

MA3001 Differential Equations
Semester B, 2018-19

Test 2

Time allowed: 60 minutes

Instructions to candidates:

This paper has TEN pages (including this cover page).

A table of Laplace transform is shown on page 2.

This paper has FIVE questions.

The questions will begin on page 3.

Write your answers in the space provided under each question.

Answer ALL questions. Show ALL steps in order to get full marks.

Question	Marks
1	13
2	13
3	20
4	18
5 (a)	9
5 (b)	19
Total mark:	92

NOT TO BE TAKEN AWAY

Laplace Transform Table

$f(t)$	$\mathcal{L}[f(t)] = F(s)$
1	$\frac{1}{s} \quad (s > 0)$
t	$\frac{1}{s^2} \quad (s > 0)$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}} \quad (s > 0)$
e^{at}	$\frac{1}{s-a} \quad (s > a)$
$\cosh at$	$\frac{s}{s^2 - a^2} \quad (s > a)$
$\sinh at$	$\frac{a}{s^2 - a^2} \quad (s > a)$
$\cos at$	$\frac{s}{s^2 + a^2} \quad (s > 0)$
$\sin at$	$\frac{a}{s^2 + a^2} \quad (s > 0)$
te^{at}	$\frac{1}{(s-a)^2} \quad (s > a)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$
$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$
$e^{at} f(t)$	$F(s-a)$
$f(t-a) u_a(t)$	$e^{-as} F(s)$
$t^n f(t)$	$(-1)^n \frac{d^n[F(s)]}{ds^n}, \quad n = 0, 1, 2, \dots$
$f(t)$ is periodic with period p	$\frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt$
$(f * g)(t) = \int_0^t f(t-u)g(u) du$	$F(s) G(s)$ where $F(s) = \mathcal{L}[f(t)]$, $G(s) = \mathcal{L}[g(t)]$

6.2-14

$$f * g = \int_0^t f(u)g(t-u)du$$

Question 1

Find the Laplace transform of the function

$$g(t) = \begin{cases} t^2 & \text{if } 0 \leq t < 3 \\ 4t & \text{if } t \geq 3 \end{cases} \quad [15 \text{ marks}]$$

$$g(t) = t^2 u_0(t) - t^2 u_3(t) + 4t u_3(t)$$

$$\text{Let } f(t-a) = t^2$$

$$\text{For } a=0 \quad f(t) = t^2 \quad \checkmark \quad \mathcal{L} t^2 = \frac{2}{s^3}$$

$$\text{For } a=3 \quad f(t) = (t+3)^2 \quad \checkmark \quad \mathcal{L} (t+3)^2 = t^2 + 6t + 9$$

$$h(t-a) = t \quad \checkmark \quad = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \quad \checkmark$$

$$\text{For } a=3 \quad h(t) = t+3 \quad \mathcal{L} (t+3) = \frac{1}{s^2} + \frac{3}{s}$$

$$> \mathcal{L} g(t) = \frac{2}{s^3} \cancel{t} e^{-3s} \left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \right) + \cancel{t} e^{-3s} \left(\frac{1}{s^2} + \frac{3}{s} \right)$$

$$\mathcal{L} g(t) = \frac{2}{s^3} + e^{-3s} \left(\frac{2}{s^3} + \frac{7}{s^2} + \frac{12}{s} \right)$$

X

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Question 2

Find the inverse Laplace transform of the function

$$\frac{17s^2 - 2s + 7}{(s^2 + 4)(3s - 1)}$$

[15 marks]

Partial Fraction

$$\frac{17s^2 - 2s + 7}{(s^2 + 4)(3s - 1)} = \frac{As + B}{s^2 + 4} + \frac{C}{3s - 1} \quad \checkmark$$

$$17s^2 - 2s + 7 = (3s - 1)(As + B) + (s^2 + 4)C$$

$$\text{Let } s = \frac{1}{3} \quad \frac{17}{9} - \frac{2}{3} + 7 = \frac{37}{9} C$$

$$C = 2 \quad \checkmark$$

$$\text{Let } s = 0 \quad 7 = A - B + 8 \Rightarrow A - B = -1$$

$$\text{Let } s = 1 \quad 22 = 2A + 2B + 10 \Rightarrow A + B = 6$$

$$\begin{cases} A = \left(\frac{5}{2}\right) \\ B = \left(\frac{7}{2}\right) \end{cases} \quad \times$$

$$\mathcal{L}^{-1} \frac{17s^2 - 2s + 7}{(s^2 + 4)(3s - 1)}$$

$$= \frac{1}{2} \mathcal{L}^{-1} \frac{5s + 7}{s^2 + 4} + \mathcal{L}^{-1} \frac{2}{3s - 1}$$

$$= \frac{5}{2} \mathcal{L}^{-1} \frac{s}{s^2 + 4} + \frac{7}{2} \frac{1}{2} \mathcal{L}^{-1} \frac{2}{s^2 + 4} + \frac{2}{3} \mathcal{L}^{-1} \frac{1}{s - \frac{1}{3}}$$

$$= \left(\frac{5}{2}\right) \cos(2t) + \left(\frac{7}{4}\right) \sin(2t) + \frac{2}{3} e^{\frac{1}{3}t}$$

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(Please turn over for Question 3)

Question 3

Solve the following initial value problem using Laplace transform:

$$\begin{cases} \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + y = t^2 e^{-t} \\ y(0) = 3, \quad y'(0) = 1 \end{cases}$$

