

MA 1201 Semester B 2018/19

Assignment 2 — Due at 5 pm, 21/3/2019 (Thursday) in tutorial classes

Instructions:

- Please show your work. Unsupported answers will receive **NO** credits.
- Make sure you write down the correct lecture session (A/B/C/D/E/F/G/H) you have registered for, together with your full name and student ID on the front page of your answer script. Hand your solution to your TA in tutorial class before the deadline.
- In case your tutorial class is scheduled on days other than Thursday, you may choose to hand your solution to your TA in one of the tutorial classes scheduled on Thursday.
- **NO** late homework will be accepted. Homework submitted to wrong tutorial sessions will **NOT** be graded and will receive **0 POINTS**.

1. (50 points) Evaluate the following integrals.

(a) (10 points) $\int (\sin^{-1} x)^2 dx$.

Solution. In view of the presence of the inverse trig function, the idea is to apply integration by parts. Let (see page 81 of the supplementary notes for the ILATE rule for choosing u)

$$u = (\sin^{-1} x)^2, \quad dv = dx.$$

Direct integration yields

$$v = \int dv = \int dx = x,$$

so

$$\begin{aligned} \int (\sin^{-1} x)^2 dx &= \int (\sin^{-1} x)^2 d(x) = (\sin^{-1} x)^2 \cdot x - \int x d[(\sin^{-1} x)^2] \\ &= x(\sin^{-1} x)^2 - \int x \cdot \left(2 \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} dx \right) = x(\sin^{-1} x)^2 - 2 \int \frac{x}{\sqrt{1-x^2}} \sin^{-1} x dx. \end{aligned}$$

For the integral on the right side, apply integration by parts one more time with

$$u = \sin^{-1} x, \quad dv = \frac{x}{\sqrt{1-x^2}} dx,$$

$$v = \int dv = \int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} \cdot x dx \stackrel{w=1-x^2}{dw=-2x dx} \int w^{-1/2} \left(-\frac{1}{2} dw \right) = -\frac{1}{2} \cdot 2w^{1/2} = -\sqrt{1-x^2},$$

to deduce

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}} \sin^{-1} x dx &= \int \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} dx = \int \sin^{-1} x d(-\sqrt{1-x^2}) \\ &= \sin^{-1} x \cdot (-\sqrt{1-x^2}) - \int (-\sqrt{1-x^2}) d(\sin^{-1} x) \\ &= -\sqrt{1-x^2} \sin^{-1} x + \int \sqrt{1-x^2} \cdot \left(\frac{1}{\sqrt{1-x^2}} dx \right) \\ &= -\sqrt{1-x^2} \sin^{-1} x + \int 1 dx = -\sqrt{1-x^2} \sin^{-1} x + x + C. \end{aligned}$$

Thus

$$\begin{aligned}\int (\sin^{-1} x)^2 dx &= x(\sin^{-1} x)^2 - 2(-\sqrt{1-x^2} \sin^{-1} x + x) + C \\ &= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2} \sin^{-1} x - 2x + C.\end{aligned}$$

Remark. An alternative way to solve the problem is to introduce the (trig) substitution

$$\begin{aligned}x &= \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad \text{so that} \quad \sin^{-1} x = \sin^{-1}(\sin \theta) = \theta, \\ dx &= (\sin \theta)' d\theta = \cos \theta d\theta.\end{aligned}$$

This leads to a simpler integral

$$\int (\sin^{-1} x)^2 dx = \int \theta^2 \cdot (\cos \theta d\theta),$$

to which the (repeated) integration by parts is applied more easily:

$$\begin{aligned}\int \theta^2 \cos \theta d\theta &= \int \theta^2 d(\sin \theta) = \theta^2 \cdot \sin \theta - \int \sin \theta d(\theta^2) = \theta^2 \sin \theta - 2 \int \theta \sin \theta d\theta \\ &= \theta^2 \sin \theta + 2 \int \theta d(\cos \theta) = \theta^2 \sin \theta + 2 \left(\theta \cdot \cos \theta - \int \cos \theta d\theta \right) \\ &= \theta^2 \sin \theta + 2\theta \cos \theta - 2 \sin \theta + C.\end{aligned}$$

