
EE 3121
Differential Equations for Electrical Engineering

Lecture 9: Partial Differential Equations



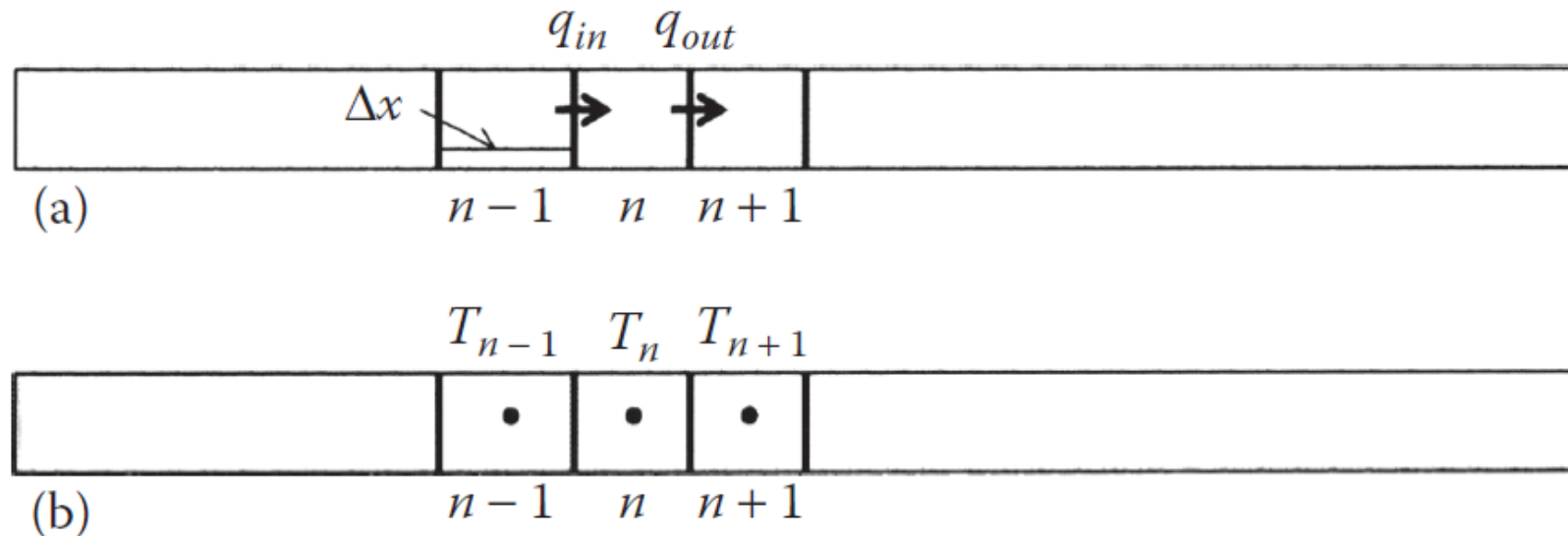
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Framework of Lectures

- Our march toward higher and higher dimensions reaches an end: systems modeled by partial differential equations (PDEs) can be considered infinite dimensional.
- Introducing different forms that PDEs and their solutions can take depending on the geometry of the problem.

9.1 Heat Equation

Figure 9.1 depicts a long rod of uniform cross section of area A through which heat is flowing. Assume that the rod is insulated against heat loss to the environment along its length.



- Figure 9.1 Heat transfer in a long rod: (a) heat flowing in rod and (b) temperature variation around segment n .

9.1 Heat Equation

- We divide the rod into segments of length Δx . Consider the interaction of segments $n - 1$, n , and $n + 1$.
- Energy is conserved, so in time Δt the energy flowing into segment n from segment $n - 1$ equals the energy flowing out of segment n into segment $n + 1$ plus an addition to the energy stored in segment n :

$$q_{\text{in}}\Delta t = q_{\text{out}}\Delta t + \Delta e_{\text{stored}}.$$

Rearranging:

$$q_{\text{in}} - q_{\text{out}} = \frac{\Delta e_{\text{stored}}}{\Delta t}. \quad (1)$$

9.1 Heat Equation

Fouriers law of heat conduction is

$$q = kA \frac{\Delta T}{\Delta x}, \quad (2)$$

where:

T is the temperature

k is the thermal conductivity, a property of the rod material Here

$$q_{\text{in}} = kA \frac{T_{n-1} - T_n}{\Delta x} \quad (3)$$

$$q_{\text{out}} = kA \frac{T_n - T_{n+1}}{\Delta x}. \quad (4)$$

9.1 Heat Equation

The change in energy stored in segment n results in a temperature change there:

$$\Delta e_{\text{stored}} = \rho c A \Delta x \Delta T_n, \quad (5)$$

where:

ρ is the density of the rod material

c is its specific heat Combining Equations (1) through (5) and rearranging we have

$$\frac{T_{n+1} - 2T_n + T_{n-1}}{\Delta x^2} = \left(\frac{\rho c}{k} \right) \frac{\Delta T_n}{\Delta t}. \quad (6)$$

9.1 Heat Equation

In the limit as Δx and Δt go to zero, Equation (6) becomes

$$\frac{\partial^2 T}{\partial x^2} = \left(\frac{\rho c}{k} \right) \frac{\partial T}{\partial t}. \quad (7)$$

Equation (7) is heat equation, one of the fundamental equations of the subject of heat transfer. It is usually written as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \quad (8)$$

where α is called the *thermal diffusivity*.

9.1 Heat Equation

- The heat equation is classified as a *parabolic* PDE, which for our purposes distinguishes it from hyperbolic PDEs and elliptic PDEs (we will meet later).
- For now it suffices to say that solutions to parabolic, hyperbolic, and elliptic equations have very different properties.

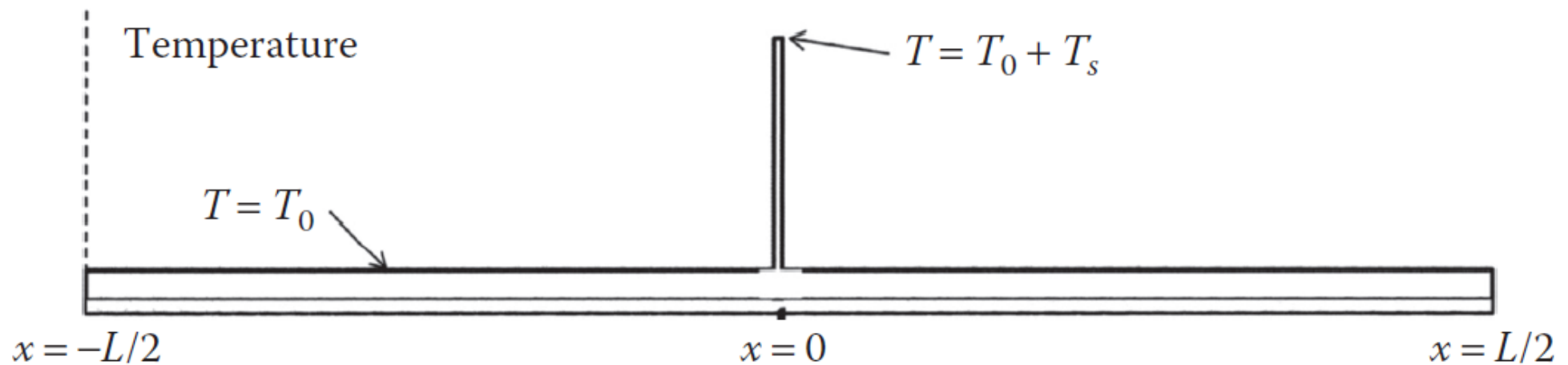
9.1.1 Heat Equation in Initial Value Problems

- An initial value problem is a differential equation combined only with initial conditions;
- In the case of the heat equation, a specification of the spatial distribution of the temperature at $t = 0$.

9.1.1.1 Heat Equation in An Example Initial Value Problems

- Consider a very long rod initially at a uniform temperature T_0 , to the center of which is applied a large amount of heat in a very short time, so that the temperature spikes in a very small region Δx around the center and is essentially unchanged everywhere else.
- Assume that the rod is insulated against heat loss to the environment along its length.
- Figure 9.2 graphs the temperature distribution across the bar at $t = 0^+$.

9.1.1.1 Heat Equation in An Example Initial Value Problems



- **Figure 9.2** Delta function initial temperature distribution across a very long rod.

9.1.1.1 Heat Equation in An Example Initial Value Problems

For simplicity let $T_0 = 0$, and the problem can be stated using the delta function:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad T(x, 0) = T_s \Delta x \delta(x) \quad (9)$$

- Until the temperature at the ends of the rod begins to rise appreciably, this is an initial value problem.
- For a period of time the dynamics depend only on initial conditions, not on what is happening at the boundaries.
- For all intents and purposes initially, the rod could be infinite in length.

9.1.1.1 Heat Equation in An Example Initial Value Problems

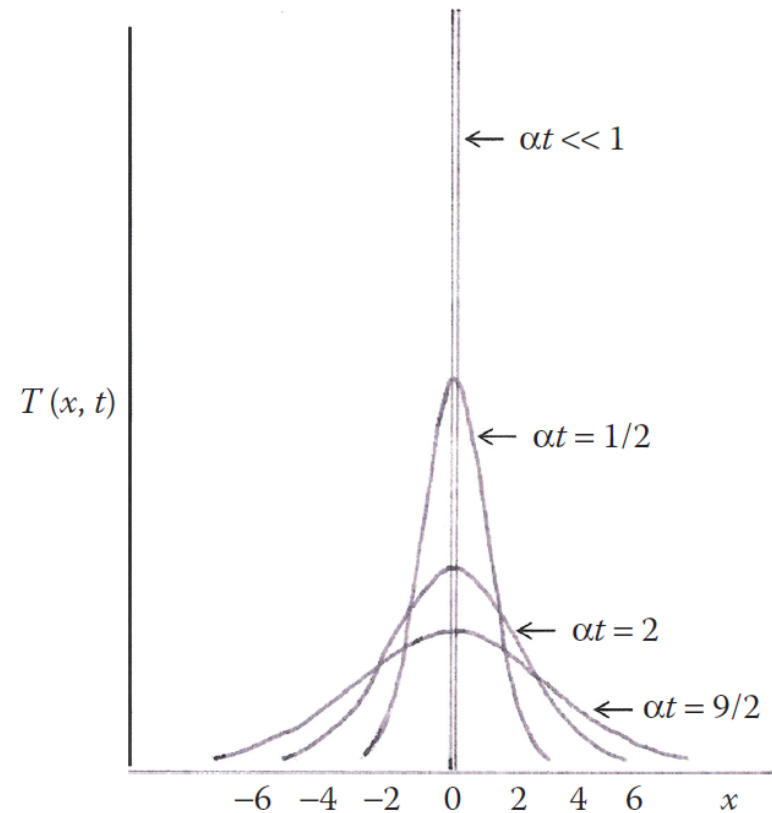
It is easy to verify by direct substitution that the solution to Equation (9) is

$$T(x, t) = \frac{1}{2\sqrt{\pi\alpha t}} \exp(-x^2/4\alpha t) (T_s \Delta x), \quad (10)$$

where $T_s \Delta x$ is just the coefficient of the delta function in the initial condition.

- We examine this problem because, as we will show, it illustrates that some initial value problems in PDEs are strongly analogous to initial value problems in ODEs.
- **Figure 9.3** graphs $T(x, t)$ as it evolves over time.

9.1.1.1 Heat Equation in An Example Initial Value Problems



- **Figure 9.3** Solution to heat equation with delta function initial condition.

9.1.1.1 Heat Equation in An Example Initial Value Problems

For an infinite length rod and an arbitrary initial temperature distribution $T(x, 0) = T_0(x)$ the solution to the heat equation is

$$T(x, t) = \int_{-\infty}^{\infty} \Gamma(x - x', t) T_0(x') dx',$$

where

$$\Gamma(x - x', t) = \frac{1}{2\sqrt{\pi\alpha t}} \exp\left(-\frac{(x - x')^2}{4\alpha t}\right). \quad (11)$$

9.1.1.1 Heat Equation in An Example Initial Value Problems

Compare Equations (10) and (11). If there are heat sources (let $q_s(x, t)$ be the energy added per unit volume and unit time), then the heat equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{q_s(x, t)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t},$$

and the solution is

$$T(x, t) = \int_{-\infty}^{\infty} \Gamma(x - x', t) T_0(x') dx' + \int_0^t \int_{-\infty}^{\infty} \Gamma(x - x', t - \tau) \left(\frac{q_s(x', \tau)}{k} \right) dx' d\tau, \quad (12)$$

where $\Gamma(x, t)$ is given by Equation (11).

9.1.1.1 Heat Equation in An Example Initial Value Problems

- Equation (12) is completely analogous to solutions of initial value problems in ODEs.
- The function $\Gamma(x, t)$ is equivalent to the state transition matrix. To demonstrate the analogy, we divide the rod into N segments and identify

$$T(x_i, t) \rightarrow T_i(t)$$

$$\Gamma(x_i - x'_i, t)\Delta x \rightarrow \Phi_{ij}(t)$$

$$\frac{q_s(x_j, \tau)}{k} \rightarrow \underline{Bu} \rightarrow u_j(\tau)$$

9.1.1.1 Heat Equation in An Example Initial Value Problems

Making these substitutions in Equation (12), we arrive at

$$T_i(t) = \sum_{j=1}^N \Phi_{ij}(t) T_{0j} + \int_0^t \sum_{j=1}^N \Phi_{ij}(t - \tau) u_j(\tau) d\tau,$$

which of course can be written in matrix notation as

$$\underline{T}(t) = \underline{\Phi}(t) \underline{T}_0 + \int_0^t \underline{\Phi}(t - \tau) \underline{u}(\tau) d\tau.$$

- Compare with Equation 8.147. Note that the additional independent variable x can be thought of as a state vector of infinite dimension.

