

EE3109 Applied Electromagnetics
2022/23 Semester B
Assignment 2 Solution

1. Given the time-domain definition of the Poynting vector is $\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)$, show that
- (a) The time-averaged power density of a general EM wave is given by $S_{av}(\mathbf{r}) = \frac{1}{2} \text{Re}\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*\}$.
- (b) The time-averaged power density of a plane wave travelling in x-direction in a lossless medium is

$$\mathbf{S}_{av}(\mathbf{r}) = \frac{|\mathbf{E}(\mathbf{r})|^2}{2\eta} \hat{\mathbf{x}}.$$

- (c) The time-averaged power density of a plane wave travelling in x-direction in a lossy medium is

$$\mathbf{S}_{av}(\mathbf{r}) = \frac{|\mathbf{E}(x=0)|^2}{2|\eta_c|} e^{-2\alpha x} \cos \theta_\eta \hat{\mathbf{x}}.$$

Solution:

- (a) The power density of a general EM-wave is

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)$$

The time-averaged power density is

$$\begin{aligned} S_{av} &= \frac{1}{T} \int_{t_0}^{t_0+T} \mathbf{S}(\mathbf{r}, t) dt = \frac{1}{T} \int_{t_0}^{t_0+T} \text{Re}\{\mathbf{E}(\mathbf{r})e^{j\omega t}\} \times \text{Re}\{\mathbf{H}(\mathbf{r})e^{j\omega t}\} dt \\ &= \frac{1}{T} \int_{t_0}^{t_0+T} \frac{1}{2} (\mathbf{E}(\mathbf{r})e^{j\omega t} + \mathbf{E}^*(\mathbf{r})e^{-j\omega t}) \times \frac{1}{2} (\mathbf{H}(\mathbf{r})e^{j\omega t} + \mathbf{H}^*(\mathbf{r})e^{-j\omega t}) dt \\ &= \frac{1}{4T} \int_{t_0}^{t_0+T} (\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r}))e^{j2\omega t} + \mathbf{E}^*(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) + \mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^* + (\mathbf{E}^*(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r}))e^{-j2\omega t} dt \end{aligned}$$

Within the integral, the first and last terms cancel out because they go through two periods of the periodic function $e^{j2\omega t}$. The middle two terms which remain and can be combined as follows.

$$\begin{aligned} S_{av} &= \frac{1}{4T} \int_{t_0}^{t_0+T} \mathbf{E}^*(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) + \mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^* dt = \frac{1}{2T} \int_{t_0}^{t_0+T} \frac{\mathbf{E}^*(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) + \mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*}{2} dt \\ &= \frac{1}{2T} \int_{t_0}^{t_0+T} \frac{\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^* + (\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*)^*}{2} dt = \frac{1}{2T} \int_{t_0}^{t_0+T} \text{Re}\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*\} dt \\ &= \frac{1}{2} \text{Re}\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*\} \left\{ \frac{1}{T} \int_{t_0}^{t_0+T} 1 dt \right\} = \frac{1}{2} \text{Re}\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*\} \end{aligned}$$

As required.

- (b) We will consider an electric field in the $\hat{\mathbf{u}}$ direction and the magnetic field in the $\hat{\mathbf{v}}$ direction. $\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{x}}$. Because we deal with the plane wave travelling in the +x-direction, the electric field can be written as

$$\mathbf{E}(\mathbf{r}) = E_0 e^{-jkx} \hat{\mathbf{u}}; \mathbf{H}(\mathbf{r}) = -\frac{1}{\eta} \mathbf{E}(\mathbf{r}) \times \hat{\mathbf{k}} = -\frac{E_0}{\eta} e^{-jkx} \hat{\mathbf{u}} \times \hat{\mathbf{x}} = \frac{E_0}{\eta} e^{-jkx} \hat{\mathbf{v}}$$

Substitution into the equation for Part (a), we have

$$S_{av} = \frac{1}{2} \text{Re}\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*\} = \frac{1}{2} \text{Re}\left\{E_0 e^{-jkx} \hat{\mathbf{u}} \times \frac{E_0^*}{\eta} e^{+jkx} \hat{\mathbf{v}}\right\} = \frac{1}{2} \text{Re}\left\{\frac{|E_0|^2}{\eta} \hat{\mathbf{u}} \times \hat{\mathbf{v}}\right\} = \frac{|E_0|^2}{2\eta} \hat{\mathbf{x}}$$