(b) (10 points) $\int \sin 3x \cos^2 2x dx.$

Motivation. The even power $\cos^2 2x$ needs to be simplified using the half-angle formula:

$$\cos^2 2x = \frac{1}{2} [1 + \cos 2(2x)] = \frac{1}{2} (1 + \cos 4x),$$

and the resulting product with $\sin 3x$ needs to be simplified using the product-to-sum formula:

$$\sin 3x \cos 4x = \frac{1}{2} [\sin(3x + 4x) + \sin(3x - 4x)] = \frac{1}{2} [\sin(7x) + \sin(-x)] = \frac{1}{2} (\sin 7x - \sin x).$$

Solution.

$$\begin{aligned}\int \sin 3x \cos^2 2x dx &= \int \sin 3x \cdot \frac{1}{2} (1 + \cos 4x) dx = \frac{1}{2} \int (\sin 3x + \sin 3x \cos 4x) dx \\ &= \frac{1}{2} \int \sin 3x dx + \frac{1}{2} \int \sin 3x \cos 4x dx = \frac{1}{2} \int \sin 3x dx + \frac{1}{2} \int \frac{1}{2} (\sin 7x - \sin x) dx \\ &= \frac{1}{2} \cdot \left(-\frac{1}{3} \cos 3x \right) + \frac{1}{4} \cdot \left(-\frac{1}{7} \cos 7x + \cos x \right) + C = -\frac{1}{6} \cos 3x - \frac{1}{28} \cos 7x + \frac{1}{4} \cos x + C.\end{aligned}$$

Remark. If the problem given is

$$\int \sin 2x \cos^2 2x dx,$$

i.e. if the arguments of the sine and cosine functions are the same, then a standard application of the procedure outlined in page 99–101 of the supplementary notes gives

$$\begin{aligned}\int \sin 2x \cos^2 2x dx &= \int \cos^2 2x d\left(-\frac{1}{2} \cos 2x\right) \stackrel{u=\cos 2x}{=} \int u^2 d\left(-\frac{1}{2} u\right) \\ &= -\frac{1}{2} \int u^2 du = -\frac{1}{2} \cdot \frac{1}{3} u^3 + C = -\frac{1}{6} \cos^3 2x + C.\end{aligned}$$

(c) (15 points) $\int_0^{\ln \sqrt{3}} \frac{e^{6x}}{\sqrt{9-e^{4x}}} dx.$

Motivation. Observe that

$$e^{4x} = (e^{2x})^2, \quad \text{and} \quad e^{6x} dx = e^{4x} \cdot e^{2x} dx = (e^{2x})^2 d\left(\frac{1}{2} e^{2x}\right).$$

This suggests that the integral can be simplified by the substitution $u = e^{2x}$.

Solution. Let

$$\begin{aligned}u &= e^{2x}, \\ du &= (e^{2x})' dx = 2e^{2x} dx, \quad \text{or} \quad e^{2x} dx = \frac{1}{2} du.\end{aligned}$$

Then

$$\int_0^{\ln \sqrt{3}} \frac{e^{6x}}{\sqrt{9-e^{4x}}} dx = \int_0^{\ln \sqrt{3}} \frac{(e^{2x})^2}{\sqrt{9-(e^{2x})^2}} \cdot e^{2x} dx = \int_{e^0}^{e^{2 \ln \sqrt{3}}} \frac{u^2}{\sqrt{9-u^2}} \cdot \left(\frac{1}{2} du\right) = \frac{1}{2} \int_1^3 \frac{u^2}{\sqrt{9-u^2}} du.$$