9.1.1.2 Existence and Uniqueness of Solutions to Parabolic Partial Differential Equations in Initial Value Problems

Consider the more general linear parabolic equation

$$\frac{\partial y}{\partial t} + \frac{\partial}{\partial x}(ay) - \frac{\partial^2}{\partial x^2}(by) + cy = f \quad 0 \leq t \leq t_F \quad y(x, 0) = p(x), \quad (13)$$

where:

$$a = a(x, t)$$

$$b = b(x, t)$$

$$c = c(x, t)$$

$$f = f(x, t)$$

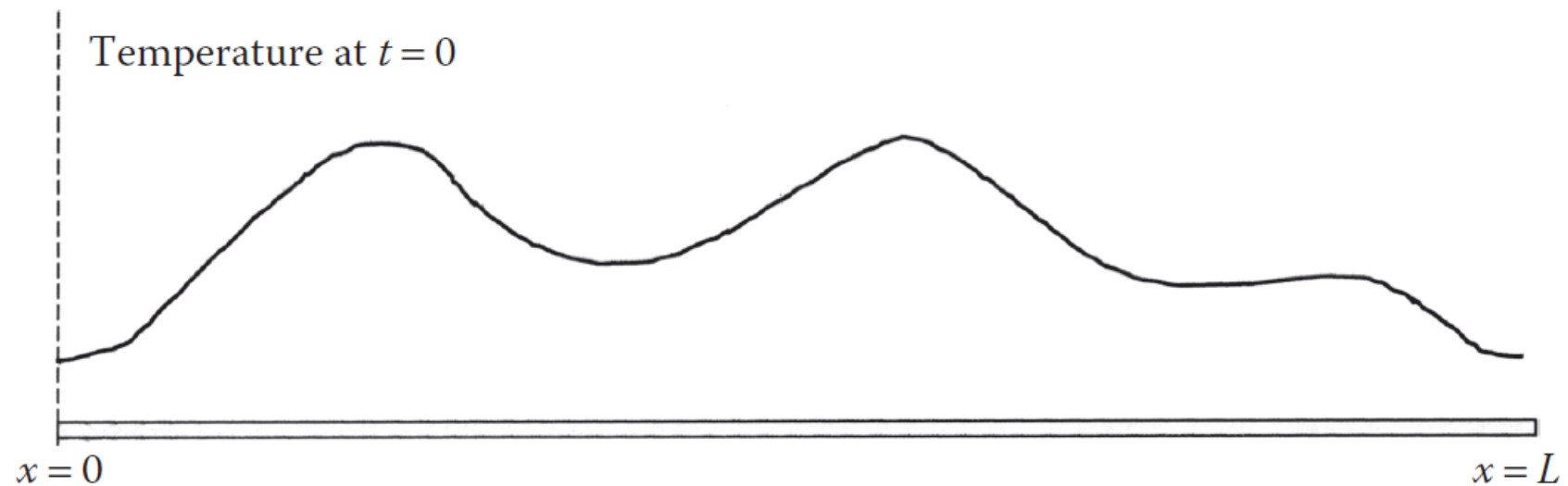
and t_F is finite and there is no boundary on x .

9.1.1.2 Existence and Uniqueness of Solutions to Parabolic Partial Differential Equations in Initial Value Problems

- Then under certain conditions on a, b, c, f , and p , smoothness conditions and bounds akin to those imposed on ODEs in Section 8.2, it can be shown that there exists a unique solution to Equation (13).

9.1.2 Heat Equations in Boundary Value Problems

- Figure 9.4 depicts a long rod with arbitrary initial temperature distribution $T_0(x)$.



- Figure 9.4 Long rod with initial temperature distribution and boundary conditions $T = 0$ at $x = 0$ and $x = L$ for $t > 0$.

9.1.2 Heat Equations in Boundary Value Problems

- Let the temperature at the rods endpoints be held at $T = 0$ for $0 < t < t_F$ and assume that the rod is insulated against heat loss to the environment along its length.
- We seek to determine the temperature distribution across the rod as a function of time.

9.1.2 Heat Equations in Boundary Value Problems

The equations are

$$\begin{aligned}\frac{\partial^2 T}{\partial x^2} &= \frac{1}{\alpha} \frac{\partial T}{\partial t} \\ T(x, 0) &= T_0(x) \\ T(0, t) &= T(L, t) = 0.\end{aligned}\tag{14}$$

We try the solution

$$T(x, t) = \Phi(x)\Theta(t).\tag{15}$$

- You may immediately object based on your physical intuition that the solution cannot have so simple a form and you would be correct.
- In the end the solution will be a combination of terms in the form of Equation (15).

9.1.2 Heat Equations in Boundary Value Problems

Now we substitute Equation (15) into the PDE in Equation (14)

$$\frac{\partial^2}{\partial x^2} (T(x, t)) = \left(\frac{d^2}{dx^2} \Phi(x) \right) \Theta(t) = \frac{1}{\alpha} \frac{\partial}{\partial t} (T(x, t)) = \frac{1}{\alpha} \Phi(x) \left(\frac{d}{dt} \Theta(t) \right).$$

Dividing through by $T(x, t) = \Phi(x)\Theta(t)$ leads to

$$\frac{1}{\Phi(x)} \frac{d^2 \Phi(x)}{dx^2} = \frac{1}{\alpha \Theta(t)} \frac{d\Theta(t)}{dt}. \quad (16)$$

The left-hand side of Equation (16) is a function only of x and the right side a function only of t , so the two sides must equal a constant.

9.1.2 Heat Equations in Boundary Value Problems

We will later determine that this constant must be a negative number:

$$\frac{1}{\Phi(x)} \frac{d^2\Phi(x)}{dx^2} = \frac{1}{\alpha\Theta(t)} \frac{d\Theta(t)}{dt} = -k \quad k > 0$$

Then we have two *ordinary* differential equations:

$$\frac{d^2\Phi(x)}{dx^2} + k\Phi(x) = 0, \tag{17}$$

$$\frac{d\Theta(t)}{dt} + \alpha k\Theta(t) = 0. \tag{18}$$

9.1.2 Heat Equations in Boundary Value Problems

We know that the general solutions to these two equations are

$$\Phi(x) = c_1 \cos \sqrt{k}x + c_2 \sin \sqrt{k}x, \quad (19)$$

$$\Theta(t) = c_3 e^{-\alpha k t}. \quad (20)$$

To meet the boundary condition $T(0, t) = T(L, t) = 0$ we must have

$$\Phi(0) = c_1 \cos(0) + c_2 \sin(0) = 0, \quad (21)$$

$$\Phi(L) = c_1 \cos(\sqrt{k}L) + c_2 \sin(\sqrt{k}L) = 0. \quad (22)$$

9.1.2 Heat Equations in Boundary Value Problems

From Equation (21), $c_1 = 0$.

From Equation (22), since $c_2 = 0$ cannot be zero,

$$\sin \sqrt{k}L = 0. \quad (23)$$

Note that if we had chosen k to be negative in Equation (17) then Equations (21) and (22) would have become

$$\Phi(0) = c_1 e^{\sqrt{k}(0)} + c_2 e^{-\sqrt{k}(0)} = c_1 + c_2 = 0,$$

$$\Phi(L) = c_1 e^{-\sqrt{k}L} + c_2 e^{\sqrt{k}L} = 0,$$

which would have required both c_1 and c_2 to be zero, which is not a viable solution.

9.1.2 Heat Equations in Boundary Value Problems

Returning to Equation (23), we see that we must have

$$\begin{aligned}\sqrt{k}L &= n\pi \\ k &= \left(\frac{n\pi}{L}\right)^2.\end{aligned}\tag{24}$$

These values of k are the *eigenvalues* of the problem. This is more evidence that the additional independent variable x can be considered a state vector of infinite dimension.

9.1.2 Heat Equations in Boundary Value Problems

Given Equation (15) and Equations (19 through (24), the *eigenfunctions* of the problem stated by Equation (14 are

$$\Theta_n(t)\Phi_n(x) = c_{2n}c_{3n}e^{-\alpha(n\pi/L)^2t}\sin(n\pi x/L) = c_n e^{-\alpha(n\pi/L)^2t}\sin(n\pi x/L). \quad (25)$$

We have no reason to choose one value of n over another. Therefore we try a sum over all n :

$$T(x, t) = \sum_{n=1}^{\infty} c_n e^{-\alpha(n\pi/L)^2t} \sin(n\pi x/L). \quad (26)$$

9.1.2 Heat Equations in Boundary Value Problems

The final condition we must meet is $T(x, 0) = T_0(x)$, or

$$\sum_{n=1}^{\infty} c_n \sin(n\pi x/L) = T_0(x). \quad (27)$$

Q: Can the coefficients c_n be chosen to satisfy Equation 9.27?

A: Yes. Under relatively weak conditions on the initial distribution $T_0(x)$ (a sufficient condition is that it and its derivative be piecewise continuous).

9.1.2 Heat Equations in Boundary Value Problems

To determine the coefficient c_m we multiply Equation (27) by $\sin(m\pi x/L)$ and integrate from $x = 0$ to $x = L$:

$$\int_0^L \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) \sin(m\pi x/L) dx = \int_0^L T_0(x) \sin(m\pi x/L) dx. \quad (28)$$

The set of functions $\sin(n\pi x/L)$ is orthogonal, meaning that

$$\int_0^L \sin(n\pi x/L) \sin(m\pi x/L) dx = 0, m \neq n. \quad (29)$$

9.1.2 Heat Equations in Boundary Value Problems

Here

$$\int_0^L \sin(m\pi x/L) \sin(m\pi x/L) dx = L/2. \quad (30)$$

From Equations (28) to (30),

$$c_m = \frac{2}{L} \int_0^L T_0(x) \sin(m\pi x/L) dx. \quad (31)$$

Equations (27) and (31) together define a *Fourier series*.

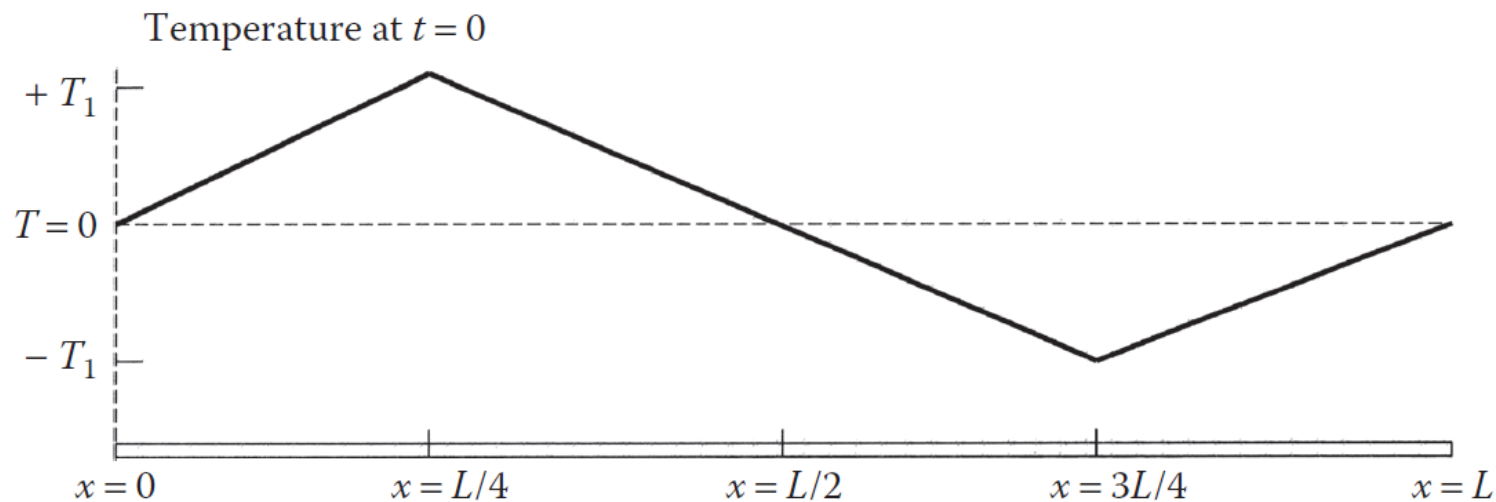
9.1.2 Heat Equations in Boundary Value Problems

Example: Consider a long rod of length L whose ends are controlled to temperature $T = 0$ and to which is applied a point heat source at $x = L/4$, raising the temperature there to $+T_1$, and a point heat sink at $x = 3L/4$, reducing the temperature there to $-T_1$. We state without proof that the steady-state solution to the heat equation in this case is as shown in [Figure 9.5](#).

- At time $t = 0$, the heat source and sink are removed, whereas the ends of the rod continue to be controlled to a temperature of zero.
- The steady-state temperature profile in the instants before $t = 0$ becomes the initial condition for the dynamic behavior we want to analyze.

9.1.2 Heat Equations in Boundary Value Problems

- Our task: To determine the temperature distribution across the rod as a function of time for $t \geq 0$.



- **Figure 9.5** Rod with heat source and sink in steady state prior to $t = 0$.

