

EE2104 Introduction to Electromagnetics
Assignment 2 Solution

Deadline 5 Oct 2023 at 1:00 pm

Please put down your name and university ID.

1. Find the values of the following mathematical expressions:

$$(a) P = \int_{z=1}^2 \int_{\varphi=0}^{\pi} \sin(\varphi) z^2 d\varphi dz$$

$$(b) Q = \frac{\partial S}{\partial x} \text{ at } (x, y) = (\pi, 0), \text{ where } S(x, y) = \cos(x^2 + y^2)$$

$$(c) R = \frac{\partial^2 T}{\partial y^2} \text{ at } (x, y) = (\pi, 0), \text{ where } T(x, y) = x^{21}(2 + 3y)$$

Solution:

For (a),

$$P = \int_{z=1}^2 z^2 dz \int_{\varphi=0}^{\pi} \sin \varphi d\varphi = \int_{z=1}^2 z^2 dz \cdot [-\cos \varphi]_0^{\pi} = 2 \int_{z=1}^2 z^2 dz = \left[\frac{2z^3}{3} \right]_1^2 = \frac{14}{3} \approx 4.667$$

For (b),

Replace $S(x, y) = \cos(x^2 + y^2)$ in Q we have:

$$Q = \frac{\partial \cos(x^2 + y^2)}{\partial x} = -\sin(x^2 + y^2) \cdot d(x^2) = -2x \cdot \sin(x^2 + y^2)$$

Since $(x, y) = (\pi, 0)$,

$$Q = -2x \cdot \sin(x^2 + y^2) = -2 * (\pi) \cdot \sin(\pi^2 + 0^2) \approx 2.704$$

For (c),

$$R = \frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial y} \cdot \frac{\partial [x^{21}(2 + 3y)]}{\partial y} = \frac{\partial}{\partial y} \cdot [3x^{21}] = 0$$

2. Consider a line along the y-axis at $y = 0$ to 23 (in meters). It contains electric charges of density $\rho_L = \rho_{L0} (1 + y/L)$ for some constant $\rho_{L0} = 5nC/m$. Find the total charge.

Solution:

The total charge is:

$$Q = \int_0^{23} \rho_L dy$$

$$= \rho_{L0} \int_0^{23} \left(1 + \frac{y}{L}\right) dy$$

$$= \rho_{L0} \left[y + \frac{1}{L} \cdot \frac{y^2}{2} \right]_0^{23}$$

$$= 5 \times 10^{-9} \times \left[\left(23 + \frac{1}{23} \cdot \frac{23^2}{2}\right) - 0 \right]$$
$$= 1.725 \times 10^{-7} C$$

3. On the xy -plane, a circular disc centered at the origin has a radius $b = 23$ cm. It contains electric charge with a surface charge density of $\rho_s(\rho, \phi)$.
- (a) Find the total charge when $\rho_s = 2.1$ pC/m².
- (b) Find the total charge when $\rho_s = A \exp(\rho^2/b^2)$ for $A = 0.4$ nC/m².
- (c) Find the total charge when $\rho_s = B \exp(\rho^2/b^2) \sin^2 \phi$ for $B = 0.8$ nC/m².
- Remember to use the correct unit(s).*

Solution:

(a) $\rho_s(\rho, \phi) = 2.1$ pC/m² = 2.1×10^{-12} C/m², $b = 23$ cm = 23×10^{-2} m, $ds = \rho d\rho d\phi$

$$Q = \int_{Disc} \rho_s ds = \rho_s \int_0^{2\pi} d\phi \int_0^b \rho d\rho = 2\pi \cdot \rho_s \cdot \left[\frac{\rho^2}{2} \right]_0^b$$

$$= 2\pi \cdot (2.1 \times 10^{-12}) \cdot \left[\frac{(23 \times 10^{-2})^2}{2} - 0 \right] = 3.490 \times 10^{-13} \text{ C}$$

(b) $\rho_s = A \exp(\rho^2/b^2)$, $A = 0.4$ nC/m² = 0.4×10^{-9} C/m², $b = 23$ cm = 23×10^{-2} m, $ds = \rho d\rho d\phi$

$$Q = \int_{Disc} \rho_s ds = A \int_0^{2\pi} d\phi \int_0^b \exp(\rho^2/b^2) \cdot \rho d\rho = 2\pi A \cdot \left[\frac{b^2}{2} \exp\left(\frac{\rho^2}{b^2}\right) \right]_0^b$$

$$= 2\pi \cdot (0.4 \times 10^{-9}) \cdot \frac{b^2}{2} \cdot (e - 1) = 1.1422 \times 10^{-10} \text{ C}$$