To proceed, recall the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ or $9 - 9 \sin^2 \theta = 9 \cos^2 \theta$, which motivates the (trig) substitution

$$\begin{aligned}u^2 &= 9 \sin^2 \theta, \quad \text{or} \quad u = 3 \sin \theta, \quad \text{so that} \quad \sqrt{9-u^2} = \sqrt{9-9 \sin^2 \theta} = \sqrt{9 \cos^2 \theta} = 3 \cos \theta, \\ du &= (3 \sin \theta)' d\theta = 3 \cos \theta d\theta.\end{aligned}$$

It then follows that (neglecting the bounds of integration for now)

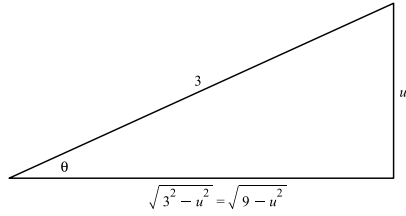
$$\begin{aligned}\frac{1}{2} \int \frac{u^2}{\sqrt{9-u^2}} du &= \frac{1}{2} \int \frac{9 \sin^2 \theta}{3 \cos \theta} \cdot (3 \cos \theta d\theta) = \frac{9}{2} \int \sin^2 \theta d\theta = \frac{9}{2} \int \frac{1}{2} (1 - \cos 2\theta) d\theta \\ &= \frac{9}{4} \left(\theta - \frac{1}{2} \sin 2\theta \right) = \frac{9}{4} (\theta - \sin \theta \cos \theta) = \frac{9}{4} \left\{ \sin^{-1} \left(\frac{1}{3} u \right) - \frac{1}{3} u \cdot \frac{1}{3} \sqrt{9-u^2} \right\},\end{aligned}$$

and so

$$\frac{1}{2} \int_1^3 \frac{u^2}{\sqrt{9-u^2}} du = \frac{9}{4} \left\{ \sin^{-1} \left(\frac{1}{3} u \right) - \frac{1}{9} u \sqrt{9-u^2} \right\} \Big|_1^3 = \frac{9}{4} \left(\sin^{-1} 1 - \sin^{-1} \frac{1}{3} + \frac{2}{9} \sqrt{2} \right).$$

Remark. 1. To express $\cos \theta$ as a function of u , recall from the equation of substitution and the definition of the sine function that

$$u = 3 \sin \theta, \quad \text{or} \quad \sin \theta = \frac{u}{3} = \frac{\text{opp}}{\text{hyp}}.$$



This motivates the construction of the following right triangle, where the adjacent side to the angle θ is determined using the Pythagorean theorem. It then follows that

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{3} \sqrt{9 - u^2}.$$

2. The integral

$$\frac{1}{2} \int_1^3 \frac{u^2}{\sqrt{9 - u^2}} du$$

resulting from the first substitution *cannot* be solved using the simple substitution

$$\begin{aligned} v = 9 - u^2, \quad \text{or} \quad u = \sqrt{9 - v}, \\ dv = (9 - u^2)' du = -2u du, \quad \text{or} \quad du = -\frac{1}{2u} dv = -\frac{1}{2\sqrt{9 - v}} dv, \end{aligned}$$

because it leads to an integral (neglecting the bounds of integration)

$$\frac{1}{2} \int \frac{u^2}{\sqrt{9 - u^2}} du = \frac{1}{2} \int \frac{9 - v}{\sqrt{v}} \cdot \left(-\frac{1}{2\sqrt{9 - v}} dv \right) = -\frac{1}{4} \int \frac{\sqrt{9 - v}}{\sqrt{v}} dv,$$

which is as difficult to evaluate as the original one. On the other hand, if the integral given is

$$\frac{1}{2} \int_1^3 \frac{u^3}{\sqrt{9 - u^2}} du,$$

then the above simple substitution works perfectly:

$$\begin{aligned} v = 9 - u^2, \quad \text{or} \quad u^2 = 9 - v, \\ dv = -2u du, \quad \text{or} \quad u du = -\frac{1}{2} dv, \end{aligned}$$