9.1.2 Heat Equations in Boundary Value Problems

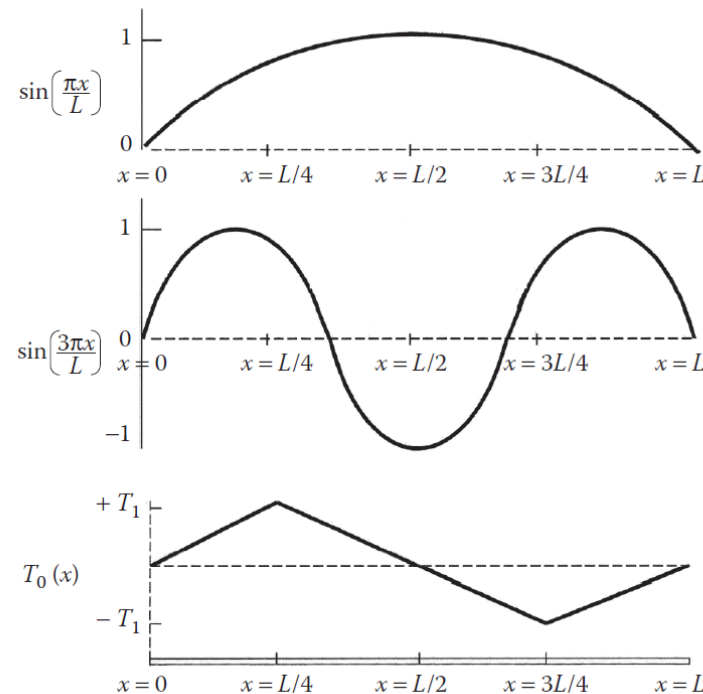
Solution:

$$T(x, t) = \sum_{n=1}^{\infty} c_n e^{-\alpha(n\pi/L)^2 t} \sin(n\pi x/L), \quad (32)$$

$$c_n = \frac{2}{L} \int_0^L T_0(x) \sin(n\pi x/L) dx. \quad (33)$$

- All that remains is to compute the integral in Equation (33).
- Now $T_0(x)$, the temperature distribution at time $t = 0$, is an odd function about the midpoint of the rod.
- $\sin(\pi x/L)$, $\sin(3\pi x/L)$, $\sin(5\pi x/L)$, and so on are even functions about the midpoint, $c_n = 0$ for odd n . [Figure 9.6](#) helps us see this.

9.1.2 Heat Equations in Boundary Value Problems



- **Figure 9.6** Comparing sine functions even about $x = L/2$ with initial temperature distribution $T_0(x)$ odd about $x = L/2$.

9.1.2 Heat Equations in Boundary Value Problems

For instance,

- when $x = \frac{L}{4}$

$$T_0(x) \sin(\pi x / L) = \frac{T_1}{\sqrt{2}},$$

- when $x = \frac{3L}{4}$

$$T_0(x) \sin(\pi x / L) = -\frac{T_1}{\sqrt{2}}.$$

- For every product at $x = \frac{L}{2} - \omega$, there is another of the opposite sign at $x = \frac{L}{2} + \omega$.

9.1.2 Heat Equations in Boundary Value Problems

To compute c_n for n even,

$$\begin{aligned}c_n &= \frac{L}{2} \int_0^L T_0(x) \sin(n\pi x/L) dx \\&= 2T_1 \int_0^{L/4} (4x/L) \sin(n\pi x/L) dx / L \\&\quad + 2T_1 \int_{L/4}^{3L/4} (2 - 4x/L) \sin(n\pi x/L) dx / L \\&\quad + 2T_1 \int_{3L/4}^L (4x/L - 4) \sin(n\pi x/L) dx / L.\end{aligned}$$

$c_n = 0$ for n any multiple of four. The only nonzero terms are

$$c_{4n+2} = (-1)^n \frac{8}{\pi^2} T_1 \frac{1}{(2n+1)^2} \quad n = 0, 1, 2, \dots \quad (34)$$

9.1.2 Heat Equations in Boundary Value Problems

From Equations (32) and (34),

$$T(x, t) = \frac{8}{\pi^2} T_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{-\alpha((4n+2)\pi/L)^2 t} \sin((4n+2)\pi x/L). \quad (35)$$

Let us evaluate Equation (35) at the point $x = L/4$, $t = 0$, where we should find $T(L/4, 0) = T_1$.

$$T(L/4, 0) = \frac{8}{\pi^2} T_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}. \quad (36)$$

9.1.2 Heat Equations in Boundary Value Problems

In yet another result attributed to **Leonhard Euler**, the sum of the inverse squares of the odd integers is

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}, \quad (37)$$

hence $T(L/4, 0) = T_1$.

- It takes quite a few terms of the infinite series to achieve an accurate approximation when $t = 0$. However, for even a relatively small lapse of time the first term in the series dominates all others.
- Consider just the first three terms at $x = L/4$:

9.1.2 Heat Equations in Boundary Value Problems

$$T(L/4, t) \cong \frac{8}{\pi^2} T_1 \left[e^{-4\pi^2 \alpha t / L^2} + \frac{1}{9} e^{-36\pi^2 \alpha t / L^2} + \frac{1}{25} e^{-100\pi^2 \alpha t / L^2} \right]. \quad (38)$$

If only enough time has elapsed to reduce the first exponential term to

$$e^{-4\pi^2 \alpha t / L^2} = 0.8,$$

then,

$$\frac{1}{9} e^{-36\pi^2 \alpha t / L^2} = \frac{1}{9} (0.8)^9 = 0.0149,$$

and

$$\frac{1}{25} e^{-100\pi^2 \alpha t / L^2} = \frac{1}{25} (0.8)^{25} = 0.0002.$$

9.1.2 Heat Equations in Boundary Value Problems

- The spatial distribution across the bar is then well represented by $\sin(2\pi x/L)$.
- In the mechanical engineering discipline of heat transfer, solving the heat equation often involves separation of variables and expansion in a Fourier series.
- It is typically the case that, after just a brief lapse of time, engineers can employ only the first term in the infinite series as an accurate representation of the temperature distribution.

9.2 Wave Equation and Power Series Solutions

The wave equation in three-dimensional Cartesian coordinates is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad (39)$$

where a is the speed of propagation.

- It describes oscillations propagating through space such as electromagnetic waves, flexure of membranes, waves on water surfaces, and acoustic vibrations.
- The wave equation is called a hyperbolic PDE.

9.2 Wave Equation and Power Series Solutions

- The terms parabolic, hyperbolic, and elliptic were originally assigned to classify PDEs with two independent variables written in the form

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial t} + C \frac{\partial^2 u}{\partial t^2} + D \frac{\partial u}{\partial t} + E \frac{\partial u}{\partial x} + F = 0 \quad (40)$$

9.2.1 Wave Equation in Initial Value Problems

- This section presents examples of the wave equation in initial value problems; that is, when boundary conditions do not affect the solution.

9.2.1.1 Traveling Plane Wave

Consider an electromagnetic wave propagating in free space (a vacuum). Maxwells equations show that each component of the electric field vector satisfies

$$\frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} - \frac{1}{a^2} \frac{\partial^2 E_i}{\partial t^2} = 0 \quad (41)$$

where

x, y and z are the spatial coordinates,

t is time,

c is a constant of nature equal to 3×10^8 m/s.

9.2.1.1 Traveling Plane Wave

Assume that the wave is far from its transmitter and any other boundary and try the form

$$E_i(x, y, z, t) = f(w) \quad (42)$$

$$w = k_1x + k_2y + k_3z + k_4t.$$

Then

$$\frac{\partial E_i}{\partial x} = \frac{df}{dw} \cdot \frac{\partial w}{\partial x} = k_1 \frac{df}{dw}$$

$$\frac{\partial^2 E_i}{\partial x^2} = k_1^2 \frac{d^2 f}{dw^2}.$$

Similarly,

$$\frac{\partial^2 E_i}{\partial y^2} = k_2^2 \frac{d^2 f}{dw^2}, \quad \frac{\partial^2 E_i}{\partial z^2} = k_3^2 \frac{d^2 f}{dw^2}, \quad \frac{\partial^2 E_i}{\partial t^2} = k_4^2 \frac{d^2 f}{dw^2}.$$

9.2.1.1 Traveling Plane Wave

Then Equation (41) becomes

$$\left(k_1^2 + k_2^2 + k_3^2 - \frac{k_4^2}{c^2} \right) \frac{d^2 f}{dw^2} = 0.$$

The function $f(w)$ can be of any arbitrary form if

$$k_1^2 + k_2^2 + k_3^2 - \frac{k_4^2}{c^2} = 0. \quad (43)$$

If we set $k_4 = \pm c$ then Equation (43) becomes

$$k_1^2 + k_2^2 + k_3^2 = 1.$$

9.2.1.1 Traveling Plane Wave

And the argument w becomes either

$$w = k_1x + k_2y + k_3z - ct, \quad (44)$$

or

$$w = k_1x + k_2y + k_3z + ct. \quad (45)$$

- Equations (44) and (45) define *plane waves* traveling at speed c either in the direction of the unit vector

$$\underline{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix},$$

or opposite it.

9.2.1.1 Traveling Plane Wave

If the electric field component has the waveform

$$E_i(x, y, z, 0) = f(k_1x + k_2y + k_3z),$$

at time $t = 0$ it satisfies

$$E_i(x, y, z, t) = f_1(k_1x + k_2y + k_3z - ct) + f_2(k_1x + k_2y + k_3z + ct), \quad (46)$$

for $t > 0$, where

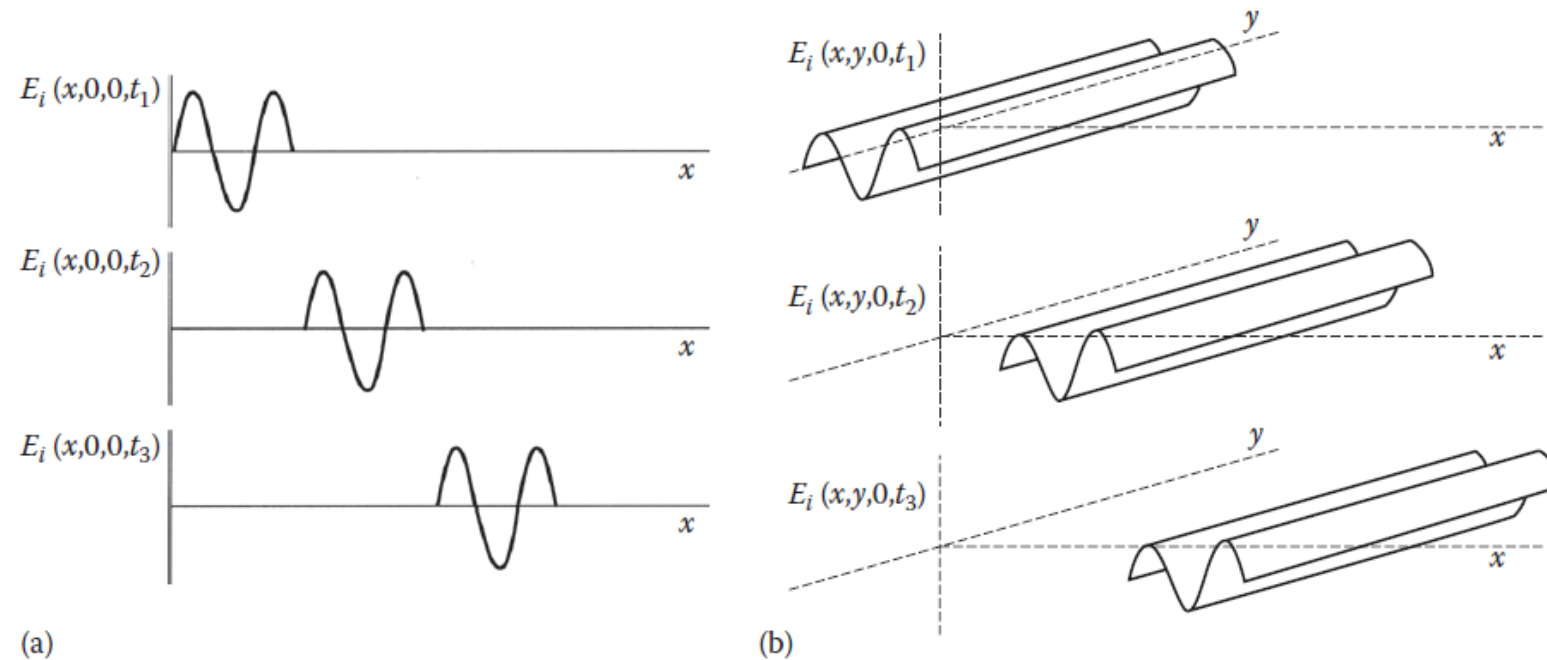
$$f_1(w) + f_2(w) = f(w)$$

- The locus of points where the argument w equals a constant is called a *wavefront*.

9.2.1.1 Traveling Plane Wave

- Figure 9.7 depicts a case where there is only an outgoing wave($f_2 = 0$).
- Graphing the electric field as the plane wave travels along the spatial x-axis in time (i.e., $k_1 = 1$, $k_2 = 0$, and $k_3 = 0$).
- Showing the waves presence along the spatial y -axis. The waves presence along the spatial z -axis is not shown.
- The vertical axis in Figure 9.7 is not the spatial z -axis; it is the scale for the value of the electric field.

