

This is a open-book exam. Submission due date is **12 pm, noon, March. 9th, 2021**. Late submission will not be accepted. If you need more space, please feel free to attach additional papers. Once you're finished, scan and upload it to Canvas course website.

Honor Pledge

Please review the following honor code, then sign your name and write down the date.

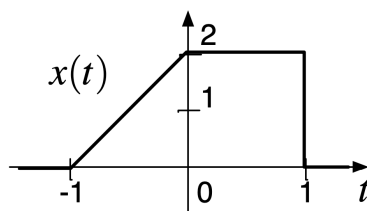
1. I pledge that the answers in this exam are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,
 - (a) I will not plagiarize (copy without citation) from any source;
 - (b) I will not communicate or attempt to communicate with any other person during the exam;
 - (c) neither will I give or attempt to give assistance to another student taking the exam; and
 - (d) I will use only approved devices (e.g., calculators) and/or approved device models.

2. I understand that any act of academic dishonesty can lead to disciplinary action.

Signature

Date

1. (10 points) Consider signal $x(t)$ as plotted in the figure below.

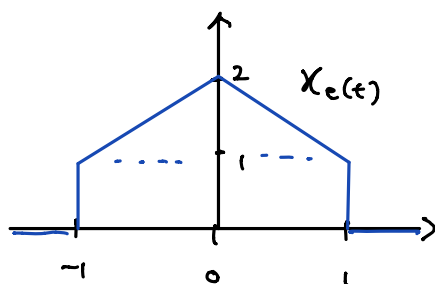


- (4 pts) (a) Plot the even part $x_e(t)$ and odd part $x_o(t)$ of the given signal $x(t)$.

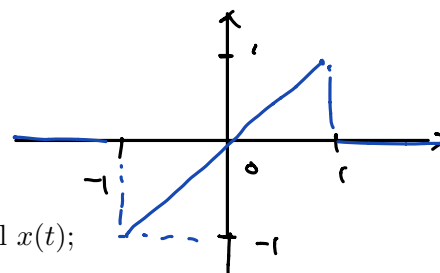
Solution)

a) Since $x(-t)$ is given by

$$x_e(t) = \frac{1}{2}(x(t) + x(-t))$$



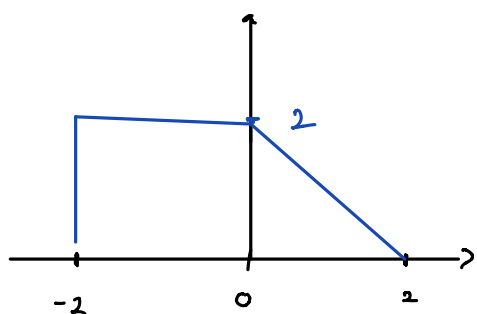
$$x_o(t) = \frac{1}{2}(x(t) - x(-t))$$



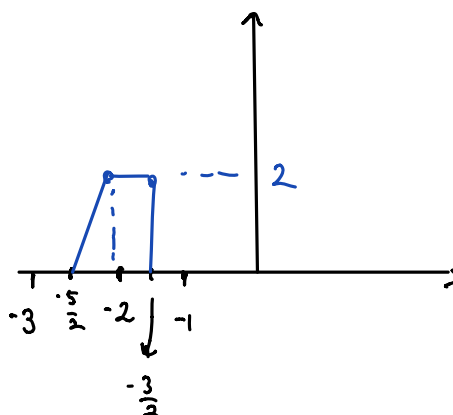
- (6 pts) (b) Plot the following functions derived from the given signal $x(t)$;

- $x(-\frac{t}{2})$
- $x(2t+4)$
- $x(-t+1)$

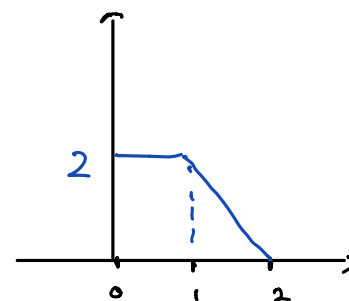
Solution) $x(-\frac{t}{2})$



$x(2t+4)$



$x(-t+1)$



2. (10 points) Determine the Fourier Series (FS) representation of the following signal $x(t)$

$$x(t) = \begin{cases} \cos(w_0 t), & \text{for } -\frac{\pi}{2w_0} < t \leq \frac{\pi}{2w_0}, \\ 0, & \text{for } -\frac{\pi}{w_0} < t \leq -\frac{\pi}{2w_0} \quad \text{or} \quad \frac{\pi}{2w_0} < t \leq \frac{\pi}{w_0}, \end{cases}$$

where $x(t + T_0) = x(t)$, $T_0 = \frac{2\pi}{w_0}$, and $w_0 > 0$ is a real-valued constant number.