(c) We follow the same procedure above but modify the plane wave fields with the lossy versions:

$$\mathbf{E}(\mathbf{r}) = E_0 e^{-\alpha x} e^{-jkx} \hat{\mathbf{u}}; H(r) = -\frac{1}{\eta_c} \mathbf{E}(\mathbf{r}) \times \hat{\mathbf{k}} = -\frac{E_0}{\eta_c} e^{-\alpha x} e^{-jkx} \hat{\mathbf{u}} \times \hat{\mathbf{x}} = \frac{E_0}{\eta_c} e^{-\alpha x} e^{-jkx} \hat{\mathbf{v}}$$

Substitution into the equation for Part (a), we have

$$\begin{aligned} S_{av} &= \frac{1}{2} \text{Re}\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*\} = \frac{1}{2} \text{Re}\left\{E_0 e^{-\alpha x} e^{-jkx} \hat{\mathbf{u}} \times \frac{E_0^*}{\eta_c} e^{-\alpha x} e^{+jkx} \hat{\mathbf{v}}\right\} = \frac{1}{2} \text{Re}\left\{\frac{|E_0|^2}{|\eta_c| e^{j\theta_\eta}} e^{-2\alpha x} \hat{\mathbf{u}} \times \hat{\mathbf{v}}\right\} \\ &= \frac{|E_0|^2}{2|\eta_c|} e^{-2\alpha x} \text{Re}\{e^{-j\theta_\eta}\} \hat{\mathbf{x}} = \frac{|E_0|^2}{2|\eta_c|} e^{-2\alpha x} \cos \theta_\eta \hat{\mathbf{x}} \end{aligned}$$

$$|\mathbf{E}(x=0)| = |E_0 e^{-\alpha(0)} e^{-jk(0)} \hat{\mathbf{u}}| = |E_0|, \text{ so } S_{av} = \frac{|\mathbf{E}(x=0)|^2}{2|\eta_c|} e^{-2\alpha x} \cos \theta_\eta \hat{\mathbf{x}} \text{ as required.}$$

2. The magnetic field of a linearly-polarized incident plane wave is

$$\mathbf{H}^i = \exp(jk_0(mx + pz)) \hat{\mathbf{y}} \text{ [A/m].}$$

The wave travels in the $\hat{\mathbf{k}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{z}})$ direction. The plane wave hits a material interface at $z = 0$, where the region $z < 0$ is filled with air, and the region $z > 0$ is filled with a material with $\epsilon_r = 9$.

- What are the values of m and p ?
- Write the wavevectors of the incident wave ($\mathbf{k}^i$) and reflected wave ($\mathbf{k}^r$).
- What is the angle of the transmitted wave in this case?
- What is the transmission coefficient in this case?
- Write the transmitted electric field in phasor form.

Solution:

2. We begin by drawing the diagram on the right.

- The incident wave is in the $\hat{\mathbf{x}} + \hat{\mathbf{z}}$ direction and the medium is air, so $\mathbf{k}^i = k_0 \left(\frac{1}{\sqrt{2}} \hat{\mathbf{x}} + \frac{1}{\sqrt{2}} \hat{\mathbf{z}} \right)$.

From the exponential term on $\mathbf{H}^i$ (given), we see:

$$jk_0(mx + pz) = -j\mathbf{k}^i \cdot \mathbf{r} \Rightarrow \mathbf{k}^i = -k_0 m \hat{\mathbf{x}} - k_0 p \hat{\mathbf{z}}$$

