{\frac{12\pi}{5}}$$

6. [15 marks] A spring with a natural length of 1 meter is stretched 10 cm by a 12 N force. Find the work done in stretching the spring from 1.2 meter to 1.3 meter. (show all steps).

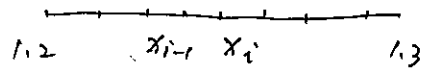
By Hooke's law $F = kx$, we have

$$12 = k \cdot 0.1 \Rightarrow k = 120 \text{ N/m.}$$

Consider the work done in stretching the spring from 1.2m to 1.3m.

Divide $[1.2, 1.3]$ into n small intervals

$$\Delta x = x_i - x_{i-1}$$



When Δx is small, the spring force for $x \in [x_{i-1}, x_i]$ can be approximated as a constant, which is

$$k \cdot (x_i^* - 1), \quad x_i^* \in [x_{i-1}, x_i]$$

Then, the work w_i done in the interval is

$$w_i \approx k \cdot (x_i^* - 1) \cdot \Delta x$$

The total work

$$W \approx \sum_{i=1}^n w_i = \sum_{i=1}^n k(x_i^* - 1) \Delta x$$

As $k \cdot (x-1)$ is a continuous function, we have

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n w_i = \int_{1.2}^{1.3} k(x-1) dx \\ &= 120 \cdot \frac{1}{2} (x-1)^2 \Big|_{x=1.2}^{1.3} \\ &= 60 (0.09 - 0.04) \\ &= 3 \text{ N}\cdot\text{m} \end{aligned}$$

End