[20 marks]

$$\mathcal{L} y''(t) = s^2 \mathcal{L} y(t) - s y(0) - y'(0) = s^2 \mathcal{L} y(t) - 3s - 1$$

$$\mathcal{L} y'(t) = s \mathcal{L} y(t) - y(0) = s \mathcal{L} y(t) - 3$$

$$\mathcal{L} y''(t) + 2 \mathcal{L} y'(t) + \mathcal{L} y = \mathcal{L} t^2 e^{-t}$$

$$(s^2 + 2s + 1) \mathcal{L} y - 3s - 1 - 6 = (-1)^2 \frac{d^2 \left(\frac{1}{s+1} \right)}{ds^2}$$

$$\boxed{\mathcal{L} e^{-t} \rightarrow \frac{1}{s+1}}$$

$$(s+1)^2 \mathcal{L} y = \frac{2}{(s+1)^3} + 3s + 7$$

$$\mathcal{L} y = \frac{2}{(s+1)^5} + \frac{3(s+1)}{(s+1)^2} + \frac{4}{(s+1)^2}$$

$$y = e^{-t} \frac{2}{4!} \mathcal{L}^{-1} \frac{4!}{s^5} + 3e^{-t} \mathcal{L}^{-1} \frac{1}{s} + 4e^{-t} \mathcal{L}^{-1} \frac{1}{s^2}$$

$$\begin{aligned} & \frac{d}{ds} (s+1)^{-1} \\ &= -(s+1)^{-2} \\ &= -\frac{d}{ds} (s+1)^{-2} \\ &= 2(s+1)^{-3} \end{aligned}$$

$$y = e^{-t} \left(\frac{1}{12} t^4 + 3 + 4t \right) \checkmark$$

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differential equations:

$$\begin{cases} \frac{dx}{dt} = 7x + y \\ \frac{dy}{dt} = -4x + 3y \end{cases}$$

[20 marks]

Let $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\frac{d\vec{u}}{dt} = A\vec{u}$

and $\vec{u} = e^{\lambda t} \vec{v}$

$$A - \lambda I = \begin{pmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (7-\lambda)(3-\lambda) + 4 = 0$$

$$21 - 3\lambda - 7\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - 10\lambda + 25 = 0$$

$$(\lambda - 5)^2 = 0$$

$\therefore$ The eigenvalue is 5 ✓

For $\lambda = 5$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$2s_1 + s_2 = 0$$

Let $s_1 = 1$, $s_2 = -2$

$\therefore \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is a solution. ✓

Try $\vec{v}_2 = \vec{w} + t\vec{v}_1$

$$(A - \lambda I)\vec{w} = -\vec{v}_1$$

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -4 & -2 & -2 \end{array} \right)$$

$$\sim \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right)$$

$$2w_1 + w_2 = 1$$

Let $w_2 = 3$, $w_1 = -1$ ✓

$\therefore \vec{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is a solution

$$\therefore \vec{u} = e^{5t} \left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} c_1 + \left[\begin{pmatrix} 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right] c_2 \right]$$

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Question 5

Let $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

(a) Given that $\lambda = -2, 3$ are the eigenvalues of matrix A , and $\vec{u}_1 = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution of the

homogeneous system $\frac{d\vec{u}}{dt} = A\vec{u}$. Find the general solution of the homogeneous system $\frac{d\vec{u}}{dt} = A\vec{u}$.

[10 marks]

$$a) \quad A - \lambda I = \begin{pmatrix} 2 - \lambda & -4 \\ -1 & -1 - \lambda \end{pmatrix}$$

For $\lambda = 3$

$$\left(\begin{array}{cc|c} -1 & -4 & 0 \\ -1 & 4 & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$s_1 + 4s_2 = 0$$

$$\text{let } s_2 = 1, s_1 = -4$$

$$\therefore \vec{v}_2 = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \text{ is a solution}$$

$\therefore$ General solution:

$$\vec{u} = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} c_1 + e^{3t} \begin{pmatrix} -4 \\ 1 \end{pmatrix} c_2$$

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(Please turn over for Question 5(b))

- (b) Using the method of variation of parameters and the result in part (a), find the general solution of the inhomogeneous system

$$\frac{d\vec{u}}{dt} = A\vec{u} + \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}.$$

[20 marks]

$$\vec{u}_h = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} c_1 + e^{3t} \begin{pmatrix} -4 \\ 1 \end{pmatrix} c_2$$

$$\vec{u}_p = h_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + h_2 e^{3t} \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\begin{cases} h_1' e^{-2t} - 4h_2' e^{3t} = e^{3t} & (1) \\ h_1' e^{-2t} + h_2' e^{3t} = 2e^{3t} & (2) \end{cases}$$

$$(2) - (1)$$

$$5h_2' e^{3t} = e^{3t}$$

$$h_2' = \frac{1}{5}$$

$$h_2 = \frac{1}{5}t$$

$$h_1' e^{-2t} + \frac{1}{5} e^{3t} = 2e^{3t}$$

$$h_1' e^{-2t} = \frac{9}{5} e^{3t}$$

$$h_1' = \frac{9}{5} e^{5t}$$

$$h_1 = \frac{9}{5} \int e^{5t} dt$$

$$h_1 = \frac{9}{25} e^{5t}$$

$$\therefore \vec{u}_p = \frac{9}{25} e^{5t} e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{5} t e^{3t} \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$= \frac{9}{25} e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{5} t e^{3t} \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

General solution:

$$\vec{u} = \vec{u}_h + \vec{u}_p$$

$$= e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} c_1 + e^{3t} \left[\begin{pmatrix} -4 \\ 1 \end{pmatrix} c_2 + \frac{9}{25} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{5} t \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right]$$

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