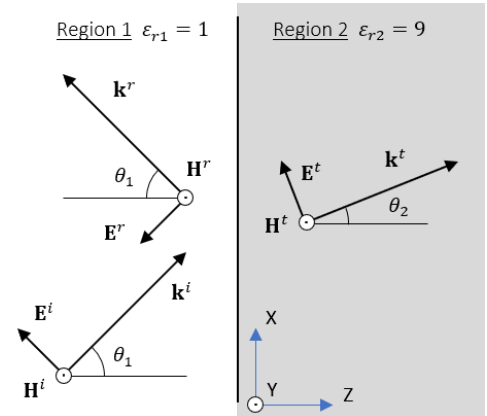
Equating the x- and z- components of the two gives

$$m = p = -\frac{1}{\sqrt{2}}.$$

- $\mathbf{k}^i = k_0 \left(\frac{1}{\sqrt{2}} \hat{\mathbf{x}} + \frac{1}{\sqrt{2}} \hat{\mathbf{z}} \right)$ as shown above.
 $\mathbf{k}^r = k_0 \left(\frac{1}{\sqrt{2}} \hat{\mathbf{x}} - \frac{1}{\sqrt{2}} \hat{\mathbf{z}} \right)$, with the x-component reversed.

- $n_1 = 1; n_2 = \sqrt{\epsilon_r} = 3$. Using Snell's Law:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \Rightarrow \theta_2 = \sin^{-1} \left(\frac{n_1 \sin \theta_1}{n_2} \right) = \sin^{-1} \left(\frac{1}{3\sqrt{2}} \right) = 13.63^\circ$$



(d) $\theta_1 = 45^\circ$ and $\mathbf{H}^i$ is normal to the plane of incidence, so the incident wave is TM-polarized.

$$\eta_1 = 120\pi; \quad \eta_2 = \frac{120\pi}{\sqrt{\epsilon_r}} = 40\pi; \quad \cos \theta_1 = \frac{1}{\sqrt{2}} = 0.7071; \quad \cos \theta_2 = 0.9718$$

$$\tau_{TM} = \frac{2\eta_2 \cos \theta_1}{\eta_2 \cos \theta_2 + \eta_1 \cos \theta_1} = \frac{2(0.7071)}{1(0.9718) + 3(0.7071)} = 0.4572$$

(e) The incident electric field is:

$$\begin{aligned} \mathbf{E}^i &= \eta_1 \mathbf{H}^i \times \hat{\mathbf{k}} = 120\pi \exp\left(-j \frac{k_0}{\sqrt{2}}(x+z)\right) \hat{\mathbf{y}} \times \left(\frac{1}{\sqrt{2}}\hat{\mathbf{x}} + \frac{1}{\sqrt{2}}\hat{\mathbf{z}}\right) \\ &= 120\pi \exp\left(-j \frac{k_0}{\sqrt{2}}(x+z)\right) \cdot \left(\frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - \hat{\mathbf{z}})\right) \end{aligned}$$

From the figure, we can find the x and z components of k^t as:

$$k_x^t = k^t \sin \theta_2 \quad \text{and} \quad k_z^t = k^t \cos \theta_2$$

Also, $k^t = k_2 = k^i \sqrt{\epsilon_{r2}} = 3k_0$

The transmitted electric field is then

$$\begin{aligned} \mathbf{E}^t &= \tau_{TM} 120\pi \exp(-jk^t(\sin \theta_2 x + \cos \theta_2 z)) \cdot (\cos \theta_2 \hat{\mathbf{x}} - \sin \theta_2 \hat{\mathbf{z}}) \\ &= 172.4 \exp(-jk_0(0.7070 x + 2.9154 z)) \cdot (0.9718\hat{\mathbf{x}} - 0.2357\hat{\mathbf{z}}) \end{aligned}$$

3. A 2.4 GHz plane wave with a power density of 1 [W/m²] travels in air the +z-direction and is normally incident onto a copper surface at $z = 0$. For this frequency, assume that copper has electromagnetic parameters $\epsilon_r = \mu_r = 1, \sigma = 5.8 \times 10^7$ [S/m]. The incident electric field is expressed as $\mathbf{E} = E_0 e^{-jk_0 z} (\hat{\mathbf{x}} + j\hat{\mathbf{y}})$, where E_0 is real and positive.

- What are k_0 and E_0 ?
- What are the reflection and transmission coefficients in this case?
- What are the wave vectors of the incident, reflected and transmitted waves?
- Write the magnetic fields of the incident, reflected, and transmitted waves.
- Determine the polarization of the incident, reflected and transmitted waves.
- At what distance into the copper does the power density become 1[nW/m²]?