(c) $\rho_s = B \exp(\rho^2/b^2) \sin^2 \phi$, $B = 0.8$ nC/m² = 0.8×10^{-9} C/m², $b = 23$ cm = 23×10^{-2} m, $ds = \rho d\rho d\phi$

$$Q = \int_{Disc} \rho_s ds = B \int_0^{2\pi} \sin^2 \phi d\phi \int_0^b \exp\left(\frac{\rho^2}{b^2}\right) \rho d\rho = 1.1422 \times 10^{-10} \text{ C}$$

since

$$\int_0^b \exp\left(\frac{\rho^2}{b^2}\right) \rho d\rho = \left[\frac{b^2}{2} \exp\left(\frac{\rho^2}{b^2}\right) \right]_0^b = \frac{b^2}{2} (e - 1)$$

$$\int_0^{2\pi} \sin^2 \phi d\phi = \int_0^{2\pi} \frac{1 - \cos(2\phi)}{2} d\phi = \pi$$

4. Consider a voltage $V(x, y) = bV_0/x$ for $V_0 = 3$ (in volts) and $b = 3$ (in meters).
- Is $V(x, y)$ a scalar field or a vector field?
 - Plot V versus x at $y = b, 2b$, and $3b$. Label the axes accurately.
 - Plot V versus y at $x = b, 2b$, and $3b$. Label the axes accurately.
 - Plot the contour lines on the xy -plane for $V = 0.25V_0, 0.50V_0, 0.75V_0$, and V_0 .
 - Given that the electric field $\mathbf{E} = -\text{grad}(V)$, find $\mathbf{E}(x, y)$.
- Remember to indicate the vectors using proper notation(s).

Solution:

- $V(x, y)$ is a scalar field.
- Since V only depends on x , the changes of y will not change V , plot is shown in Fig 1, where set $x \in [0, 1]$.

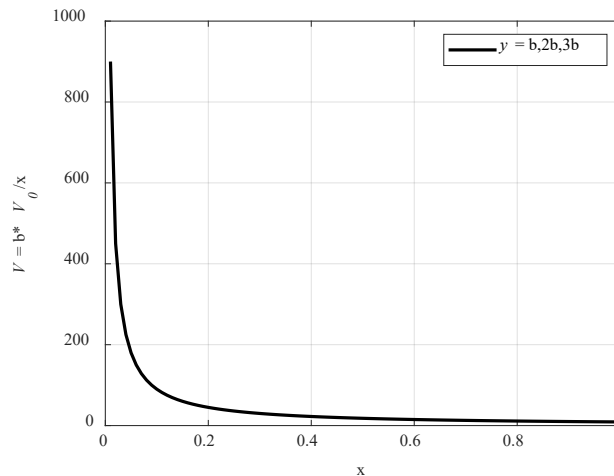


Figure 1

- V only depends on x , set $y \in [0, 1]$, plot is shown in Figure 2.

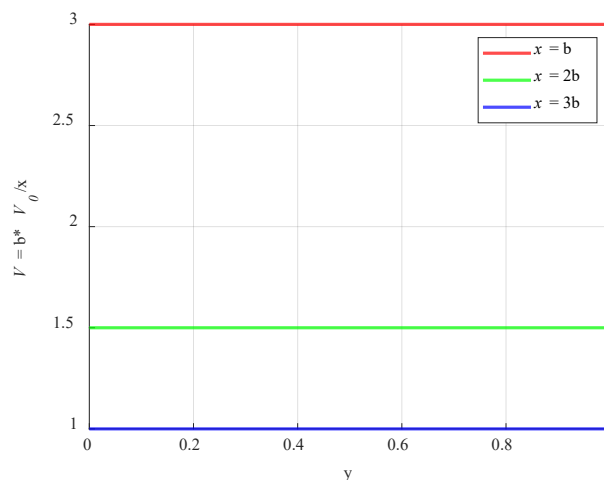


Figure 2

- Replace V by $0.25V_0, 0.50V_0, 0.75V_0$, and V_0 , we have:

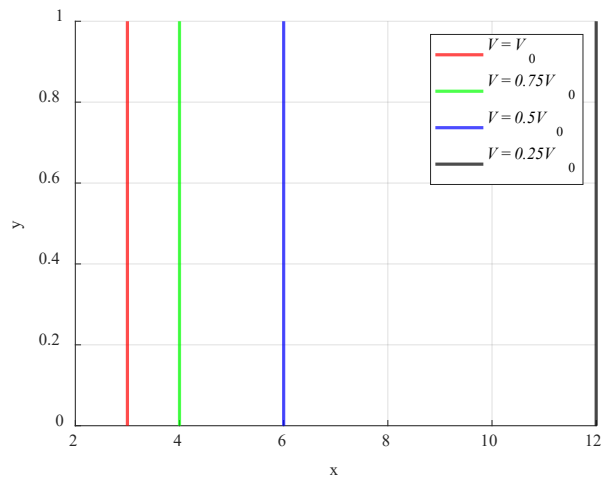
$$0.25V_0 = \frac{bV_0}{x} \rightarrow x = 12$$

$$0.50V_0 = \frac{bV_0}{x} \rightarrow x = 6$$

$$0.75V_0 = \frac{bV_0}{x} \rightarrow x = 4$$

$$V_0 = \frac{bV_0}{x} \rightarrow x = 3$$

Curves are given in Figure 3, where $y \in [0, 1]$.