which yields (again, neglecting the bounds of integration)

$$\begin{aligned} \frac{1}{2} \int \frac{u^3}{\sqrt{9 - u^2}} du &= \frac{1}{2} \int \frac{u^2}{\sqrt{9 - u^2}} \cdot u du = \frac{1}{2} \int \frac{9 - v}{\sqrt{v}} \cdot \left(-\frac{1}{2} dv \right) = -\frac{1}{4} \int \frac{9 - v}{\sqrt{v}} dv \\ &= -\frac{1}{4} \int \left(\frac{9}{v^{1/2}} - \frac{v}{v^{1/2}} \right) dv = -\frac{1}{4} \int (9v^{-1/2} - v^{1/2}) dv = -\frac{1}{4} \left(9 \cdot 2v^{1/2} - \frac{2}{3} v^{3/2} \right). \end{aligned}$$

Clearly, in this latter case a trig substitution is not necessary.

(d) (15 points) $\int \frac{x^3 - 5x^2 + 14x - 7}{x^2 - 4x + 9} dx.$

Solution. The idea is to decompose the integrand, a rational function, into partial fractions. To begin with, note that (see page 121–123 of the supplementary notes for the general procedure for finding partial fraction decompositions of rational functions)

$$\deg(\text{numerator}) = 3 > \deg(\text{denominator}) = 2.$$

Thus a long division is needed to decompose the integrand, which gives

$$x^3 - 5x^2 + 14x - 7 = (x - 1)(x^2 - 4x + 9) + (x + 2),$$

or

$$\frac{x^3 - 5x^2 + 14x - 7}{x^2 - 4x + 9} = (x - 1) + \frac{x + 2}{x^2 - 4x + 9}.$$

Since the quadratic polynomial $x^2 - 4x + 9$ is irreducible ($\Delta = (-4)^2 - 4 \cdot 1 \cdot 9 < 0$), the proper rational function on the right side is already a partial fraction (and hence requires no further decomposition). It then follows that

$$\int \frac{x^3 - 5x^2 + 14x - 7}{x^2 - 4x + 9} dx = \int (x - 1) dx + \int \frac{x + 2}{x^2 - 4x + 9} dx = \frac{1}{2}x^2 - x + \int \frac{x + 2}{x^2 - 4x + 9} dx.$$

To evaluate the last integral on the right side, observe that the substitution

$$\begin{aligned} u &= x^2 - 4x + 9, \\ du &= (x^2 - 4x + 9)' dx = (2x - 4) dx = 2(x - 2) dx, \end{aligned}$$

indicates that an extra factor of $(x - 2)$ (= a constant multiple of u') is needed to simplify the integral. This suggests the decomposition of the numerator:

$$x + 2 = (x - 2 + 2) + 2 = (x - 2) + 4,$$

and hence

$$\int \frac{x + 2}{x^2 - 4x + 9} dx = \int \left(\frac{x - 2}{x^2 - 4x + 9} + \frac{4}{x^2 - 4x + 9} \right) dx = \int \frac{x - 2}{x^2 - 4x + 9} dx + \int \frac{4}{x^2 - 4x + 9} dx.$$

The first integral on the right side can be directly solved using the substitution

$$\begin{aligned} u &= x^2 - 4x + 9, \\ du &= 2(x - 2) dx, \quad \text{or} \quad (x - 2) dx = \frac{1}{2} du, \end{aligned}$$

which gives

$$\begin{aligned} \int \frac{x - 2}{x^2 - 4x + 9} dx &= \int \frac{1}{x^2 - 4x + 9} \cdot (x - 2) dx \\ &= \int \frac{1}{u} \cdot \left(\frac{1}{2} du \right) = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|x^2 - 4x + 9| + C. \end{aligned}$$