Solution:

First we write the material parameters of copper at $f = 2.4$ GHz, which is a good conductor (since $\sigma \gg \omega\epsilon$) :

$$\begin{aligned} \eta_c &= (1+j) \sqrt{\frac{\pi f \mu}{\sigma}} = (1+j) \sqrt{\frac{\pi(2.4 \times 10^9)(4\pi \times 10^{-7})}{(5.8 \times 10^7)}} = 12.78(1+j) \text{ [m}\Omega\text{]}; \\ \alpha_c &= \beta_c = \sqrt{\pi f \mu \sigma} = \sqrt{\pi(2.4 \times 10^9)(4\pi \times 10^{-7})(5.8 \times 10^7)} = 7.413 \times 10^5 \text{ [m}^{-1}\text{]}. \end{aligned}$$

For air:

$$\eta_{air} = \eta_0 = 377 \text{ [}\Omega\text{]}; \quad \alpha_{air} = 0; \quad \beta_{air} = k_0 = \frac{\omega}{c} = \frac{2\pi \times 2.4 \times 10^9}{3 \times 10^8} = 16\pi = 50.27 \text{ [rad/m]}$$

- $k_0 = 16\pi$ [rad/m] as found above.

$$S_0 = \frac{|E_0|^2}{\eta_0} \Rightarrow E_0 = |E_0| = \sqrt{\eta_0 S_0} = \sqrt{(377)(1)} = 19.42 \text{ [V/m]}.$$

(b)

$$\Gamma = \frac{\eta_c - \eta_{air}}{\eta_c + \eta_{air}} = 0.9999322 \angle 179.9961^\circ; \quad \tau = 1 + \Gamma = 9.588 \times 10^{-5} \angle 45.00^\circ.$$

(Many digits were kept for Γ to show that the value is very close to, but slightly deviated from, $\Gamma = -1$, which would be the reflection coefficient of the interface with a perfect conductor.)

- (c) Direction: the question says that the wave is **travelling in the +z-direction** and **normally incident** on interface at $z = 0$. It directly follows that the reflected wave is in the -z-direction and the transmitted wave is in the +z direction.

Magnitude: As this is the **normal incidence** case, the magnitude of the wavevector is the entire wave number. The incident and reflected waves are in Medium 1 (air), so their magnitudes are β_{air} ; the transmitted wave is in Medium 2 (copper) so its magnitude is β_c .

Putting things together, we have

$$\mathbf{k}^i = \beta_{air} \hat{\mathbf{z}}; \quad \mathbf{k}^r = -\beta_{air} \hat{\mathbf{z}}; \quad \mathbf{k}^t = \beta_c \hat{\mathbf{z}}.$$

(Substitute in the betas above to get the numerical answers.)

(d)

$$\begin{aligned} \mathbf{H}^i &= \frac{1}{\eta_0} \hat{\mathbf{k}}^i \times \hat{\mathbf{E}}^i = \frac{E_0}{\eta_0} e^{-jk_0 z} \hat{\mathbf{z}} \times (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) = \left(\frac{19.42}{377}\right) e^{-j16\pi z} (\hat{\mathbf{y}} - j\hat{\mathbf{x}}) \\ &= 51.51 e^{-\frac{j\pi}{2}} e^{-j16\pi z} (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) \text{ [mA/m]}; \end{aligned}$$

$$\mathbf{H}^r = -\frac{\Gamma \mathbf{E}^i}{\eta_0} e^{-\frac{j\pi}{2}} e^{j16\pi z} (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) = 51.50 e^{-\frac{j\pi}{2}} e^{j16\pi z} (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) \text{ [mA/m]};$$

$$\mathbf{H}^t = \frac{\tau \mathbf{E}^i}{\eta_c} e^{-\frac{j\pi}{2}} e^{-\alpha_c z} e^{-j\beta_c z} (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) = 103.0 e^{-\frac{j\pi}{2}} e^{-(7.413 \times 10^5)(1+j)z} (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) \text{ [mA/m]}.$$

- (e) The x- and y- components of the electric fields are of the same amplitude, and have phase differences of $\pm\pi/2$, so all three components are circularly polarized plane waves. The reflected wave is in the -z-direction, and $\Delta\phi = \angle E_y^r - \angle E_x^r = \frac{\pi}{2}$, so it is RHC polarized. For the incident and transmitted waves, the $\Delta\phi$ remains the same but the travel direction is opposite, so they are LHC polarized.