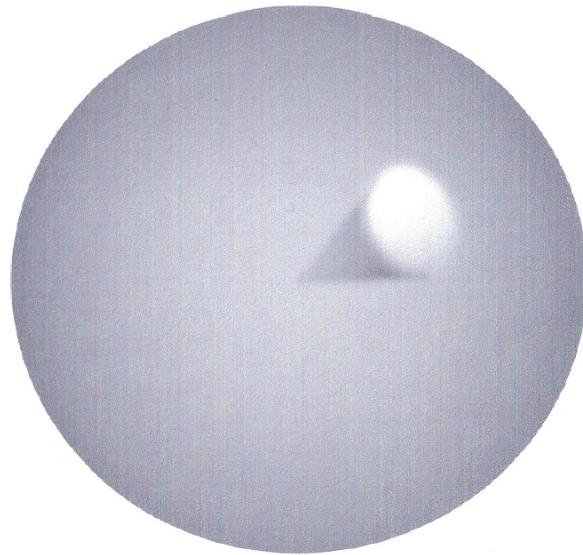
9.2.1.1 Traveling Plane Wave



- **Figure 9.7** A traveling plane wave at three different instants in time ($t_1 < t_2 < t_3$): (a) showing the wave along the x -axis only and (b) showing the wave along the x - and y - axes.

9.2.1.2 Traveling Wave in Spherical Coordinates

Figure 9.8 depicts a wavefront in an electromagnetic wave emanating from a distant transmitter.



- Figure 9.8 An electromagnetic wavefront at a large distance from its transmitter.

9.2.1.2 Traveling Wave in Spherical Coordinates

- At a large distance the transmitters antenna pattern is not locally observable;
- The wave appears to be uniform in azimuth and elevation. Hence, we can describe the value of the electric field locally by the wave equation in spherical coordinates assuming radial symmetry.
- The wave equation is

$$\frac{\partial^2 E_i}{\partial R^2} + \frac{2}{R} \frac{\partial E_i}{\partial R} - \frac{1}{c^2} \frac{\partial^2 E_i}{\partial t^2} = 0. \quad (47)$$

9.2.1.2 Traveling Wave in Spherical Coordinates

We assume that the transmitter is emitting a sinusoidal signal of radian frequency ω , and hence that the electric field can be represented at a distance R from the transmitter as

$$E_i(R, t) = \Phi_1(R) \cos \omega t + \Phi_2(R) \sin \omega t \quad (48)$$

Substituting Equation (48) into Equation (47) we find that

$$\left(\frac{d^2 \Phi_1}{dR^2} + \frac{2}{R} \frac{d\Phi_1}{dR} + \left(\frac{\omega}{c} \right)^2 \Phi_1 \right) \cos \omega t + \left(\frac{d^2 \Phi_2}{dR^2} + \frac{2}{R} \frac{d\Phi_2}{dR} + \left(\frac{\omega}{c} \right)^2 \Phi_2 \right) \sin \omega t = 0 \quad (49)$$

9.2.1.2 Traveling Wave in Spherical Coordinates

Since Equation (49) must be true for all t , it implies that

$$\frac{d^2\Phi_n}{dR^2} + \frac{2}{R} \frac{d\Phi_n}{dR} + k^2\Phi_n = 0, \quad (50)$$

for $n = 1, 2$, where we have substituted $k = \omega/c$. Letting

$$\Phi_n(R) = \frac{\Psi_n(R)}{R}, \quad (51)$$

we have

$$\frac{d\Phi_n}{dR} = -\frac{1}{R^2}\Psi_n + \frac{1}{R} \frac{d\Psi_n}{dR}, \quad (52)$$

9.2.1.2 Traveling Wave in Spherical Coordinates

$$\frac{d^2\Phi_n}{dR^2} = \frac{2}{R^3}\Psi_n - \frac{2}{R^2}\frac{d\Psi_n}{dR} + \frac{1}{R}\frac{d^2\Psi_n}{dR^2}. \quad (53)$$

Substituting Equations (51), (52), and (53) into Equation (50) we find,

$$\frac{d^2\Psi_n}{dR^2} + k^2\Psi_n = 0. \quad (54)$$

Hence

$$\Psi_1 = A_1 \cos kR + B_1 \sin kR \quad (55)$$

$$\Psi_2 = A_2 \cos kR + B_2 \sin kR$$

9.2.1.2 Traveling Wave in Spherical Coordinates

Using Equations (48), (51), and (55)

$$E_i(R, t) = ((A_1 \cos kR + B_1 \sin kR) \cos \omega t + (A_2 \cos kR + B_2 \sin kR) \sin \omega t) / R \quad (56)$$

Recalling the trigonometric identities

$$\cos kR \cos \omega t = (\cos(kR + \omega t) + \cos(kR - \omega t)) / 2$$

$$\sin kR \cos \omega t = (\sin(kR + \omega t) + \sin(kR - \omega t)) / 2$$

$$\cos kR \sin \omega t = (\sin(kR + \omega t) - \sin(kR - \omega t)) / 2$$

$$\sin kR \sin \omega t = (-\cos(kR + \omega t) + \cos(kR - \omega t)) / 2,$$

9.2.1.2 Traveling Wave in Spherical Coordinates

and using, as we have often done before,

$$c_1 \cos \alpha + c_2 \sin \alpha = G \sin(\alpha - \theta),$$

where

$$G = \sqrt{c_1^2 + c_2^2}$$

$$\theta = \arctan(c_1, c_2).$$

9.2.1.2 Traveling Wave in Spherical Coordinates

Equation (56) becomes

$$E_i(R, t) = G_1 \cos(kR - \omega t - \theta_1)/R + G_2 \cos(kR + \omega t - \theta_2)/R, \quad (57)$$

where

$$G_1 = \sqrt{(A_1 + B_2)^2 + (B_1 - A_2)^2} / 2$$

$$G_2 = \sqrt{(A_1 - B_2)^2 + (B_1 + A_2)^2} / 2$$

$$\theta_1 = \arctan(B_1 - A_2, A_1 + B_2)$$

$$\theta_2 = \arctan(B_1 + A_2, A_1 - B_2).$$

9.2.1.2 Traveling Wave in Spherical Coordinates

- Equation (57) represents two traveling waves,
 - One headed away from the transmitter and the other toward.
 - Choosing $A_1 = B_2$ and $B_1 = -A_2$ nulls G_2 , leaving only the outbound wave, which is the correct result for a signal from a distant transmitter.
 - This can be proved by solving the nonhomogeneous problem, considering the radiation source, which is how this problem is normally posed.
 - For simplicity we have posed it as homogeneous; considering the physics of the antenna and its radiation properties is beyond the scope of this text.

9.2.2 Wave Equation in Boundary Value Problem

- Here we consider a thin elastic membrane under tension, fixed securely around its circumference.
- Membrane is a sheet with no resistance to bending. (A steel plate, for example, is not a membrane.) The membrane is distorted by a force and then the force is removed.
- What are the subsequent vertical motions?

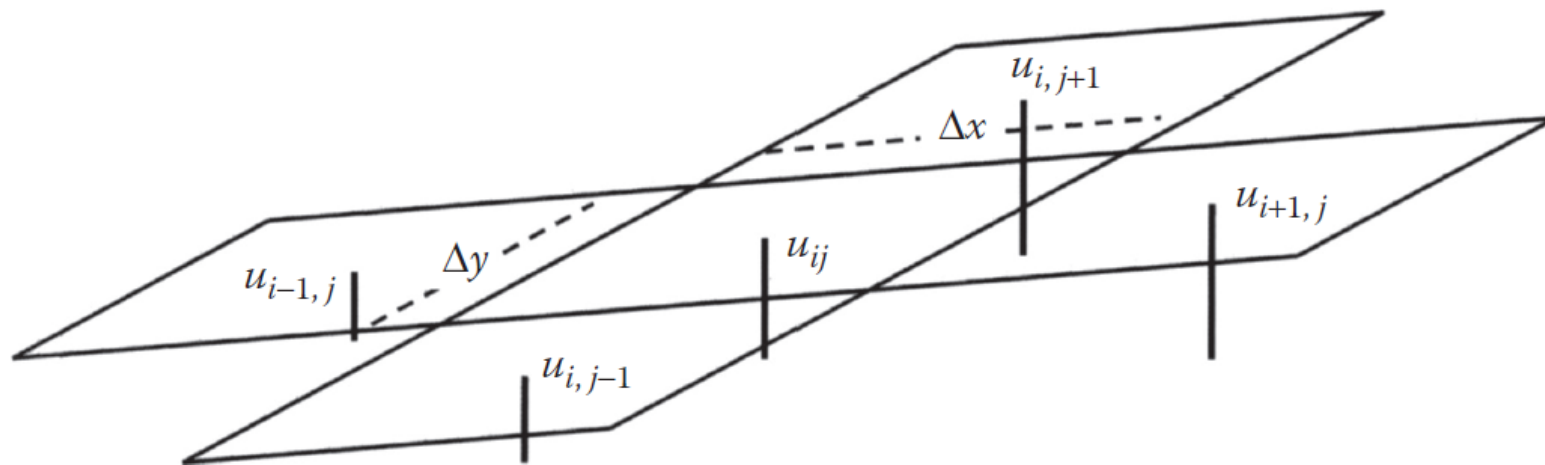
9.2.2.1 Derivation of the Wave Equation for a Taut Membrane

- The vertical displacement of the membrane is given by $u(x, y, t)$. We divide the membrane into infinitesimally small cells of width Δx and length Δy .
- The mass of each cell is $\rho \Delta x \Delta y$.
- By Newton's law the displacement of the cell at location x, y is given by

$$\rho \Delta x \Delta y \frac{\partial^2 u}{\partial t^2} = F(x, y). \quad (58)$$

9.2.2.1 Derivation of the Wave Equation for a Taut Membrane

The vertical force on the cell is the sum of the vertical forces on its four sides, each due to the tension T in the membrane having a vertical component. See [Figure 9.9](#).



■ [Figure 9.9](#) Displacement of adjoining cells in membrane.

9.2.2.1 Derivation of the Wave Equation for a Taut Membrane

- The force due to tension on the left side of cell i, j , at location x, y , is $T\Delta y$.
- The vertical component of this force (assuming small angles) is

$$-T\Delta y \left(\frac{u_{i,j} - u_{i-1,j}}{\Delta x} \right).$$

Continuing in this way around the four sides of cell i, j we arrive at

$$\begin{aligned} F(x, y) = & T\Delta y \left(\frac{u_{i+1,j} - u_{i,j}}{\Delta x} - \frac{u_{i,j} - u_{i-1,j}}{\Delta x} \right) \\ & + T\Delta x \left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y} - \frac{u_{i,j} - u_{i,j-1}}{\Delta y} \right). \end{aligned} \quad (59)$$

9.2.2.1 Derivation of the Wave Equation for a Taut Membrane

Combining Equations (58) and (59), dividing by $\Delta x \Delta y$ and letting Δx and Δy go to zero, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \left(\frac{\rho}{T}\right) \frac{\partial^2 u}{\partial t^2} = 0. \quad (60)$$

- This is the wave equation in two dimensions and Cartesian coordinates.
- The ratio $T/\rho = a^2$, which we will take to be constant, is the square of the speed of propagation of vertical displacements across the membrane.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

The motions of a taut *circular* membrane fixed at its boundary r_0 are best described by the wave equation in cylindrical coordinates:

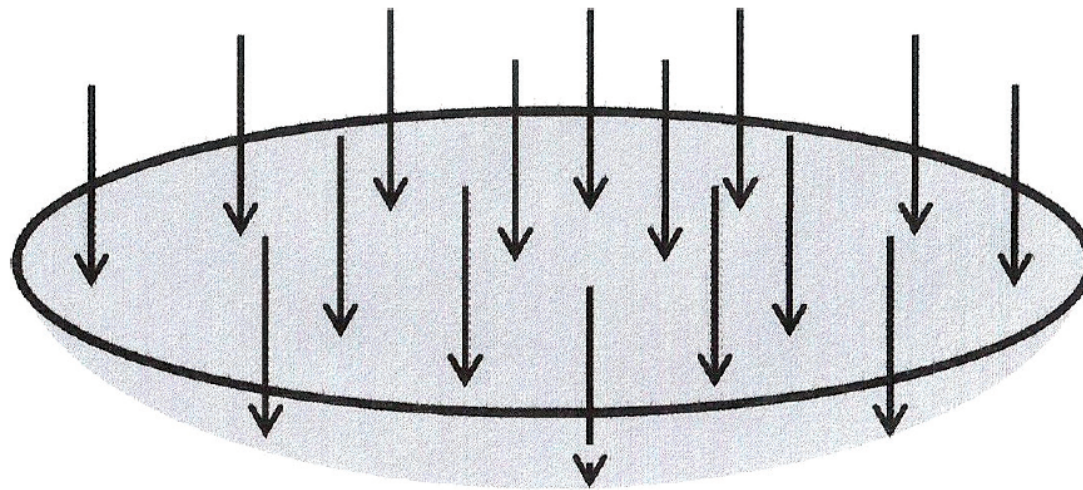
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = 0. \quad (61)$$

We consider a membrane initially subject to a uniform force across it that causes the steadystate displacement

$$u(r, 0) = h(1 - r^2/r_0^2). \quad (62)$$

See [Figure 9.10](#)

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates



- **Figure 9.10** Membrane under steady-state uniform force.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

Since the membrane is in steady state prior to $t = 0$ we have

$$\frac{\partial u}{\partial t}(r, 0) = 0. \quad (63)$$

- If the force immediately ceases, subsequent motion of the membrane is governed by Equation (61).
- The wave equation in cylindrical coordinates, with Equations (62) and (63) as initial conditions and

$$u(r_0, \varphi, t) = 0, \quad (64)$$

as boundary condition.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

We can see that in this case there is no dependence on azimuth, so

$$\frac{\partial^2 u}{\partial \varphi^2} = 0. \quad (65)$$

- To demonstrate the form of the equations when there is an initial variation of the displacement with azimuth we will not immediately invoke Equation (65) here.