As for the second integral, the fact that the quadratic polynomial $x^2 - 4x + 9$ is irreducible implies that

$$\begin{aligned}\int \frac{4}{x^2 - 4x + 9} dx &= \int \frac{4}{(x^2 - 2 \cdot x \cdot 2) + 9} dx = \int \frac{4}{(x^2 - 2 \cdot x \cdot 2 + 2^2 - 2^2) + 9} dx \\ &= \int \frac{4}{(x-2)^2 + 5} dx = \frac{4}{5} \int \frac{1}{(1/5)(x-2)^2 + 1} dx = \frac{4}{5} \cdot \frac{\tan^{-1}[\sqrt{1/5}(x-2)]}{\sqrt{1/5}} + C.\end{aligned}$$

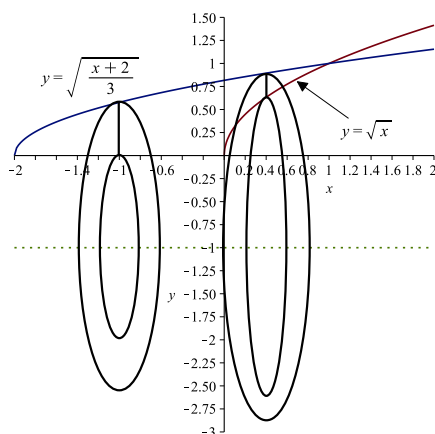
In conclusion,

$$\int \frac{x^3 - 5x^2 + 14x - 7}{x^2 - 4x + 9} dx = \frac{1}{2}x^2 - x + \frac{1}{2} \ln|x^2 - 4x + 9| + \frac{4}{\sqrt{5}} \tan^{-1}[\sqrt{1/5}(x-2)] + C.$$

2. (15 points) Find the volume of the solid generated by revolving the region bounded by the curves $x = 3y^2 - 2$, $x = y^2$ in the *upper* half plane

(a) (5 points) about the line $y = -1$, using the washer method (see page 157–164 of the supplementary notes for a description of the washer method).

Solution. As shown in the following figure, the washer method requires that the chosen slice be



perpendicular to the axis of rotation. Thus the slice is vertical, and the integration has to be carried out in x on two separate subintervals (on which the inner radii of the washers differ). On the first subinterval $[-2, 0]$, the inner and outer radii $r_1(x)$ and $R_1(x)$ of the washer, *expressed as functions of x* , are given by

$$r_1(x) = 0 - (-1) = 1, \quad R_1(x) = \sqrt{\frac{1}{3}(x+2)} - (-1) = \sqrt{\frac{1}{3}(x+2)} + 1.$$

On the second subinterval $[0, 1]$, the inner and outer radii $r_2(x)$ and $R_2(x)$ of the washer are given by

$$r_2(x) = \sqrt{x} - (-1) = \sqrt{x} + 1, \quad R_2(x) = \sqrt{\frac{1}{3}(x+2)} - (-1) = \sqrt{\frac{1}{3}(x+2)} + 1.$$

Note that the right end point of the second subinterval is determined by equating $y = \sqrt{\frac{1}{3}(x+2)}$ to $y = \sqrt{x}$ and solving for x , which yields

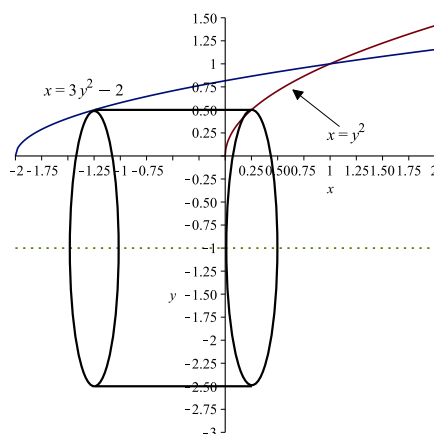
$$\sqrt{\frac{1}{3}(x+2)} = \sqrt{x} \implies \frac{1}{3}(x+2) = x \implies x+2 = 3x \implies x = 1.$$