- (f) The corresponding magnetic field for this power density is

$$S(z_d) = \frac{1}{2} |\eta_c| |H(z_d)|^2 \Rightarrow |H(z_d)| = \sqrt{\frac{2S(z_d)}{|\eta_c|}} = \sqrt{\frac{2(1 \times 10^{-9})}{|0.01278(1+j)|}} = 332.7 \text{ [}\mu\text{A/m]}.$$

The magnetic field magnitude just into the copper is:

$$H(z = 0^+) = |\mathbf{H}^t(z = 0)| = 332.7 \text{ [}\mu\text{A/m]}.$$

The distance after which the magnetic field decays to this level is:

$$\begin{aligned} H(z_d) &= H(z = 0^+) e^{-\alpha_c z_d} \Rightarrow z_d = -\frac{1}{\alpha_c} \ln\left(\frac{H(z_d)}{H(z = 0^+)}\right) = \frac{1}{7.413 \times 10^5} \ln\left(\frac{0.1030}{332.7 \times 10^{-6}}\right) \\ &= 7.737 \text{ [}\mu\text{m]} . \end{aligned}$$

Remark: The amplitude of the H-field will change from (i) the presence of the interface and (ii) the attenuation loss. Remember to account for both effects!

4. Consider a plane wave travelling in the metamaterial with $\mu_r = \epsilon_r = -1$. For simplicity, we define the propagation direction to be along the z-axis, and the electric field to be along the x-axis. The electric field is written below.

$$\mathbf{E}(z, t) = E_{x0} \cos(\omega t - kz + \phi) \hat{\mathbf{x}}$$

- Find the corresponding magnetic field from Maxwell's equations.
- Using Maxwell's equations, show that in this material $\hat{\mathbf{E}} \times \hat{\mathbf{H}} = -\hat{\mathbf{k}}$. (It is due to this relationship that this material is called a "left-handed" metamaterial.)
- Find the refractive index of the plane wave – that is, the phase velocity of an EM wave in free space divided by the phase velocity of the EM wave in the material.
- Find the wavevector of the plane wave.
- Find the Poynting vector (phasor form) of the plane wave.
- If a plane wave in air is incident upon a slab of the material, at 30° from normal, how much of the wave is transmitted into the material, and what will the transmitted angle be? Draw a diagram showing the plane of incidence, the incident wave, the transmitted wave and the angles the two waves make with the surface normal.

Solution:

- (a) The electric field phasor is $\mathbf{E} = E_{x0}^c \exp(-jkz) \hat{\mathbf{x}}$ where $E_{x0}^c = E_{x0} e^{j\phi}$

$$\begin{aligned} \mathbf{H} &= -\frac{1}{j\omega\mu_0\mu_r} \nabla \times \mathbf{E} \\ &= \frac{1}{j\omega\mu_0} \nabla \times (E_{x0}^c \exp(-jkz) \hat{\mathbf{x}}) \\ &= \frac{E_{x0}^c}{j\omega\mu_0} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (\exp(-jkz), 0, 0) \\ &= \frac{E_{x0}^c}{j\omega\mu_0} (0, -jk \exp(-jkz), 0) \\ &= -\frac{E_{x0}^c k}{\omega\mu_0} \exp(-jkz) \hat{\mathbf{y}} \end{aligned}$$

- (b) The corresponding electric field $\mathbf{E}'$ to $\mathbf{H}$ is:

$$\begin{aligned} \mathbf{E}' &= \frac{1}{j\omega\epsilon_0\epsilon_r} \nabla \times \mathbf{H} \\ &= -\frac{1}{j\omega\epsilon_0} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times \left(0, -\frac{E_{x0}^c k}{\omega\mu_0} \exp(-jkz), 0 \right) \\ &= -\frac{1}{j\omega\epsilon_0} \left(jk \left(-\frac{E_{x0}^c k}{\omega\mu_0} \exp(-jkz) \right), 0, 0 \right) \\ &= -\frac{jk}{j\omega\epsilon_0} \left(-\frac{E_{x0}^c k}{\omega\mu_0} \right) \exp(-jkz) \hat{\mathbf{x}} \\ &= \frac{k^2}{\omega^2\mu_0\epsilon_0} E_{x0}^c \exp(-jkz) \hat{\mathbf{x}} \\ &= \frac{k^2}{\omega^2\mu_0\epsilon_0} \mathbf{E} \end{aligned}$$