Let

$$u(r, \varphi, t) = R(r)\Phi(\varphi)T(t), \quad (66)$$

and try to separate variables.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

Substituting Equation (66) into Equation (61) we have

$$\Phi T \frac{d^2 R}{dr^2} + \Phi T \frac{1}{r} \frac{dR}{dr} + RT \frac{1}{r^2} \frac{d^2 \Phi}{d\varphi^2} = R\Phi \frac{1}{a^2} \frac{d^2 T}{dt^2}.$$

Dividing through by $R\Phi T$ leaves

$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{Rr} \frac{dR}{dr} + \frac{1}{\Phi r^2} \frac{d^2 \Phi}{d\varphi^2} = \frac{1}{T a^2} \frac{d^2 T}{dt^2}. \quad (67)$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

- The right side of Equation (67) is a function of t only and the left side a function of r and φ only, hence both sides must equal a constant.
- It will turn out that to satisfy boundary conditions this constant must be negative. Call it $-k^2$.

The right side of Equation (67) becomes

$$\frac{d^2T}{dt^2} + a^2k^2T = 0. \quad (68)$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

We know that the solution of Equation (68) is

$$T(t) = c_1 \cos(akt) + c_2 \sin(akt). \quad (69)$$

- The membrane is initially at rest, from Equation (63) there is no time derivative at $t = 0$ so we must have $c_2 = 0$.
- We will be solving for constants to meet initial conditions in the equations for $R(r)$ and $\Phi(\varphi)$ we can.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

Set $c_1 = 1$

$$T(t) = \cos(akt). \quad (70)$$

■ a is given but we do not yet know what k is.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

The left-hand side of Equation (66) must also equal $-k^2$:

$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{Rr} \frac{dR}{dr} + \frac{1}{\Phi r^2} \frac{d^2 \Phi}{d\varphi^2} = -k^2,$$

$$r^2 \frac{1}{R} \frac{d^2 R}{dr^2} + r \frac{1}{R} \frac{dR}{dr} + \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = -k^2 r^2,$$

$$r^2 \frac{1}{R} \frac{d^2 R}{dr^2} + r \frac{1}{R} \frac{dR}{dr} + k^2 r^2 = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2}. \quad (71)$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

- The right-hand side of Equation (71) is a function only of φ .
- The left-hand side is a function only of r .
- Both sides must equal a constant.

We find that to satisfy boundary conditions it must be positive. Call it b . Then

$$-\frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = b,$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

$$\frac{d^2\Phi}{d\varphi^2} + b\Phi = 0. \quad (72)$$

Equation (72) has solution

$$\Phi(\varphi) = c_3 \cos(\sqrt{b}\varphi) + c_4 \sin(\sqrt{b}\varphi). \quad (73)$$

But by the nature of the azimuth coordinate φ we must have

$$\Phi(0) = \Phi(2\pi). \quad (74)$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

From Equations (73) and (74), we must have $\sqrt{b} = m$, an integer. Then the left-hand side of Equation (71) becomes

$$\begin{aligned} r^2 \frac{1}{R} \frac{d^2 R}{dr^2} + r \frac{1}{R} \frac{dR}{dr} + k^2 r^2 &= m^2, \\ r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (k^2 r^2 - m^2) R &= 0, \\ \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(k^2 - \frac{m^2}{r^2}\right) R &= 0. \end{aligned} \tag{75}$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

We now introduce a new dimensionless independent variable $x = kr$. Equation (75) becomes

$$\frac{d^2 R}{dr^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{m^2}{x^2}\right) R = 0. \quad (76)$$

- Equation (76) is known as *Bessels equation of order m* .
- The solution with finite value and finite derivative at the origin is called the *Bessel function of the first kind of order m* and denoted as $J_m(x)$.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

- Higher-order ($m > 0$) Bessel equations arise when we have cylindrical geometry but not azimuthal symmetry.

In Equations (61), (62), and (63) there is no dependence on azimuth, so Equation (76) becomes *Bessels equation of order zero*:

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + R = 0. \quad (77)$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

- The solution to Equation (77) with finite value and derivative at the origin is the *Bessel function of the first kind of order zero*:

$$R(x) = J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}. \quad (78)$$

- There is also a boundary condition. We must have $R(kr_0) = 0$ that implies

$$J_0(kr_0) = 0. \quad (79)$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

- Not all values of k will work for Equation (79).
- k_n are called *eigenvalues* and the corresponding functions $J_0(k_n r)$ called *eigenfunctions*.
- The first three zeros of $J_0(x)$ are approximately $x_1 = 2.405$, $x_2 = 5.520$, and $x_3 = 8.654$.
- For $n > 3$ the zeros are given, approximately but accurately, by $x_n = n\pi - \frac{\pi}{4}$.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

If we had selected a positive number ($+k^2$), Equation (77) would have read

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - R = 0.$$

- Its solutions could not have met the condition $R(kr_0) = 0$.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

Recall $u(r, t) = R(r)T(t)$ and $T(t) = \cos(akt)$ and recognize that an infinite number of eigenvalues $k_n = x_n/r_0$ exist,

$$u(r, t) = \sum_{n=0}^{\infty} c_n J_0(x_n r / r_0) \cos(x_n a t / r_0), \quad (80)$$

determine the coefficients c_n from the initial condition

$$u(r, 0) = h(1 - r^2/r_0^2),$$

$$\sum_{n=0}^{\infty} c_n J_0(x_n r / r_0) = h(1 - r^2/r_0^2). \quad (81)$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

Recall that in [Section 9.1.2](#), in a problem defined along one Cartesian spatial axis with boundary conditions at each end, we sought coefficients c_n to satisfy

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f_0(x),$$

where $f_0(x)$ was a given function.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

- Under relatively weak smoothness conditions on $f_0(x)$ we could indeed accomplish this, because the sine functions are orthogonal under a certain integral

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0 \quad m \neq n,$$
$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2} \quad m = n.$$

- That is also the case in this boundary value problem with cylindrical geometry.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

We can find c_n to satisfy Equation (81) because Bessel functions also form an orthogonal set:

$$\int_0^{r_0} J_0(x_n r/r_0) J_0(x_m r/r_0) r dr = 0 \quad m \neq n,$$

$$\int_0^{r_0} J_0(x_n r/r_0) J_0(x_m r/r_0) r dr = \frac{r_0^2}{2} J_1^2(x_n) \quad m = n,$$

where $J_1(x)$ is the Bessel function of the first kind of order 1.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

To find c_n we multiply Equation (81) by $J_0(x_m r/r_0)$ and integrate:

$$\sum_{n=0}^{\infty} c_n J_0(x_n r/r_0) J_0(x_m r/r_0) = h(1 - r^2/r_0^2) J_0(x_m r/r_0),$$

$$\sum_{n=0}^{\infty} c_n \int_0^{r_0} J_0(x_n r/r_0) J_0(x_m r/r_0) r dr = \int_0^{r_0} h(1 - r^2/r_0^2) J_0(x_m r/r_0) r dr,$$

$$c_m \left(\frac{r_0^2}{2} J_1^2(x_m) \right) = \int_0^{r_0} h(1 - r^2/r_0^2) J_0(x_m r/r_0) r dr.$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

$$c_m = \frac{\int_0^{r_0} h(1 - r^2/r_0^2) J_0(x_m r/r_0) r dr}{\left(\frac{r_0^2}{2} J_1^2(x_m)\right)}. \quad (82)$$

The result is

$$c_m = h \frac{8}{x_m^3 J_1(x_m)}. \quad (83)$$

Using the membrane displacement (Equations (80) and (83)), the solution to Equations (61) through (65), is

$$u(r, t) = h \sum_{n=0}^{\infty} \left(\frac{8}{x_n^3 J_1(x_n)} \right) J_0(x_n r/r_0) \cos(x_n a t/r_0). \quad (84)$$

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

- We can check this result by evaluating the solution at $r = 0$ and $t = 0$. According to the initial condition Equation (62), we have $u(0, 0) = h$. We expect

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n c_m = h. \quad (85)$$

Table 9.1 records the partial sums. It appears the series is indeed converging to the correct result.

9.2.2.2 Solving the Wave Equation in a Boundary Value Problem in Cylindrical Coordinates

TABLE 9.1

Checking the Accuracy of the Wave Equation Solution

n	x_n	$J_1(x_n)$	c_n/h	$\sum_{m=1}^n c_m/h$
1	2.405	0.5191	1.1079	1.1079
2	5.520	-0.3403	-0.1398	0.9681
3	8.654	0.2714	0.0455	1.0136
4	11.792	-0.2325	-0.0210	0.9926
5	14.931	0.2065	0.0116	1.0042
6	18.071	-0.1877	-0.0072	0.9970
7	21.212	0.1733	0.0048	1.0018
8	24.352	-0.1617	-0.0034	0.9984
9	27.493	0.1522	0.0025	1.0009
10	30.635	-0.1442	-0.0019	0.9990

9.2.3 Power Series Solutions: An Introduction

Our objective here is to show how to solve problems exemplified by Bessels equation of order zero:

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = 0. \quad (86)$$

- Equation (86) is a linear ODE with variable coefficients.
- Time is not the independent variable.

9.2.3 Power Series Solutions: An Introduction

- In general we are unable to solve equations with variable coefficients in terms of standard functions like the sine and exponential, as we were able to do with linear equations with constant coefficients.
- We solve Equation (86) in terms of an infinite series:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n. \quad (87)$$

The initial condition $y(0) = 1$ implies that

$$c_0 = 1. \quad (88)$$

9.2.3 Power Series Solutions: An Introduction

Differentiating Equation (88) term-by-term,

$$\frac{1}{x} \frac{dy}{dx} = \frac{1}{x} \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-2}, \quad (89)$$

$$\frac{d^2 y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}. \quad (90)$$

Substituting Equations (87), (89), and (90) into Equation (86) we have

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

9.2.3 Power Series Solutions: An Introduction

$$\begin{aligned} \sum_{n=2}^{\infty} n^2 c_n x^{n-2} - \sum_{n=2}^{\infty} n c_n x^{n-2} + \left(\sum_{n=2}^{\infty} n c_n x^{n-2} + c_1 x^{-1} \right) + \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \sum_{n=2}^{\infty} n^2 c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n + c_1 x^{-1} &= 0 \end{aligned} \quad (91)$$

- For Equation (91) to be true for all x , the coefficients of each power must be zero.

$$c_1 = 0, \quad (92)$$

which implies

$$\frac{dy}{dx}(0) = 0.$$

9.2.3 Power Series Solutions: An Introduction

We are left with

$$\sum_{n=2}^{\infty} n^2 c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0. \quad (93)$$

We must manipulate the indices so that Equation (93) can be written as a single sum:

$$\sum_{m=0}^{\infty} k_m x^m = 0.$$

Replacing $n - 2$ with m in the first sum in Equation (93) and n with m in the second.

9.2.3 Power Series Solutions: An Introduction

The result is

$$\sum_{m=0}^{\infty} \left((m+2)^2 c_{m+2} + c_m \right) x^m = 0. \quad (94)$$

- For this equation to be true for all x we must have the coefficient of each power of x equal to zero. This implies the *recursion relation*

$$c_{m+2} = -\frac{1}{(m+2)^2} c_m. \quad (95)$$

9.2.3 Power Series Solutions: An Introduction

Using Equations (88), (92) , and (95), we find that $c_m = 0$ for odd m and

$$c_2 = -\frac{1}{(0+2)^2}c_0 = -\frac{1}{2^2},$$

$$c_4 = -\frac{1}{(2+2)^2}c_2 = \frac{1}{2^4 2^2}, \quad (96)$$

$$c_6 = -\frac{1}{(4+2)^2}c_4 = -\frac{1}{2^6 (3 \cdot 2)^2}.$$

In general, the solution to Equation (86) with $y(0) = 1$ can be written as

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \quad (97)$$

9.2.3 Power Series Solutions: An Introduction

or as

$$y(x) = J_0(x). \quad (98)$$

- $J_0(x)$ is the Bessel function of the first kind of order zero (In Equation (97)).

9.2.4 Bessel Equations And Bessel Functions

Bessels equation of order m is

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{m^2}{x^2}\right) y = 0. \quad (99)$$

- It is a second-order linear ODE and has two fundamental solutions, called *Bessel functions of the first and second kinds*.

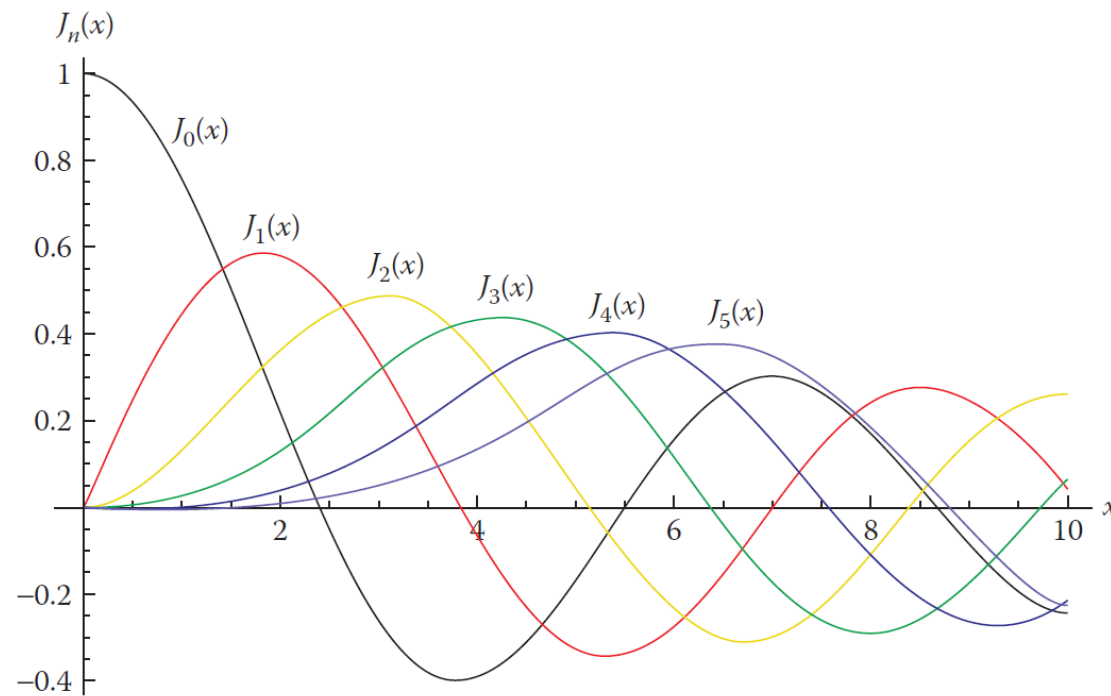
9.2.4.1 Bessel Functions of the First Kind

The solution of Equation (99) having finite value at the origin is called the Bessel function of the first kind:

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+m)!n!} \left(\frac{x}{2}\right)^{2n+m}. \quad (100)$$

Figure 9.11 presents a graph of $J_0(x)$ through $J_5(x)$.