The volume of the solid is then given by

$$\begin{aligned}
 V &= \int_{-2}^0 \pi [R_1^2(x) - r_1^2(x)] dx + \int_0^1 \pi [R_2^2(x) - r_2^2(x)] dx \\
 &= \int_{-2}^0 \pi \left\{ \left(\sqrt{\frac{1}{3}(x+2)} + 1 \right)^2 - 1^2 \right\} dx + \int_0^1 \pi \left\{ \left(\sqrt{\frac{1}{3}(x+2)} + 1 \right)^2 - (\sqrt{x} + 1)^2 \right\} dx \\
 &= \pi \int_{-2}^0 \left(\frac{1}{3}(x+2) + \frac{2}{\sqrt{3}}\sqrt{x+2} \right) dx + \pi \int_0^1 \left(\frac{1}{3}(x+2) + \frac{2}{\sqrt{3}}\sqrt{x+2} - x - 2\sqrt{x} \right) dx \\
 &= \frac{1}{3} \pi \int_{-2}^1 (x+2) dx + \frac{2}{\sqrt{3}} \pi \int_{-2}^1 \sqrt{x+2} dx - \pi \int_0^1 (x + 2\sqrt{x}) dx \\
 &= \frac{1}{3} \pi \cdot \left(\frac{1}{2}x^2 + 2x \right) \Big|_{-2}^1 + \frac{2}{\sqrt{3}} \pi \cdot \frac{2}{3} (x+2)^{3/2} \Big|_{-2}^1 - \pi \cdot \left(\frac{1}{2}x^2 + 2 \cdot \frac{2}{3}x^{3/2} \right) \Big|_0^1 \\
 &= \frac{1}{3} \pi \cdot \left(-\frac{3}{2} + 6 \right) + \frac{2}{\sqrt{3}} \pi \cdot \frac{2}{3} \cdot 3^{3/2} - \pi \cdot \left(\frac{1}{2} + \frac{4}{3} \right) = \frac{3}{2} \pi + 4\pi - \frac{11}{6} \pi = \frac{11}{3} \pi.
 \end{aligned}$$

- (b) (5 points) about the line $y = -1$, using the shell method (see page 165–176 of the supplementary notes for a description of the shell method).

Solution. As shown in the following figure, the shell method requires that the chosen slice be parallel



to the axis of rotation. Thus the slice is horizontal, and the integration has to be carried out in y (note, in particular, that there is no need to consider multiple subintervals). The radius $r(y)$ and height $h(y)$ of the shell, *expressed as functions of y* , are given by

$$r(y) = y - (-1) = y + 1, \quad h(y) = y^2 - (3y^2 - 2) = -2y^2 + 2.$$

The bounds of integration, on the other hand, are determined by equating $x = 3y^2 - 2$ to $x = y^2$ and solving for y , which yields

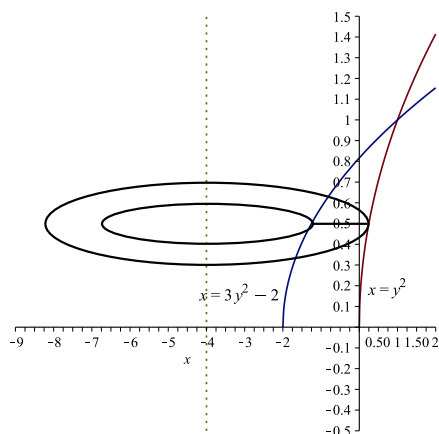
$$3y^2 - 2 = y^2 \implies y^2 = 1 \implies y = -1 \text{ (discard)}, y = 1.$$

The volume of the solid is then given by

$$\begin{aligned} V &= \int_0^1 2\pi r(y)h(y) dy = \int_0^1 2\pi(y+1)(-2y^2+2) dy = 2\pi \int_0^1 (-2y^3 - 2y^2 + 2y + 2) dy \\ &= 2\pi \cdot \left(-2 \cdot \frac{1}{4} y^4 - 2 \cdot \frac{1}{3} y^3 + 2 \cdot \frac{1}{2} y^2 + 2y \right) \Big|_0^1 = 2\pi \cdot \left(-\frac{1}{2} - \frac{2}{3} + 1 + 2 \right) = \frac{11}{3} \pi. \end{aligned}$$

(c) (5 points) about the line $x = -4$, using any appropriate method.

