

MA 1201 Semester B 2018/19

Extra Review Problems

1. Evaluate the following integrals.

(a) $\int \sin^5 x \cos^3 x dx.$

Sketch of Solution. (See page 99–101 of the supplementary notes.)

$$\begin{aligned} \int \sin^5 x \cos^3 x dx &= \int (1 - \cos^2 x)^2 \cos^3 x d(-\cos x) \\ &\stackrel{u=\cos x}{=} - \int (1 - u^2)^2 u^3 du = -\frac{1}{4} \cos^4 x + \frac{1}{3} \cos^6 x - \frac{1}{8} \cos^8 x + C. \end{aligned}$$

Or, alternatively,

$$\begin{aligned} \int \sin^5 x \cos^3 x dx &= \int \sin^5 x (1 - \sin^2 x) d(\sin x) \\ &\stackrel{u=\sin x}{=} \int u^5 (1 - u^2) du = \frac{1}{6} \sin^6 x - \frac{1}{8} \sin^8 x + C. \end{aligned}$$

(b) $\int \sin^2 x \cos^2 x dx.$

Answer. (See page 99–101 of the supplementary notes.)

$$\int \sin^2 x \cos^2 x dx = \frac{1}{8} x - \frac{1}{32} \sin 4x + C.$$

(c) $\int \sec^3 x dx.$

Answer. (See page 102–104 of the supplementary notes.)

$$\int \sec^3 x dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C.$$

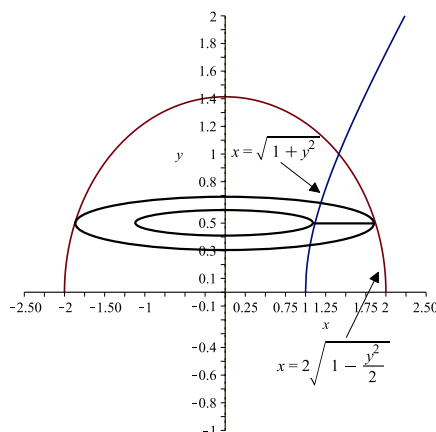
(d) $\int \tan^4 x dx.$

Answer. (See page 102–104 of the supplementary notes.)

$$\int \tan^4 x dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

2. Find the volume of the solid generated by revolving the region bounded by the upper branch of the ellipse $(x^2/4) + (y^2/2) = 1$, the right branch of the hyperbola $x^2 - y^2 = 1$, and the x -axis about the y -axis.

Solution. As shown in the following figure, the slice is chosen to be horizontal (and hence the integration has to be carried out in y), since this avoids partitioning the interval of integration into multiple subintervals. Since the slice is perpendicular to the axis of rotation, the revolution of the slice generates a washer, which



then suggests that the washer method has to be used. Now the inner and outer radii $r(y)$ and $R(y)$ of the washer, *expressed as functions of y* , are given by

$$\begin{aligned} r(y) &= +\sqrt{1+y^2} - 0 = \sqrt{1+y^2}, & (\text{right branch of hyperbola, so } x > 0), \\ R(y) &= +2\sqrt{1-\frac{1}{2}y^2} - 0 = 2\sqrt{1-\frac{1}{2}y^2}, & (\text{right half of the upper branch of ellipse, so } x > 0). \end{aligned}$$

The bounds of integration, on the other hand, are determined by equating $x = 2\sqrt{1-\frac{1}{2}y^2}$ to $x = \sqrt{1+y^2}$ and solving for y , which yields

$$2\sqrt{1-\frac{1}{2}y^2} = \sqrt{1+y^2} \implies 4\left(1-\frac{1}{2}y^2\right) = 1+y^2 \implies 3 = 3y^2 \implies y = -1 \text{ (discard)}, y = 1.$$

The volume of the solid is then given by

$$\begin{aligned} V &= \int_0^1 \pi [R^2(y) - r^2(y)] dy = \int_0^1 \pi \left\{ \left(2\sqrt{1-\frac{1}{2}y^2}\right)^2 - \left(\sqrt{1+y^2}\right)^2 \right\} dy = \pi \int_0^1 (3-3y^2) dy \\ &= \pi \cdot (3y - y^3) \Big|_0^1 = 2\pi. \end{aligned}$$