9.2.4.1 Bessel Functions of the First Kind



■ Figure 9.11 Bessel functions of the first kind.

9.2.4.1 Bessel Functions of the First Kind

For small x the first few terms of the infinite series in Equation (100) suffice. For example,

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{4}\left(\frac{x}{2}\right)^4 - \frac{1}{36}\left(\frac{x}{2}\right)^6 + \dots$$

$$J_1(x) = \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^3 + \frac{1}{12}\left(\frac{x}{2}\right)^5 - \frac{1}{144}\left(\frac{x}{2}\right)^7 + \dots$$

For large x the approximation

$$J_m(x) \cong \sqrt{\frac{2}{\pi x}} \cos \left(x - m\frac{\pi}{2} - \frac{\pi}{4} \right), \quad (101)$$

is quite accurate.

9.2.4.2 Bessel Functions of the Second Kind

From [Sections 5.2](#) and [5.4](#) we learned that the equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \quad (102)$$

with specified initial conditions

$$\begin{aligned} y(t_0) &= y_0 \\ \dot{y}(t_0) &= \dot{y}_0 \end{aligned} \quad (103)$$

has a solution of the form

$$y(t) = c_1y_1(t) + c_2y_2(t) \quad (104)$$

9.2.4.2 Bessel Functions of the Second Kind

■ c_1 and c_2 given by the solution to

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$

$$c_1 \dot{y}_1(t_0) + c_2 \dot{y}_2(t_0) = \dot{y}_0$$

provided that $p(t)$ and $q(t)$ are continuous in some open interval I containing $t = t_0$.

9.2.4.2 Bessel Functions of the Second Kind

- The Bessel equation does not meet this proviso when $t_0 = 0$.
 - Writing the independent variable as t instead of x in Equation (99) and comparing to Equation (102), $p(t) = 1/t$ is discontinuous at $t = 0$ for all orders and $q(t) = 1 - m^2/t^2$ is discontinuous at $t = 0$ for all except order zero.
 - In fact it is not possible to solve the Bessel equation with arbitrary initial conditions as in Equation (104) when $t_0 = 0$.

9.2.4.2 Bessel Functions of the Second Kind

- We already saw in Section 9.2.3 that a solution $y(t)$ of the Bessel equation of order zero that is valid at $t = 0$ **must satisfy $dy/dt = 0$ at $t = 0$.**
- A solution to the Bessel equation with arbitrary initial conditions is possible if those conditions are specified at a point $t_0 > 0$ and we are interested in a solution only for $t \geq t_0$.

9.2.4.2 Bessel Functions of the Second Kind

Returning to the independent variable x , the Bessel equation

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{m^2}{x^2}\right) y = 0.$$

has a second solution valid for $x > 0$

$$y_m(t) = J_m(x) \ln x + \sum_{n=1}^{\infty} c_n x^n. \quad (105)$$

9.2.4.2 Bessel Functions of the Second Kind

Rather than use Equation (105) as the second solution, it is traditional to employ instead a certain linear combination of it and $J_m(x)$. The traditional form for order zero is

$$Y_0(x) = \frac{2}{\pi} \left[\left(\gamma + \ln \left(\frac{x}{2} \right) \right) J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n!)^2} \left(\sum_{m=1}^n \frac{1}{m} \right) \left(\frac{x}{2} \right)^{2n} \right],$$

where

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{m=1}^n \frac{1}{m} - \ln n \right) \cong 0.5772.$$

9.2.4.2 Bessel Functions of the Second Kind

- The higher-order functions $Y_m(x)$ are defined similarly and as a result the asymptotic relationships are complementary. For large x ,

$$J_m(x) \cong \sqrt{\frac{2}{\pi x}} \cos \left(x - m\frac{\pi}{2} - \frac{\pi}{4} \right),$$

$$Y_m(x) \cong \sqrt{\frac{2}{\pi x}} \sin \left(x - m\frac{\pi}{2} - \frac{\pi}{4} \right).$$

9.2.5 Power Series Solutions: Another Example

Considers Bessels equation of order m , where m is an integer greater than zero:

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{m^2}{x^2}\right) y = 0. \quad (106)$$

The solution with finite value and derivative at $x = 0$ is

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+m)!n!} \left(\frac{x}{2}\right)^{2n+m}. \quad (107)$$

We seek to derive Equation (107) from first principles.

9.2.5 Power Series Solutions: Another Example

If we were to try the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n,$$

it fails for order m . We try, instead,

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} c_n x^n. \quad (108)$$

We must solve for the exponent α as well as the coefficients c_n .

9.2.5 Power Series Solutions: Another Example

Substituting Equation (108) into Equation (106) we find

$$\begin{aligned} \sum_{n=0}^{\infty} (n + \alpha)(n + \alpha - 1)c_n x^{n+\alpha-2} + \sum_{n=0}^{\infty} (n + \alpha)c_n x^{n+\alpha-2} \\ + \sum_{n=0}^{\infty} c_n x^{n+\alpha} - \sum_{n=0}^{\infty} m^2 c_n x^{n+\alpha-2} = 0. \end{aligned} \quad (109)$$

- The coefficient of $x^{\alpha-2}$ in Equation (109) is $(\alpha(\alpha - 1) + \alpha - m^2)c_0 = (\alpha^2 - m^2)c_0$, which must equal zero, as must all coefficients. Therefore,

$$\alpha^2 - m^2 = 0. \quad (110)$$

Equation (110) is the *indicial equation* for the ODE Equation (106).

9.2.5 Power Series Solutions: Another Example

For $y(x)$ to be finite at the origin we must have $\alpha = +m$. The coefficient of $x^{\alpha-1}$ in Equation (109) is

$$\begin{aligned} & ((1 + \alpha)(1 + \alpha - 1) + (1 - \alpha) - m^2)c_1 \\ &= ((1 + m)(1 + m - 1) + (1 + m) - m^2)c_1 \\ &= (2m + 1)c_1, \end{aligned}$$

the coefficient must be zero, which implies that $c_1 = 0$.

9.2.5 Power Series Solutions: Another Example

To evaluate the higher-order coefficients we rearrange the remaining terms of Equation (109) into a more convenient form:

$$\begin{aligned} \sum_{n=0}^{\infty} (n + \alpha)(n + \alpha - 1)c_n x^{n+\alpha-2} + \sum_{n=0}^{\infty} (n + \alpha)c_n x^{n+\alpha-2} \\ + \sum_{n=0}^{\infty} c_n x^{n+\alpha} - \sum_{n=0}^{\infty} m^2 c_n x^{n+\alpha-2} = 0. \end{aligned}$$

$$\sum_{n=2}^{\infty} ((n + m)^2 - m^2)c_n x^{n+m-2} + \sum_{n=0}^{\infty} c_n x^{n+m} = 0.$$

$$\sum_{n=0}^{\infty} (n^2 + 2nm)c_n x^{n+m-2} + \sum_{n=0}^{\infty} c_n x^{n+m} = 0. \quad (111)$$

9.2.5 Power Series Solutions: Another Example

We need to represent Equation (111) in the form

$$\sum_{k=0}^{\infty} a_k x^{k+m} = 0.$$

Replace n in the first sum with $k + 2$ and replace n in the second sum with k .

$$\sum_{k=0}^{\infty} (((k + 2)^2 + 2(k + 2)m)c_{k+2} + c_k)x^{k+m} = 0.$$

9.2.5 Power Series Solutions: Another Example

The recursion relation

$$c_{k+2} = -\frac{c_k}{(k+2)^2 + 2(k+2)m}. \quad (112)$$

- Letting $k = 1$ in Equation (112) and recalling that $c_1 = 0$ shows us that $c_k = 0$ for odd k .
- Letting $k = 0$ in Equation (112) shows us that to have a nonzero solution to the ODE we must have $c_0 \neq 0$.

9.2.5 Power Series Solutions: Another Example

We have

$$c_2 = -\frac{1}{2^2 + 2 \cdot 2 \cdot m} c_0 = (-1) \left(\frac{1}{2}\right)^2 \frac{1}{m+1} c_0$$

$$c_4 = -\frac{1}{4^2 + 2 \cdot 4 \cdot m} c_2 = (-1)^2 \left(\frac{1}{2}\right)^4 \left(\frac{1}{(m+2)(m+1)}\right) \left(\frac{1}{2}\right) c_0$$

$$c_6 = -\frac{1}{6^2 + 2 \cdot 6 \cdot m} c_4 = (-1)^3 \left(\frac{1}{2}\right)^6 \left(\frac{1}{(m+3)(m+2)(m+1)}\right) \left(\frac{1}{3 \cdot 2}\right) c_0$$

It is convenient and useful to let

$$c_0 = \frac{1}{2^m m!}.$$

9.2.5 Power Series Solutions: Another Example

Then the general coefficient can be written as

$$c_{2k} = (-1)^k \left(\frac{1}{2}\right)^{2k+m} \frac{1}{(k+m)!k!},$$

and the solution is

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+m)!n!} \left(\frac{1}{2}\right)^{2n+m} x^{2n+m},$$

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+m)!n!} \left(\frac{1}{2}\right)^{2n+m},$$

as was to be shown.

9.2.6 Power Series Solutions: Some General Rules

Three different forms of power series that satisfy different second-order linear ODEs:

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n,$$

$$y_2(x) = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n,$$

$$y_3(x) = x^{\alpha} \sum_{n=0}^{\infty} c_n x^n.$$

- Given a specific form of ODE, what form of solution should we try?

9.2.6.1 A Few Definitions

A function $f(x)$ is said to be analytic at $x = x_0$ if it can be expanded in a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

that converges in some interval around x_0 . Consider the ODE

$$\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0. \quad (113)$$

- The point $x = x_0$ is said to be an *ordinary point* of the ODE if $p(x)$ and $q(x)$ are analytic at x_0 .
- If the point $x = x_0$ is not an ordinary point it is called a *singular point*.

9.2.6.1 A Few Definitions

A singular point is called a *regular singular point* of the ODE (Equation (113)) if

$$\lim_{x \rightarrow x_0} (x - x_0)p(x) = p_0, \quad (114)$$

$$\lim_{x \rightarrow x_0} (x - x_0)^2 q(x) = q_0, \quad (115)$$

and p_0 and q_0 are finite. The indicial equation is

$$\alpha(\alpha - 1) + p_0\alpha + q_0 = 0. \quad (116)$$

9.2.6.2 Some General Rules

- First divide by the coefficient of the highest derivative in the ODE to make that coefficient equal to one.
- Then determine if the coefficients of $\frac{dy}{dx}$ and y are analytic.
- If $x = x_0$ is an ordinary point of the ODE (Equation (113)) then each of the two fundamental solutions can be written in the form

$$y_i = \sum_{n=0}^{\infty} c_n (x - x_0)^n. \quad (117)$$

- The series in Equation (117) and in equations written below converge in some region around x_0 .

9.2.6.2 Some General Rules

- If $x = x_0$ is a regular singular point and α_1 and α_2 are real solutions to the indicial equation with $\alpha_1 \geq \alpha_2$, then one solution of the ODE can be written in the form

$$y_1 = (x - x_0)^{\alpha_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

- If $\alpha_1 = \alpha_2$ then the second fundamental solution of the ODE can be written in the form

$$y_2 = y_1 \ln(x - x_0) + (x - x_0)^{\alpha_1} \sum_{n=1}^{\infty} b_n (x - x_0)^n.$$

9.2.6.2 Some General Rules

■ If $\alpha_1 \neq \alpha_2$ then:

- If $\alpha_1 - \alpha_2$ is not an integer then the second ODE solution can be written in the form

$$y_2 = (x - x_0)^{\alpha_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n.$$

- If $\alpha_1 - \alpha_2$ is an integer then the second solution can be written in the form

$$y_2 = ay_1 \ln(x - x_0) + (x - x_0)^{\alpha_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

where the coefficient a may or may not be zero.

9.2.6.3 Special Cases

The CauchyEuler equation has the form

$$x^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = 0$$

- If the two solutions α_1 and α_2 to the indicial equation are real and $\alpha_1 \geq \alpha_2$ then one of the solutions is simply $y_1(x) = x^{\alpha_1}$.