Solution. As shown in the following figure, the slice is chosen to be horizontal (and hence the integra-



tion has to be carried out in y), since this avoids partitioning the interval of integration into multiple subintervals. Since the slice is perpendicular to the axis of rotation, the revolution of the slice generates a washer, which then suggests that the washer method has to be used. Now the inner and outer radii $r(y)$ and $R(y)$ of the washer, *expressed as functions of y* , are given by

$$r(y) = (3y^2 - 2) - (-4) = 3y^2 + 2, \quad R(y) = y^2 - (-4) = y^2 + 4.$$

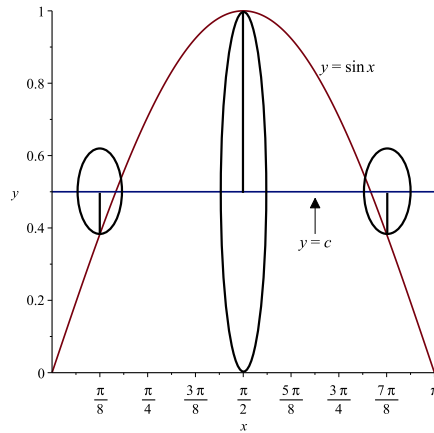
The bounds of integration, on the other hand, are determined by equating $x = 3y^2 - 2$ to $x = y^2$ and solving for y , which yields

$$3y^2 - 2 = y^2 \implies y^2 = 1 \implies y = -1 \text{ (discard)}, y = 1.$$

The volume of the solid is then given by

$$\begin{aligned} V &= \int_0^1 \pi [R^2(y) - r^2(y)] dy = \int_0^1 \pi [(y^2 + 4)^2 - (3y^2 + 2)^2] dy = \pi \int_0^1 (-8y^4 - 4y^2 + 12) dy \\ &= \pi \cdot \left(-8 \cdot \frac{1}{5} y^5 - 4 \cdot \frac{1}{3} y^3 + 12y \right) \Big|_0^1 = \pi \cdot \left(-\frac{8}{5} - \frac{4}{3} + 12 \right) = \frac{136}{15} \pi. \end{aligned}$$

3. (20 points) The region bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$, $y = c$ ($0 \leq c \leq 1$) is revolved about the line $y = c$ to generate a solid.



- (a) (10 points) Find the value of c that minimizes the volume of the solid. What is the minimum volume?

Solution. As shown in the following figure, the slice is chosen to be vertical (and hence the integration has to be carried out in x), since this avoids partitioning the interval of integration into multiple subintervals. Since the slice is perpendicular to the axis of rotation, the revolution of the slice generates a disk, which then suggests that the disk method has to be used. Now the radius $R(x, c)$ of the disk, expressed as a function of x and c , is given by

$$R(x, c) = |c - \sin x|.$$

The bounds of integration, on the other hand, are trivially given by $x = 0$ and $x = \pi$. The volume of the solid is then given by

$$\begin{aligned} V(c) &= \int_0^\pi \pi R^2(x, c) dx = \int_0^\pi \pi [c - \sin x]^2 dx = \pi \int_0^\pi (c - \sin x)^2 dx = \pi \int_0^\pi (c^2 - 2c \sin x + \sin^2 x) dx \\ &= \pi \int_0^\pi \left(c^2 - 2c \sin x + \frac{1}{2} (1 - \cos 2x) \right) dx = \pi \cdot \left\{ c^2 x + 2c \cos x + \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) \right\} \Big|_0^\pi \\ &= \pi \cdot \left\{ c^2 \pi + 2c \cdot (-2) + \frac{1}{2} \left(\pi - \frac{1}{2} \cdot 0 \right) \right\} = c^2 \pi^2 - 4c\pi + \frac{1}{2} \pi^2. \end{aligned}$$