3. Solve the complex equation $z^4 - 7z^3 + 27z^2 - 47z + 26 = 0$, given that $(2 + 3i)$ is one of its roots.

Solution. Both $(2 + 3i)$ and $\overline{(2 + 3i)} = (2 - 3i)$ are roots of the real-coefficient polynomial $z^4 - 7z^3 + 27z^2 - 47z + 26$, which means

$$[z - (2 + 3i)][z - (2 - 3i)] = [(z - 2) - 3i][(z - 2) + 3i] = (z - 2)^2 - (3i)^2 = z^2 - 4z + 13$$

is a factor of it. A long division shows

$$\frac{z^4 - 7z^3 + 27z^2 - 47z + 26}{z^2 - 4z + 13} = z^2 - 3z + 2 = (z - 1)(z - 2).$$

Thus

$$z^4 - 7z^3 + 27z^2 - 47z + 26 = (z^2 - 4z + 13)(z - 1)(z - 2) = [z - (2 + 3i)][z - (2 - 3i)](z - 1)(z - 2),$$

and the four solutions of the equation are given by

$$z = 2 \pm 3i, \quad z = 1, \quad z = 2.$$

4. Consider the subset $S = \{(2, -1, 0, 5)^T, (1, -1, 2, 0)^T, (-1, 0, 1, 1)^T\}$ of $\mathbb{R}^4$.

(a) Determine whether S is linearly independent or dependent.

Solution. Denote the three vectors in S by v_1, v_2, v_3 . Then (see page 271–274 of the supplementary notes)

$$\begin{aligned} A = [v_1 | v_2 | v_3] &= \begin{pmatrix} 2 & 1 & -1 \\ -1 & -1 & 0 \\ 0 & 2 & 1 \\ 5 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & -1 & 0 \\ 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{R_2 + 2R_1 \\ R_4 + 5R_1}} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -5 & 1 \end{pmatrix} \\ &\xrightarrow{\substack{R_3 + 2R_2 \\ R_4 - 5R_2}} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{R_4 + 6R_3} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Since $\text{rank}(A) = 3$, i.e. the number of pivots (independent equations) is the same as the number of columns (unknowns), the homogeneous system $Ax = 0$ has only the trivial solution. Thus the columns of A , namely the vectors v_1, v_2, v_3 comprising the set S , are linearly independent.

(b) Find a linear combination of vectors in S that produces $(-2, 0, 3, -4)^T$ (an element of $\text{span}(S)$).

Solution. Let $b = (-2, 0, 3, -4)^T$. It amounts to finding $\alpha_1, \alpha_2, \alpha_3$ such that (see page 271 of the supplementary notes)

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = b, \quad \text{or} \quad [v_1 | v_2 | v_3] \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = b.$$

The same elimination procedure as employed in (a) applied to $[A|b] = [v_1 | v_2 | v_3 | b]$ yields

$$[A|b] = \left(\begin{array}{ccc|c} 2 & 1 & -1 & -2 \\ -1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 5 & 0 & 1 & -4 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Back substitution then gives

$$\alpha_3 = 1, \quad \alpha_2 = 1, \quad \alpha_1 = -1.$$

Therefore, the linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = -v_1 + v_2 + v_3$$

produces the vector $b = (-2, 0, 3, -4)^T$.

- (c) Is there a different linear combination of the elements of S that yields $(-2, 0, 3, -4)^T$? If so, find one. If not, why not?

Solution. No. If there is another linear combination

$$\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$$

that produces the vector $(-2, 0, 3, -4)^T$, then $A(\alpha - \beta) = 0$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ is the vector determined in (b) and $\beta = (\beta_1, \beta_2, \beta_3)^T$. But (a) implies that the homogeneous system $Ax = 0$ has only the trivial solution, thus $\alpha = \beta$, i.e. the only linear combination that produces $(-2, 0, 3, -4)^T$ is

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = -v_1 + v_2 + v_3.$$

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