Since $\mathbf{E}' = \mathbf{E}$, we have $k^2 = \omega^2\mu_0\epsilon_0$ or $k = \pm\omega\sqrt{\mu_0\epsilon_0}$. Choosing the positive root will lead to $k = k_0$, which means the $\mathbf{k}$ is in the +z-direction and $\mathbf{H}$ is in the -y-direction (see the exponent and H-field in the solution of (a)). On the other hand, choosing the negative root means $k = -k_0$, which means $\mathbf{k}$ is in the negative z-direction and $\mathbf{H}$ is in the +y-direction. In both cases, $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{k}$ form a left-hand triplet, i.e. $\hat{\mathbf{E}} \times \hat{\mathbf{H}} = -\hat{\mathbf{k}}$.

- (c) We choose the negative root ($k = -k_0$) in (b) to continue the explanation. This leads to an +x-directed $\mathbf{E}$, +y-directed $\mathbf{H}$ and -z-directed $\hat{\mathbf{k}}$ in the metamaterial, whereas similarly directed $\mathbf{E}$ and $\mathbf{H}$ fields will lead to +z-directed $\hat{\mathbf{k}}$ in free-space. Comparing the velocity of both waves, we have

$$v_{p,LHM} = \frac{\omega}{k} = \frac{\omega}{-k_0} = -\frac{\omega}{k_0} = -v_{p,free-space} = -c$$

So the refractive index is (LHM stands for left-handed metamaterial)

$$n = \frac{v_{p,free-space}}{v_{p,LHM}} = -1$$

- (d) As shown above, $|k_{LHM}| = |k_0|$, but $\hat{\mathbf{k}}_{LHM} = -\hat{\mathbf{k}}_0$, so $\mathbf{k}_{LHM} = -\mathbf{k}_0$.

(e)

$$\begin{aligned} \mathbf{S} &= \frac{1}{2} \mathbf{E} \times \mathbf{H}^* = \frac{1}{2} (E_{x0}^c \exp(-jkz) \hat{\mathbf{x}}) \times \left(-\frac{E_{x0}^c k}{\omega \mu_0} \exp(-jkz) \hat{\mathbf{y}} \right)^* \\ &= -\frac{1}{2} \frac{|E_{x0}^c|^2 k}{\omega \mu_0} (\hat{\mathbf{x}} \times \hat{\mathbf{y}}) \\ &= \frac{1}{2} \frac{E_{x0}^2 k_0}{\omega \mu_0} \hat{\mathbf{z}} \\ &= \frac{1}{2} \frac{E_{x0}^2}{\eta_0} \hat{\mathbf{z}} \end{aligned}$$

- (f) Note that the intrinsic impedance of the metamaterial is $\eta_{LHM} = \sqrt{\mu_0 \mu_r / \epsilon_0 \epsilon_r} = \sqrt{\mu_0 (-1) / \epsilon_0 (-1)} = \eta_0$. The transmission angle can be calculated as

$$n_0 \sin \theta_i = n_{LHM} \sin \theta_t \Rightarrow \theta_t = \sin^{-1} \left(\left(\frac{n_0}{n_{LHM}} \right) \sin \theta_i \right) = \sin^{-1}(-\sin \theta_i) = -\theta_i$$

Which means $\cos \theta_t = \cos \theta_i$.