To find the extreme values of $V(c)$, compute

$$V'(c) = 2c\pi^2 - 4\pi,$$

and solve the equation $V'(c) = 0$ for c . This yields the critical point $c = 2/\pi$. Evaluating $V(c)$ at the critical point and the end points (recall that $0 \leq c \leq 1$), one finds

$$V(2/\pi) = \frac{1}{2} \pi^2 - 4 \approx 0.9348, \quad V(0) = \frac{1}{2} \pi^2 \approx 4.9348, \quad V(1) = \frac{3}{2} \pi^2 - 4\pi \approx 2.238.$$

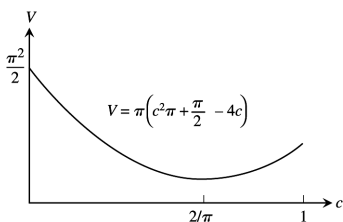
Comparing these values then shows that $V(c)$ attains its global minimum on $[0, 1]$ at $c = 2/\pi$, with a global minimum value of $(\frac{1}{2} \pi^2 - 4)$.

- (b) (5 points) What value of c in $[0, 1]$ maximizes the volume of the solid?

Solution. Based on the results obtained in (a), it can be observed that $V(c)$ attains its global maximum on $[0, 1]$ at $c = 0$, with a global maximum value of $\frac{1}{2} \pi^2$.

- (c) (5 points) Graph the solid's volume as a function of c , first for $0 \leq c \leq 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.

Solution. The graph of $V(c)$ as a function of c for $0 \leq c \leq 1$ is shown below. As c moves away from $[0, 1]$, the volume of the solid increases without bound. If one approximates the solid as a set of solid disks, one can see that the radius of a typical disk increases without bound as c moves away from $[0, 1]$.



4. (15 points) Find the length of the curve $x = (y^3/12) + (1/y)$, $1 \leq y \leq 2$.

Solution. Since the curve is described by a function of y , the curve length should be expressed as an integral of y (see page 189–191 of the supplementary notes for the differential form of the arc length formula):

$$L = \int_1^2 ds = \int_1^2 \sqrt{1 + [x'(y)]^2} dy,$$

where

$$\begin{aligned} 1 + [x'(y)]^2 &= 1 + \left(\frac{1}{4}y^2 - \frac{1}{y^2}\right)^2 = 1 + \left\{\left(\frac{1}{4}y^2\right)^2 - 2 \cdot \left(\frac{1}{4}y^2\right) \cdot \frac{1}{y^2} + \left(\frac{1}{y^2}\right)^2\right\} \\ &= 1 + \left(\frac{1}{4}y^2\right)^2 - \frac{1}{2} + \left(\frac{1}{y^2}\right)^2 = \left(\frac{1}{4}y^2\right)^2 + \frac{1}{2} + \left(\frac{1}{y^2}\right)^2 \\ &= \left(\frac{1}{4}y^2\right)^2 + 2 \cdot \left(\frac{1}{4}y^2\right) \cdot \frac{1}{y^2} + \left(\frac{1}{y^2}\right)^2 = \left(\frac{1}{4}y^2 + \frac{1}{y^2}\right)^2. \end{aligned}$$

It then follows that

$$L = \int_1^2 \left(\frac{1}{4}y^2 + \frac{1}{y^2}\right) dy = \left(\frac{1}{4} \cdot \frac{1}{3}y^3 - \frac{1}{y}\right) \Big|_1^2 = \frac{7}{12} - \left(-\frac{1}{2}\right) = \frac{13}{12}.$$

— THE END —