9.3 Laplace's Equation

- In three-dimensional Cartesian coordinates ([Figure 9.12a](#)) the Laplace's Equation is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

- In three-dimensional cylindrical coordinates ([Figure 9.12b](#)) the Laplace's Equation is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

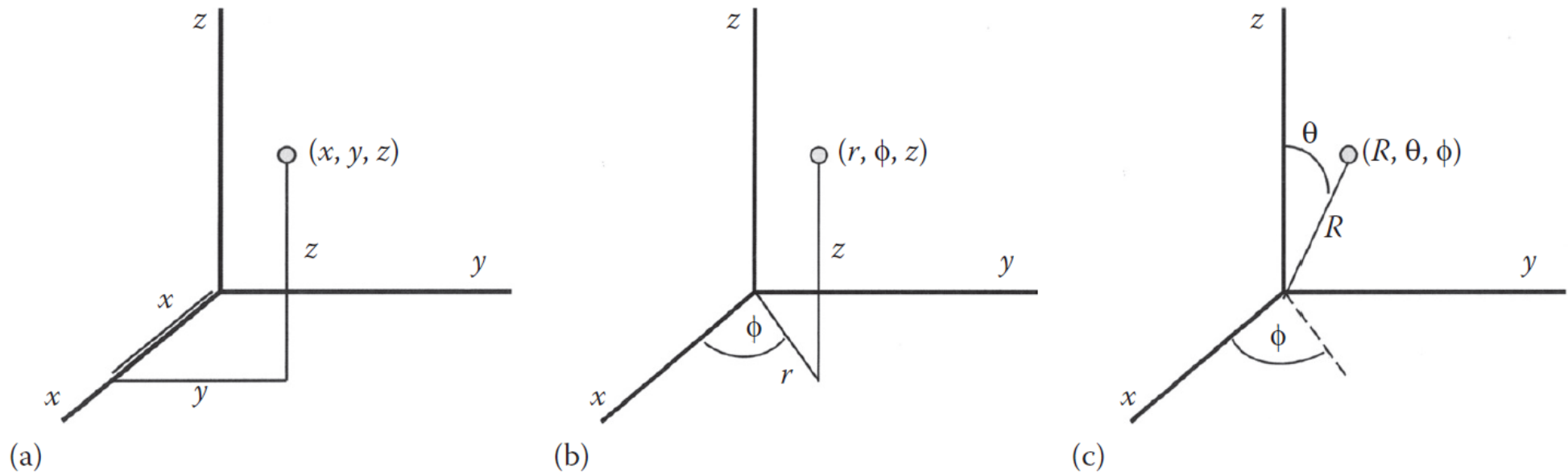
9.3 Laplace's Equation

- In spherical coordinates ([Figure 9.12c](#)) the Laplace's Equation is

$$\frac{\partial^2 V}{\partial R^2} + \frac{2}{R} \frac{\partial V}{\partial R} + \frac{1}{R^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\cot \theta}{R^2} \frac{\partial V}{\partial \theta} + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \varphi^2} = 0. \quad (118)$$

- Laplace's Equation is called an elliptic PDE.
- A change in the value of V at any point on the boundary changes the value of V everywhere inside the boundary and V takes on its maximum and minimum values on the boundary.

9.3 Laplace's Equation



- **Figure 9.12** Three coordinate systems: (a) cartesian, (b) cylindrical, and (c) spherical.

9.3.1 Laplace's Equation Example in Spherical Coordinates

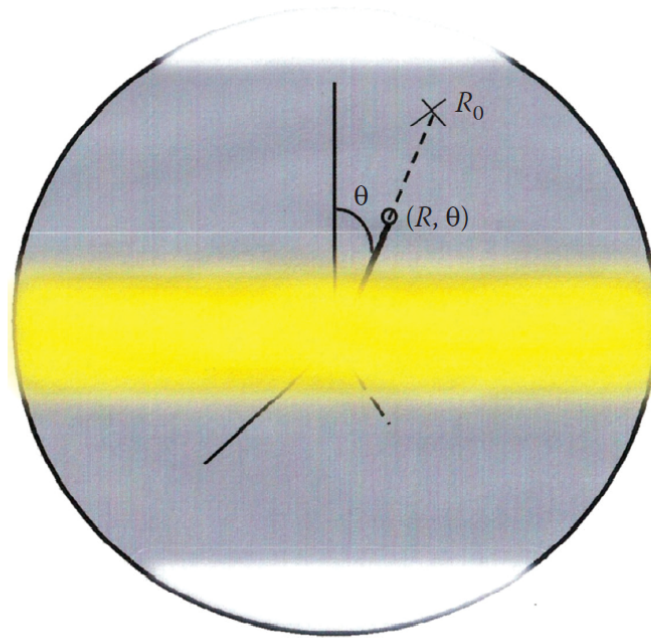
Figure 9.13 depicts a solid sphere of radius R_0 with uniform heat conduction properties whose surface is maintained at a temperature that depends on the angle θ (latitude).

What is the temperature $T(R, \theta)$ inside the sphere?

Given the azimuthal (longitudinal) symmetry, Equation (118) becomes

$$\frac{\partial^2 T}{\partial R^2} + \frac{2}{R} \frac{\partial T}{\partial R} + \frac{1}{R^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\cot \theta}{R^2} \frac{\partial T}{\partial \theta} = 0. \quad (119)$$

9.3.1 Laplace's Equation Example in Spherical Coordinates



- Figure 9.13 Temperature distribution on surface of sphere.

9.3.1 Laplace's Equation Example in Spherical Coordinates

Letting

$$T(R, \theta) = G(R)H(\theta),$$

the resulting ODEs are

$$R^2 \frac{d^2 G}{dR^2} + 2R \frac{dG}{dR} - n(n+1)G = 0, \quad \frac{d^2 H}{d\theta^2} + \cot \theta \frac{dH}{d\theta} + n(n+1)H = 0.$$

With the substitution $x = \cos \theta$ the second equation becomes

$$(1 - x^2) \frac{d^2 H}{dx^2} - 2x \frac{dH}{dx} + n(n+1)H = 0,$$

which is *Legendres equation of order n* .

9.3.1 Laplace's Equation Example in Spherical Coordinates

Following the general rules of [Section 9.2.6](#), the result is

$$T(R, \theta) = \sum_{n=1}^{\infty} c_n \left(\frac{R}{R_0} \right)^n P_n(\cos \theta),$$

where the functions $P_n(x)$ are Legendre polynomials, the first five of which are

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = (3x^2 - 1)/2$$

$$P_3(x) = (5x^3 - 3x)/3$$

$$P_4(x) = (35x^4 - 30x^2 + 3)/8.$$

9.3.1 Laplace's Equation Example in Spherical Coordinates

$$c_n = \frac{2n+1}{2} \int_0^\pi T_S(\theta) P_n(\cos \theta) \sin \theta d\theta$$

Similar to the sine and Bessel functions, Legendre polynomials are orthogonal. Here,

$$\int_0^\pi P_n(\cos \theta) P_m(\cos \theta) \sin \theta d\theta = 0 \quad m \neq n$$

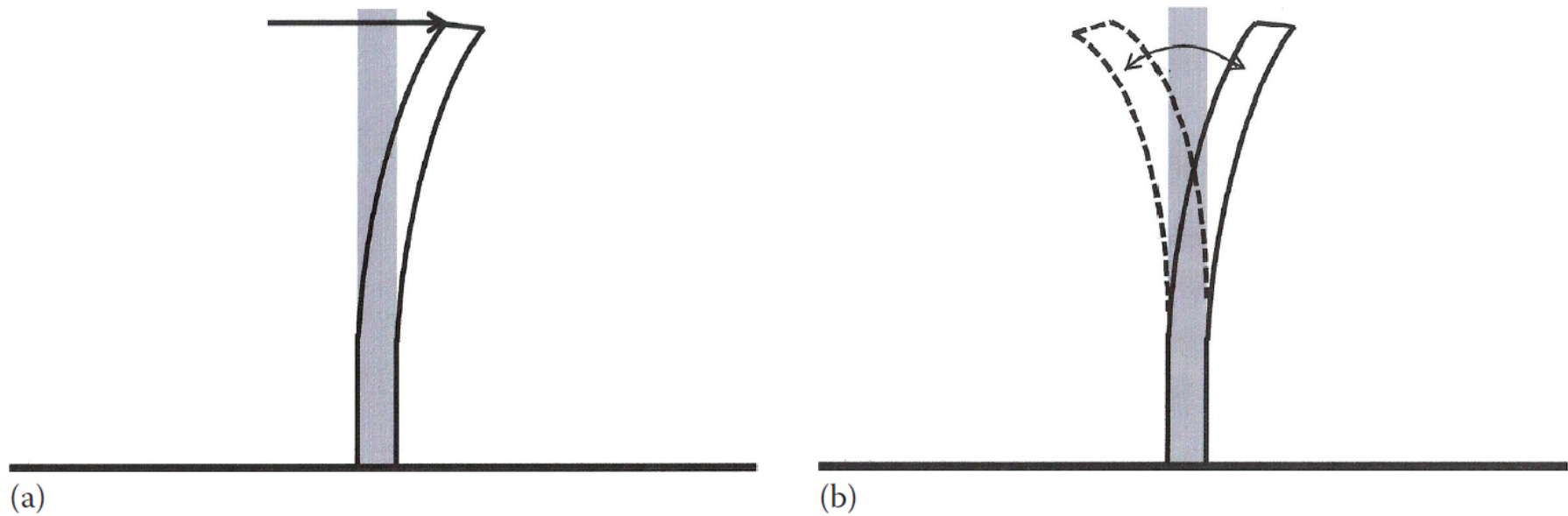
$$\int_0^\pi P_n(\cos \theta) P_m(\cos \theta) \sin \theta d\theta = \frac{2}{2n+1} \quad m = n$$

9.4 Beam Equation

- The beam equation differs from the other examples in that it is a fourth-order equation and its boundary conditions require a different solution approach.

9.4.1 Deriving the Beam Equation

Consider a cantilever beam of length h and uniform cross section initially depressed with a point load at the end and then released to vibrate freely (Figure 9.14).



- **Figure 9.14** Motion of cantilever beam: (a) beam shape prior to release and (b) beam shape after release.

9.4.1 Deriving the Beam Equation

The formulation presented here is due to Leonhard Euler and Daniel Bernoulli. The force along the beam is given by

$$\frac{\partial F}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2}, \quad (120)$$

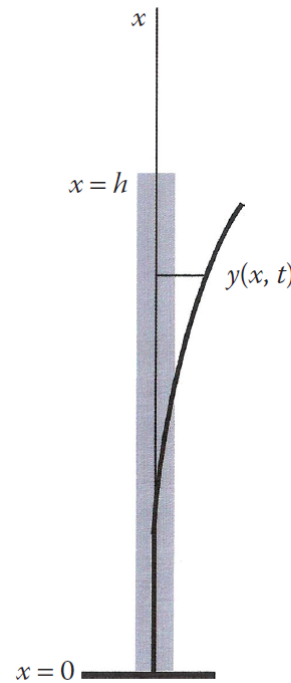
where:

y is the beam deflection (displacement of the beam from vertical)

$\rho = m/h$

m is the mass of the beam. See [Figure 9.15](#).

9.4.1 Deriving the Beam Equation



- **Figure 9.15** Definition of variables in analysis of motions of a cantilever beam.

9.4.1 Deriving the Beam Equation

The increase in torque along the beam is given by

$$\frac{\partial \tau}{\partial x} = -F. \quad (121)$$

After considerable analysis of the interior of the beam, one arrives at

$$\tau = EI \frac{\partial^2 y}{\partial x^2}, \quad (122)$$

where:

E is Youngs modulus for the beam material.

I is the beams cross-sectional moment of inertia.

9.4.1 Deriving the Beam Equation

Combining Equations (120), (121), and (122), we arrive at

$$\frac{\partial^4 y}{\partial x^4} = - \left(\frac{\rho}{EI} \right) \frac{\partial^2 y}{\partial t^2}. \quad (123)$$

- Equation (123) is the EulerBernoulli beam equation for a uniform cantilever beam in free vibration.

9.4.2 Beam Equation in A Boundary Value Problem

Suppose, as shown in [Figure 9.14](#), the initial external force on the beam is a point load at its end. Then the initial deflection along the beam is

$$y(x, 0) = \frac{P}{6EI}(3hx^2 - x^3) \quad (124)$$

We assume that the beam is in steady state prior to its release, so that

$$\frac{\partial y}{\partial t}(x, 0) = 0 \quad (125)$$

What are the motions of the beam after its release?

9.4.2 Beam Equation in A Boundary Value Problem

Equation (123) with initial conditions Equations (124) and (125) together with boundary conditions (for $t > 0$) whose statement requires a knowledge of solid mechanics:

- The base is held stationary:

$$y(0, t) = 0. \quad (126)$$

- The beam remains vertical at its base:

$$\frac{\partial y}{\partial x}(0, t) = 0. \quad (127)$$

9.4.2 Beam Equation in A Boundary Value Problem

- There is no torque at the tip:

$$\frac{\partial^2 y}{\partial x^2}(h, t) = 0. \quad (128)$$

- There is no shear stress at the tip:

$$\frac{\partial^3 y}{\partial x^3}(h, t) = 0. \quad (129)$$

9.4.2 Beam Equation in A Boundary Value Problem

To solve Equations (123) through (129) we try separation of variables:

$$y(x, t) = w(x)u(t), \quad (130)$$

and then Equation (123) becomes

$$u \frac{d^4 w}{dx^4} = -w \left(\frac{\rho}{EI} \right) \frac{d^2 u}{dt^2}.$$

Dividing through by $w(x)u(t)$ we have

$$\frac{1}{w} \frac{d^4 w}{dx^4} = -\frac{1}{u} \left(\frac{\rho}{EI} \right) \frac{d^2 u}{dt^2}. \quad (131)$$

9.4.2 Beam Equation in A Boundary Value Problem

- The left side of Equation (131) is a function only of x and the right side a function only of t , hence both sides must equal a constant k .
- To meet boundary conditions this constant must be positive. Then we have two ODEs

$$\frac{d^4 w}{dx^4} - kw = 0. \quad (132)$$

$$\frac{d^2 u}{dt^2} + \left(\frac{kEI}{\rho} \right) u = 0. \quad (133)$$

9.4.2 Beam Equation in A Boundary Value Problem

The solution to Equation (133) is

$$u(t) = A \cos \omega t + B \sin \omega t, \quad (134)$$

where

$$\omega = \sqrt{\frac{kEI}{\rho}}. \quad (135)$$

- From Equation (125) we see that $B = 0$.
- Without loss of generality we can set $A = 1$, since we will deal with the initial condition as we solve Equation (132).