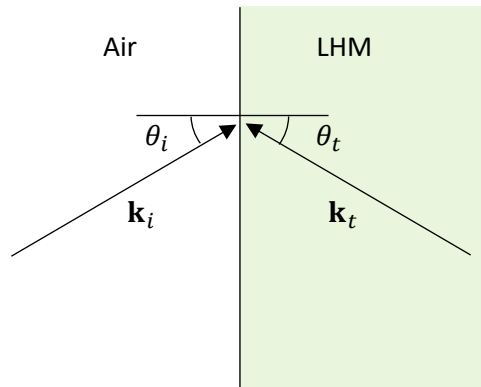
When a wave is incident from free-space in to the metamaterial,

$$\text{TE Polarization: } \tau = \frac{2\eta_{LHM} \cos \theta_i}{\eta_{LHM} \cos \theta_t + \eta_0 \cos \theta_i} = \frac{2}{1+1} = 1$$

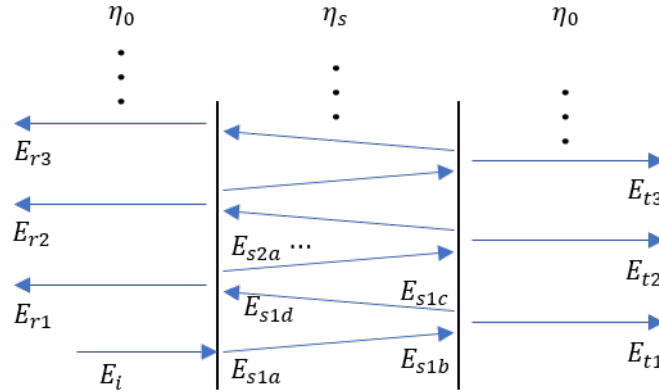
$$\text{TM Polarization: } \tau = \frac{2\eta_{LHM} \cos \theta_i}{\eta_{LHM} \cos \theta_i + \eta_0 \cos \theta_t} = \frac{2}{1+1} = 1$$

So all of the wave will be transmitted into the material.

A diagram showing the incident and transmitted waves are shown below. (Only the wavevectors are drawn. The directions of $\mathbf{E}$ and $\mathbf{H}$ depend on the polarization of the plane wave.)



5. **(BONUS: 2 points)** Consider the reflection and transmission coefficient of a slab of width d , of a material with intrinsic impedance η_s and wave number k_s . At steady state, the reflected and transmitted wave can be viewed as a summation of reflection and transmission components after multiple reflections on the internal surfaces of the slab.



In the diagram above, the electric field at different points of its first round trip is denoted as E_{s1a} , E_{s1b} , E_{s1c} and E_{s1d} . The start of the second round trip is denoted as E_{s2a} , from which point onwards the process repeats, and finding using elementary relationships one can relate one electric field value to the next.

- We will define r and t as the reflection and transmission coefficients of a wave travelling from air to the slab, and r' and t' as the reflection and transmission coefficients of a wave travelling from the slab material into air. Write an expression for all four coefficients as a function of η and η_0 .
- We will define “propagation” coefficients p and p' . p corresponds to the multiplicative change in E-field when a positively traveling plane wave propagates for a distance of d within the slab. p' corresponds to the multiplicative change in E-field when a negatively traveling plane wave propagates for a distance of d (ie. displacement of $-d$) within the slab. Write expressions for p and p' as functions of k_s and d .
- Using the above coefficients $\{r, r', t, t', p, p'\}$, write $\{E_{s1a}, E_{s1b}, E_{s1c}, E_{s1d}, E_{s2a}\}$ in terms of E_i . (For example, $E_{s1b} = pE_{s1a} = ptE_i$)
- Write $\{E_{r1}, E_{r2}, E_{r3}\}$ and $\{E_{t1}, E_{t2}, E_{t3}\}$ in terms of E_i .
- As stated in the question, the total reflected electric field is the sum of all reflected electric fields (ie. $E_r = \sum_{n=1}^{\infty} E_{rn}$), and the total transmitted electric field is the sum of all transmitted electric fields. Based on this, find the reflection coefficient $R = E_r/E_i$ and the transmission coefficient $T = E_t/E_i$ for EM wave reflection from and transmission through the slab. Leave your answer in terms of r and t : no need to substitute in the intrinsic impedances.