(e) Since $\mathbf{E} = -\text{grad}(V)$

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}\right) = -(-bV_0x^{-2})\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}} = \frac{bV_0}{x^2}\hat{\mathbf{x}} = \frac{9}{x^2}\hat{\mathbf{x}}$$

5. Consider two points $A(L, 0)$ and $B(0, L)$ on the xy -plane for some positive distance L . Find the electric field $\vec{E}(0, 2L)$ under the following conditions.
- Point charges $+Q$ and $-Q$ are put at A and B , respectively.
 - Infinite line charges of densities $+\rho_L$ and $-\rho_L$ are put at A and B , respectively. The line charges are parallel to the z -axis.

Remember to use write the rectangular unit vectors correctly with the “hats”.

Solution:

The problem is defined in Figure 2. The electric field at point P is the superposition of the electric fields from point A and point B to point P .

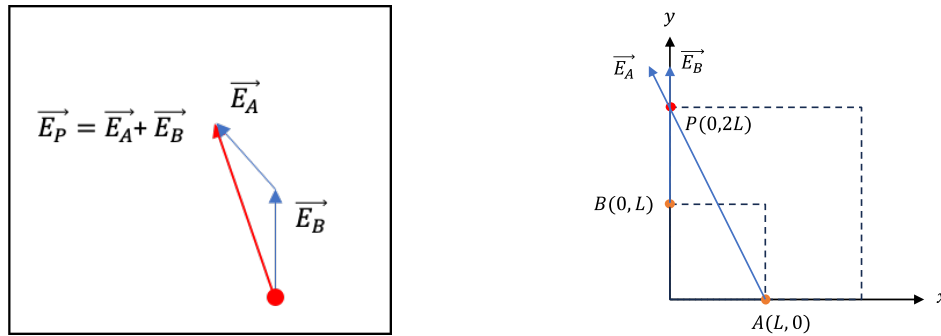


Figure 2 Problem Definition

- Two Point charges $+Q$ and $-Q$ are put at A and B , respectively.

Unit vector from A to P :

$$\hat{a}_A = \frac{\overrightarrow{AP}}{|\overrightarrow{AP}|} = \frac{-L\hat{x} + 2L\hat{y}}{[L^2 + (2L)^2]^{1/2}} = \frac{-\hat{x} + 2\hat{y}}{\sqrt{5}}$$

Unit vector from B to P :

$$\hat{a}_B = \frac{\overrightarrow{BP}}{|\overrightarrow{BP}|} = \frac{0\hat{x} + L\hat{y}}{[L^2]^{1/2}} = \hat{y}$$

When A contains charge $+Q$ and B contains $-Q$, the field $\vec{E}$ at point P can be obtained by the superposition principle and the Coulomb's Law:

$$\begin{aligned}\vec{E} &= \frac{Q}{4\pi\epsilon_0} \left[\frac{\hat{a}_A}{|\overrightarrow{AP}|^2} - \frac{\hat{a}_B}{|\overrightarrow{BP}|^2} \right] \\ &= \frac{Q}{4\pi\epsilon_0} \left[\frac{-\hat{x} + 2\hat{y}}{5\sqrt{5}L^2} - \frac{\hat{y}}{L^2} \right] \\ &= \frac{Q}{4\pi\epsilon_0 L^2} (-0.0894\hat{x} - 0.8211\hat{y})\end{aligned}$$

- Infinite line charges of densities $+\rho_L$ and $-\rho_L$ are put at A and B , respectively. The line charges are parallel to the z -axis.

$$\begin{aligned}\vec{E} &= \frac{\rho_L}{2\pi\epsilon_0} \left[\frac{\hat{a}_A}{|\vec{AP}|^1} - \frac{\hat{a}_B}{|\vec{BP}|^1} \right] = \frac{\rho_L}{2\pi\epsilon_0} \cdot \left[\frac{-\hat{x} + 2\hat{y}}{5L} - \frac{\hat{y}}{L} \right] \\ &= \frac{\rho_L}{2\pi\epsilon_0 L} (-0.2\hat{x} - 0.6\hat{y})\end{aligned}$$