- (6pts) (a) Derive the complex exponential FS coefficients c_0 , c_k of the given signal $x(t)$
(Hint: Use the integral table provided in the Lecture note at the end of Chapter 3.)

- (4pts) (b) Derive the trigonometric FS coefficients a_0 , a_k , and b_k of the signal $x(t)$

(Answer Page for Question 2)

Solution) a) $C_0 = \frac{1}{T_0} \int_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}} \cos(\omega_0 t) dt = \frac{1}{\omega_0 T_0} \sin(\omega_0 t) \Big|_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}}$

$$= \frac{1}{2\pi} \left(\underbrace{\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)}_{=1 - (-1) = 2} \right) = \frac{1}{\pi}$$

$$C_k = \frac{1}{T_0} \int_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}} \cos(\omega_0 t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T_0} \left[\underbrace{\frac{(-jk\omega_0) \cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{(-jk\omega_0)^2 + \omega_0^2} e^{-jk\omega_0 t}}_{\text{blue arrow}} \right]_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}}$$

$$\left(= \left(\frac{\omega_0}{\omega_0^2 - (\omega_0 k)^2} \right) e^{-jk\frac{\pi}{2}} - \left(\frac{-\omega_0}{\omega_0^2 - (\omega_0 k)^2} \right) e^{jk\frac{\pi}{2}} \right)$$

$$= \frac{1}{2\pi} \cdot \frac{1}{(1 - k^2)} \cdot \left(e^{-jk\frac{\pi}{2}} + e^{jk\frac{\pi}{2}} \right) = \frac{\cos\left(\frac{k\pi}{2}\right)}{\pi(1 - k^2)}$$

$$\Rightarrow \begin{cases} \text{even } k = 2m, & \frac{(-1)^m}{\pi(1 - k^2)} \\ \text{odd } k = 2m+1, & \emptyset \end{cases}$$

$$b) a_0 = 2C_0 = \frac{2}{\pi}$$

$$A_k = 2\operatorname{Re}(C_k) = \begin{cases} \text{even } k = 2m, & \frac{2(-1)^m}{\pi(1 - k^2)} \\ \text{odd } k = 2m+1, & \emptyset \end{cases} \quad \text{or} \quad \Rightarrow \frac{2\cos\left(\frac{k\pi}{2}\right)}{\pi(1 - k^2)}$$

$$b_k = -2\operatorname{Im}(C_k) = \emptyset$$

3. (30 points) Evaluate the convolution integral of the following signals. Provide the

(10 pts) (a) $x(t) = \cos(w_0 t)$, $y(t) = e^{-\alpha|t|}$, where α and w_0 are real-valued constants, $\alpha > 0$, $w_0 > 0$.

$$\Rightarrow x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau$$

(10 pts) (b) $x(t) = \sin(w_1 t) U(t)$, $y(t) = \sin(w_2 t) U(t)$, where $U(t)$ is a step function, w_1 and w_2 are real-valued constants, $w_1 > 0$, $w_2 > 0$, and $w_1 \neq w_2$.

$$\Rightarrow x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau$$

(10 pts) (c) $x(t)$ and $y(t)$ are periodic signals with the fundamental period T_0

$$x(t) = \sin(w_0 t), \quad y(t) = \cos(w_0 t) \quad \text{for} \quad -\frac{\pi}{w_0} < t \leq \frac{\pi}{w_0}, \quad w_0 = \frac{2\pi}{T_0},$$

$$x(t) = x(t + T_0), \quad y(t) = y(t + T_0),$$

where w_0 is a real-valued constant, $w_0 > 0$. Evaluate the periodic convolution $x(t) \circledast y(t)$.

$$\Rightarrow x(t) \circledast y(t) = \int_{-\frac{\pi}{w_0}}^{\frac{\pi}{w_0}} x(\tau) y(t - \tau) d\tau$$

(Hint: Use the integral table provided in the Lecture note at the end of Chapter 3.)