Solution:

- Applying the formula from the lecture notes, we have:

$$r = \frac{\eta_s - \eta_0}{\eta_s + \eta_0}; \quad r' = \frac{\eta_0 - \eta_s}{\eta_0 + \eta_s} = -r; \quad t = 1 + r = \frac{2\eta_s}{\eta_s + \eta_0}; \quad t' = 1 + r' = \frac{2\eta_0}{\eta_s + \eta_0} = \left(\frac{\eta_0}{\eta_s}\right)t$$

- We label the entrance and exit facets of the slab with locations $z_a = -d$ and $z_b = 0$ respectively. The plane wave with electric field E_{s1a} travels in the positive z -direction from the entrance to the exit, to become E_{s1b} . So, we write both as a positively travelling wave as follows:

$$E_{s1a} = E_1^+ \exp(-jk_s(-d)) = E_1^+ \exp(jk_s d); \quad E_{s1b} = E_1^+ \exp(-jk_s(0))$$

$$p = \frac{E_{s1b}}{E_{s1a}} = \exp(-jk_s d)$$

Similarly we can write p' as the multiplicative change between E_{s1c} and E_{s1d} :

$$E_{s1c} = E_1^- \exp(jk_s(0)) = E_1^-; \quad E_{s1d} = E_1^- \exp(jk_s(-d)) = E_1^- \exp(-jk_s d).$$

$$p' = \frac{E_{s1d}}{E_{s1c}} = \exp(-jk_s d) = p.$$

- Following the illustration and using reflection, transmission and propagation coefficients, we have:

$$E_{s1a} = tE_i; \quad E_{s1b} = pE_{s1a} = tpE'_i \quad E_{s1c} = r'E_{s1b} = tpr'E_i = -tprE_i; \quad E_{s1d} = p'E_{s1c} \\ = -tp^2rE_i; \quad E_{s2a} = r'E_{s1d} = tp^2r^2E_i$$

We note that $E_{s2a} = (pr)^2 E_{s1a}$. In general, if the wave travels one round-trip within the slab it will accrue a multiplicative coefficient of $(pr)^2$, due to two propagations of distance d and two internal reflections within the slab.

(d) Following a similar procedure as above and using the “round-trip” relationship, we have:

$$E_{r1} = t'E_{s1d} = -p^2rt^2 \left(\frac{\eta_0}{\eta_s} \right) E_i; \\ E_{r2} = (pr)^2 E_{r1} = -p^4r^3t^2 \left(\frac{\eta_0}{\eta_s} \right) E_i; \\ E_{r3} = (pr)^4 E_{r1} = -p^6r^5t^2 \left(\frac{\eta_0}{\eta_s} \right) E_i.$$

$$E_{t1} = t'E_{s1b} = pt^2 \left(\frac{\eta_0}{\eta_s} \right) E_i; \\ E_{t2} = (pr)^2 E_{t1} = p^3r^2t^2 \left(\frac{\eta_0}{\eta_s} \right) E_i; \\ E_{t3} = (pr)^4 E_{t1} = p^5r^4t^2 \left(\frac{\eta_0}{\eta_s} \right) E_i.$$

(e) We actually forgot to draw in $E_{r0} = rE_i$ in the diagram. Including this we have:

$$E_r = \sum_{\{n=0\}}^{\infty} E_{rn} = rE_i + \sum_{m=0}^{\infty} (pr)^{2m} E_{r1} = \left[r + \frac{-p^2rt^2 \left(\frac{\eta_0}{\eta_s} \right)}{1 - (pr)^2} \right] E_i = \left[1 - \frac{p^2t^2}{1 - (pr)^2} \left(\frac{\eta_0}{\eta_s} \right) \right] rE_i \\ E_t = \sum_{\{n=1\}}^{\infty} E_{tn} = \sum_{m=0}^{\infty} (pr)^{2m} E_{t1} = \left[\frac{pt^2 \left(\frac{\eta_0}{\eta_s} \right)}{1 - (pr)^2} \right] E_i = \left[\frac{pt^2}{1 - (pr)^2} \left(\frac{\eta_0}{\eta_s} \right) \right] E_i$$

The terms in the square brackets are the reflection and transmission coefficients through the slab, i.e.:

$$R = 1 - \frac{\eta_0 p^2 t^2}{\eta_s (1 - (pr)^2)}; \quad T = \frac{\eta_0 p t^2}{\eta_s (1 - (pr)^2)}.$$

Note: We looked at the case for normal incidence here. If we deal with oblique incidence, the same equations apply as long as we substitute the oblique incidence formulae for $\{r, r', t, t', p, p'\}$, for the corresponding polarization.