9.4.2 Beam Equation in A Boundary Value Problem

Trying $w(t) = e^{rx}$ in Equation (132) we arrive at the characteristic equation $r^4 - k = 0$, which has solutions

$$r_1 = k^{1/4}, \quad r_2 = -k^{1/4}, \quad r_3 = ik^{1/4}, \quad r_4 = -ik^{1/4}.$$

The fundamental solutions to Equation (132) are

$$\begin{aligned}\varphi_1(x) &= e^{px} \\ \varphi_2(x) &= e^{-px} \\ \varphi_3(x) &= \cos px \\ \varphi_4(x) &= \sin px\end{aligned}\tag{136}$$

where

$$p = k^{1/4}\tag{137}$$

9.4.2 Beam Equation in A Boundary Value Problem

The general solution to Equation (132) can be written as

$$w(x) = \sum_{i=1}^4 a_i \varphi_i(x) \quad (138)$$

The coefficients a_i must be chosen to satisfy the boundary conditions,

$$\begin{aligned} w(0) &= 0 \\ \frac{dw}{dx}(0) &= 0 \\ \frac{d^2w}{dx^2}(h) &= 0 \\ \frac{d^3w}{dx^3}(h) &= 0 \end{aligned} \quad (139)$$

9.4.2 Beam Equation in A Boundary Value Problem

Substituting Equations (136) and (138) into Equation (139), we arrive at the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ e^{ph} & e^{-ph} & -\cos ph & -\sin ph \\ e^{ph} & -e^{-ph} & \sin ph & -\cos ph \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (140)$$

9.4.2 Beam Equation in A Boundary Value Problem

For Equation (140) to be true for nonzero a_i we must have

$$\det \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ e^{ph} & e^{-ph} & -\cos ph & -\sin ph \\ e^{ph} & -e^{-ph} & \sin ph & -\cos ph \end{pmatrix} = 0. \quad (141)$$

After some algebra Equation (141) results in

$$-2(1 + \cos ph \cosh ph) = 0,$$

or

$$\cos ph \cosh ph = -1. \quad (142)$$

9.4.2 Beam Equation in A Boundary Value Problem

$$\cosh ph = \frac{e^{ph} + e^{-ph}}{2}. \quad (143)$$

is called the *hyperbolic cosine*.

- Equation (142) defines the eigenvalues for this problem, the particular values of ph satisfying the boundary conditions.
- There are an infinite number of eigenvalues, the first four given by $p_1h = 1.87510$, $p_2h = 4.69409$, $p_3h = 7.85476$, and $p_4h = 10.99554$.

9.4.2 Beam Equation in A Boundary Value Problem

We selected a positive number (k) as the separation constant in Equation (131). If we had selected a negative number then Equation (132) would have become

$$\frac{d^4 w}{dx^4} + kw = 0.$$

- Its solutions could not have met the boundary conditions in (139).

9.4.2 Beam Equation in A Boundary Value Problem

Substituting solutions of Equation (142) into Equation (140) and applying the results in Equation (138) provides an infinite number of eigenfunctions,

$$w_n(x) = \cosh p_n x - \cos p_n x - \sigma_n (\sinh p_n x - \sin p_n x), \quad (144)$$

where

$$\sigma_n = \frac{\cosh p_n h + \cos p_n h}{\sinh p_n h + \sin p_n h}, \quad (145)$$

and

$$\sinh z = \frac{e^z - e^{-z}}{2}. \quad (146)$$

is the *hyperbolic sine*.

9.4.2 Beam Equation in A Boundary Value Problem

Via Equations (135), (137), and (142), the allowed frequencies of vibration in time are also distinct and infinite in number:

$$\omega_n = \left(\frac{EI}{\rho h^4} \right)^{1/2} (p_n h)^2. \quad (147)$$

To summarize results to this point, we tried the solution $y(x, t) = w(x)u(t)$ and found an infinite number of solutions. Set

$$y(x, t) = \sum_{n=1}^{\infty} c_n w_n(x) u_n(t), \quad (148)$$

and determine c_n through the initial condition.

9.4.2 Beam Equation in A Boundary Value Problem

We must have

$$y(x, 0) = \sum_{n=1}^{\infty} c_n w_n(x) = \frac{P}{6EI}(3hx^2 - x^3) \quad (149)$$

The eigenfunctions $w_n(x)$ are orthogonal to one another (We will prove in the next section), so we can determine the coefficient c_n via

$$c_n = \frac{\int_0^h w_n(x) y(x, 0) dx}{\int_0^h w_n^2(x) dx} \quad (150)$$

9.4.2 Beam Equation in A Boundary Value Problem

After some simplification(Challenge Problem CP9.4 addresses), we ultimately find that

$$c_n = \left(\frac{2Ph^3}{EI} \right) \frac{(-1)^{n+1}}{(p_n h)^4} \quad (151)$$

- To check the results, we can evaluate $y(x, t)$ as given by Equations (142) through (150) at the points $x = h$ and $t = 0$ and compare to the initial condition.

At the point $x = h$ the initial condition is

$$y(h, 0) = \frac{P}{6EI}(3h(h)^2 - h^3) = \frac{Ph^3}{3EI} \quad (152)$$

9.4.2 Beam Equation in A Boundary Value Problem

Using

$$y(h, 0) = \sum_{n=1}^{\infty} c_n w_n(h) u_n(0), \quad (153)$$

we have

$$u_n(0) = 1, \quad (154)$$

and again from the results of Challenge Problem CP9.4

$$w_n(h) = 2(-1)^{n+1}. \quad (155)$$

Then

$$y(h, 0) = \sum_{n=1}^{\infty} c_n w_n(h) u_n(0) = \left(\frac{4Ph^3}{EI} \right) \sum_{n=1}^{\infty} \frac{1}{(p_n h)^4}. \quad (156)$$

9.4.2 Beam Equation in A Boundary Value Problem

Equating (152) and (156) we see that we must have

$$\sum_{n=1}^{\infty} \frac{1}{(p_n h)^4} = \frac{1}{12}. \quad (157)$$

9.4.2 Beam Equation in A Boundary Value Problem

Table 9.2 records the partial sums and shows that the series does indeed appear to be converging to the correct value.

TABLE 9.2
Checking the Accuracy of the Beam Equation Solution

n	$p_n h$	$1/(p_n h)^4$	$\sum_{m=1}^n 1/(p_m h)^4$	$\left(\sum_{m=1}^n 1/(p_m h)^4\right)^{-1}$
1	1.87510	0.0808914	0.0808914	12.3623
2	4.69409	0.0020596	0.0829510	12.0553
3	7.85476	0.0002627	0.0832137	12.0173
4	10.99554	0.0000684	0.0832821	12.0074
5	14.13717	0.0000250	0.0833071	12.0038
6	17.27876	0.0000112	0.0833183	12.0022
7	20.42035	0.0000058	0.0833241	12.0013
8	23.56194	0.0000032	0.0833273	12.0009
9	26.70354	0.0000020	0.0833293	12.0006
10	29.84513	0.0000013	0.0833306	12.0004

9.4.3 Orthogonality of The Beam Equation Eigenfunctions

Theorem:

The eigenfunctions $w_n(x)$ in the solution $y(x, t) = \sum_{n=1}^{\infty} c_n w_n(x) u_n(t)$ to the beam equation

$$\frac{\partial^4 y}{\partial x^4} = - \left(\frac{\rho}{EI} \right) \frac{\partial^2 y}{\partial t^2},$$

with boundary conditions

$$y(0, t) = 0, \quad \frac{\partial^2 y}{\partial x^2}(h, t) = 0,$$

$$\frac{\partial y}{\partial x}(0, t) = 0, \quad \frac{\partial^3 y}{\partial x^3}(h, t) = 0,$$

are orthogonal.

9.4.3 Orthogonality of The Beam Equation Eigenfunctions

Proof: Separating variables as we did in Equations (130) and (131) led to Equation (132):

$$\frac{d^4 w}{dx^4} - kw = 0.$$

For two different eigenfunction solutions we can write

$$\frac{d^4 w_n}{dx^4} = k_n w_n, \quad \frac{d^4 w_m}{dx^4} = k_m w_m.$$

9.4.3 Orthogonality of The Beam Equation Eigenfunctions

Multiplying the first equation by w_m and the second by w_n , subtracting the second equation from the first and integrating over the length of the beam we have

$$\int_0^h \left(w_m \frac{d^4 w_n}{dx^4} - w_n \frac{d^4 w_m}{dx^4} \right) dx = (k_n - k_m) \int_0^h w_n(x) w_m(x) dx. \quad (158)$$

Integrating by parts on the left-hand side, letting $w_i = u$ and $(d^4 w_j / dx^4) dx = dv$:

$$\begin{aligned} \int_0^h \left(w_m \frac{d^4 w_n}{dx^4} - w_n \frac{d^4 w_m}{dx^4} \right) dx &= \left(w_m \frac{d^3 w_n}{dx^3} - w_n \frac{d^3 w_m}{dx^3} \right) \Big|_0^h \\ &\quad - \int_0^h \left(\frac{dw_m}{dx} \frac{d^3 w_n}{dx^3} - \frac{dw_n}{dx} \frac{d^3 w_m}{dx^3} \right) dx. \end{aligned}$$

9.4.3 Orthogonality of The Beam Equation Eigenfunctions

Integrating the last term by parts, letting $dw_i/dx = u$ and $(d^3w_j/dx^3)dx = dv$:

$$\begin{aligned} \int_0^h \left(w_m \frac{d^4 w_n}{dx^4} - w_n \frac{d^4 w_m}{dx^4} \right) dx &= \left(w_m \frac{d^3 w_n}{dx^3} - w_n \frac{d^3 w_m}{dx^3} \right) \Big|_0^h \\ &- \left(\frac{dw_m}{dx} \frac{d^2 w_n}{dx^2} - \frac{dw_n}{dx} \frac{d^2 w_m}{dx^2} \right) \Big|_0^h + \int_0^h \left(\frac{d^2 w_m}{dx^2} \frac{d^2 w_n}{dx^2} - \frac{d^2 w_n}{dx^2} \frac{d^2 w_m}{dx^2} \right) dx. \end{aligned} \quad (159)$$

Now

$$\frac{d^2 w_m}{dx^2} \frac{d^2 w_n}{dx^2} - \frac{d^2 w_n}{dx^2} \frac{d^2 w_m}{dx^2} = 0. \quad (160)$$

9.4.3 Orthogonality of The Beam Equation Eigenfunctions

The boundary conditions on $y(x, t)$ imply

$$w(0) = 0, \quad \frac{d^2 w}{dx^2}(h) = 0,$$

$$\frac{dw}{dx}(0) = 0, \quad \frac{d^3 w}{dx^3}(h) = 0,$$

hence

$$\left(w_m \frac{d^3 w_n}{dx^3} - w_n \frac{d^3 w_m}{dx^3} \right) \Big|_0^h - \left(\frac{dw_m}{dx} \frac{d^2 w_n}{dx^2} - \frac{dw_n}{dx} \frac{d^2 w_m}{dx^2} \right) \Big|_0^h = 0. \quad (161)$$

9.4.3 Orthogonality of The Beam Equation Eigenfunctions

From Equations (159) through (161),

$$\int_0^h \left(w_m \frac{d^4 w_n}{dx^4} - w_n \frac{d^4 w_m}{dx^4} \right) dx = 0 \quad (162)$$

Hence from Equation (158),

$$(k_n - k_m) \int_0^h w_n(x) w_m(x) dx = 0. \quad (163)$$

Since the eigenvalues k_n and k_m are distinct,

$$\int_0^h w_n(x) w_m(x) dx = 0,$$

as was to be shown.

9.4.3 Orthogonality of The Beam Equation Eigenfunctions

- Equation 9.162 is Lagranges identity in the form it takes for the beam problem.
- For a broad class of boundary value problems Lagranges identity can be constructed and its left-hand side shown to be zero.
- As long as eigenvalues are distinct, Lagranges identity then shows that the eigenfunctions are orthogonal.
- The infinite series eigenfunction solutions for a broad class of boundary value problems can be shown to converge to the initial condition, with only mild limitations on the initial condition.

9.4.4 Nonhomogenous Boundary Value Problems

- Challenge Problems CP9.5 and CP9.6 are examples of nonhomogeneous PDEs in boundary value problems, considering cantilevered beams driven to oscillate.
- The results demonstrate that if forces are applied in the rhythm of one of the beams natural frequencies, a resonance occurs.
- A beam can be regarded as a system of infinite dimensions, responding to inputs in ways similar to that exhibited by finite dimensional systems.

9.4.5 Other Geometries

- Vibrations of a circular flat metal plate clamped at its circumference are governed by a PDE with variable coefficients that is fourth order in the radial spatial dimension and second order in time.
- The solutions involve Bessel functions.

END