(Answer Page for Question 3)

Solution) a) $x(t) = \cos(\omega_0 t)$, $y(t) = e^{-\alpha|t|}$

$$\Rightarrow x(t) * y(t) = \int_{-\infty}^{\infty} e^{-\alpha|\tau|} \underbrace{\cos(\omega_0(t-\tau))}_{= \cos(\omega_0 t) \cos(\omega_0 \tau) + \sin(\omega_0 t) \sin(\omega_0 \tau)} d\tau$$

$$= \underbrace{\cos(\omega_0 t) \int_{-\infty}^{\infty} e^{-\alpha|\tau|} \cos(\omega_0 \tau) d\tau}_{(1)} + \underbrace{\sin(\omega_0 t) \int_{-\infty}^{\infty} e^{-\alpha|\tau|} \sin(\omega_0 \tau) d\tau}_{(2)}$$

Integral (1) $\Rightarrow \int_0^{\infty} e^{-\alpha\tau} \cos(\omega_0 \tau) d\tau + \int_{-\infty}^0 e^{\alpha\tau} \cos(\omega_0 \tau) d\tau$

$$= \left(\frac{(-\alpha) \cos(\omega_0 \tau) + \omega_0 \sin(\omega_0 \tau)}{(-\alpha)^2 + \omega_0^2} \right) e^{-\alpha\tau} \Big|_0^{\infty} + \left(\frac{\alpha \cos(\omega_0 \tau) + \omega_0 \sin(\omega_0 \tau)}{\alpha^2 + \omega_0^2} \right) e^{\alpha\tau} \Big|_{-\infty}^0$$

$$= \frac{\alpha}{\alpha^2 + \omega_0^2} + \frac{\alpha}{\alpha^2 + \omega_0^2} = \frac{2\alpha}{\alpha^2 + \omega_0^2}$$

Integral (2) $\Rightarrow \int_0^{\infty} e^{-\alpha\tau} \sin(\omega_0 \tau) d\tau + \int_{-\infty}^0 e^{\alpha\tau} \sin(\omega_0 \tau) d\tau$

$$= \left(\frac{(-\alpha) \sin(\omega_0 \tau) - \omega_0 \cos(\omega_0 \tau)}{(-\alpha)^2 + \omega_0^2} \right) e^{-\alpha\tau} \Big|_0^{\infty} + \left(\frac{\alpha \sin(\omega_0 \tau) - \omega_0 \cos(\omega_0 \tau)}{\alpha^2 + \omega_0^2} \right) e^{\alpha\tau} \Big|_{-\infty}^0$$

$$= \frac{\omega_0}{\alpha^2 + \omega_0^2} - \frac{\omega_0}{\alpha^2 + \omega_0^2} = 0.$$

Therefore, $x(t) * y(t) = \frac{2\alpha}{\alpha^2 + \omega_0^2} \cos(\omega_0 t)$

Sol) Q3 - b) $x(t) = \sin(\omega_1 t) u(t)$, $y(t) = \sin(\omega_2 t) u(t)$

$$x(t) * y(t) = \int_{-\infty}^{\infty} \sin(\omega_1 \tau) u(\tau) \sin(\omega_2 (t-\tau)) u(t-\tau) d\tau$$

$$= \int_0^t \sin(\omega_1 \tau) \sin(\omega_2 (t-\tau)) d\tau$$

if $t \geq 0$

if $t < 0$,
then $x(t) * y(t) = 0$

$$= \sin(\omega_2 t) \cos(\omega_2 \tau) - \cos(\omega_2 t) \sin(\omega_2 \tau)$$

$$\Rightarrow \sin(\omega_2 t) \int_0^t \sin(\omega_1 \tau) \cos(\omega_2 \tau) d\tau - \cos(\omega_2 t) \int_0^t \sin(\omega_1 \tau) \sin(\omega_2 \tau) d\tau$$

$$= \frac{1}{2} (\sin(\omega_1 + \omega_2) \tau + \sin(\omega_1 - \omega_2) \tau) = \frac{1}{2} (\cos(\omega_1 - \omega_2) \tau - \cos(\omega_1 + \omega_2) \tau)$$

$$\Rightarrow \frac{1}{2} \sin(\omega_2 t) \left[\frac{1}{\omega_1 + \omega_2} \cos(\omega_1 + \omega_2) \tau \Big|_0^t + \frac{1}{\omega_1 - \omega_2} \cos(\omega_1 - \omega_2) \tau \Big|_0^t \right]$$

$$- \frac{1}{2} \cos(\omega_2 t) \left[\frac{1}{\omega_1 - \omega_2} \sin(\omega_1 - \omega_2) \tau \Big|_0^t - \frac{1}{\omega_1 + \omega_2} \sin(\omega_1 + \omega_2) \tau \Big|_0^t \right]$$

$$= \sin(\omega_1 - \omega_2) t = \sin \omega_1 t \cdot \cos \omega_2 t - \sin \omega_2 t \cdot \cos \omega_1 t$$

$$= \sin(\omega_1 + \omega_2) t = \sin \omega_1 t \cdot \cos \omega_2 t + \cos \omega_1 t \cdot \sin \omega_2 t$$

$$\Rightarrow \sin(\omega_2 t) \cdot \left(\frac{\omega_1}{\omega_1^2 - \omega_2^2} \right) - \cos(\omega_1 t) \sin(\omega_2 t) \cos(\omega_2 t) \left(\frac{\omega_1}{\omega_1^2 - \omega_2^2} \right)$$

$$- \sin(\omega_1 t) \sin^2(\omega_2 t) \left(\frac{\omega_2}{\omega_1^2 - \omega_2^2} \right) - \sin(\omega_1 t) \cos^2(\omega_2 t) \left(\frac{\omega_2}{\omega_1^2 - \omega_2^2} \right) \left(\left(\sin^2(\omega_2 t) + \cos^2(\omega_2 t) \right) = 1 \right)$$

$$+ \cos(\omega_1 t) \sin(\omega_2 t) \cos(\omega_2 t) \left(\frac{\omega_1}{\omega_1^2 - \omega_2^2} \right)$$

Therefore

$$\Rightarrow x(t) * y(t) = \left[\frac{\omega_1}{\omega_1^2 - \omega_2^2} \sin(\omega_2 t) - \frac{\omega_2}{\omega_1^2 - \omega_2^2} \sin(\omega_1 t) \right] u(t)$$

Sol) Q3 - c) $x(t) = \sin(\omega_0 t)$, $y(t) = \cos(\omega_0 t)$

$$x(t) \otimes y(t) = \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \sin(\omega_0 \tau) \underbrace{\cos(\omega_0(t-\tau))}_{= \cos(\omega_0 t) \cos(\omega_0 \tau) + \sin(\omega_0 t) \sin(\omega_0 \tau)} d\tau$$

$$\Rightarrow \cos(\omega_0 t) \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \underbrace{\sin(\omega_0 \tau) \cos(\omega_0 \tau)}_{= \frac{1}{2} \sin(2\omega_0 \tau)} d\tau + \sin(\omega_0 t) \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \underbrace{\sin^2(\omega_0 \tau)}_{= \frac{1}{2}(1 - \cos(2\omega_0 \tau))} d\tau$$

$$= \frac{1}{2} \cos(\omega_0 t) \left[\underbrace{\frac{1}{2\omega_0} \cos(2\omega_0 \tau)}_{\cancel{\phi}} \Big|_{\frac{\pi}{\omega_0}}^{-\frac{\pi}{\omega_0}} \right] + \frac{1}{2} \sin(\omega_0 t) \left[\frac{2\tau}{\omega_0} - \underbrace{\frac{1}{2\omega_0} \sin(2\omega_0 \tau)}_{\cancel{\phi}} \Big|_{\frac{\pi}{\omega_0}}^{-\frac{\pi}{\omega_0}} \right]$$

Therefore ,

$$\Rightarrow x(t) \otimes y(t) = \frac{\pi}{\omega_0} \sin(\omega_0 t)$$

4. (25 points) Calculate the Fourier Series coefficients of a signal $x(t)$, defined as follows

$$\begin{cases} x(t) = t - \frac{1}{\pi^2} t^3, & \text{for } 0 < t \leq \pi, \\ x(t + \pi) = x(t). \end{cases}, \quad \omega_0 = \frac{2\pi}{\pi} = 2$$

(20 pts)

- (a) Derive the complex exponential FS coefficients c_0, c_k of the given signal $x(t)$

Sol)

$$\begin{aligned} a) \quad \frac{1}{\pi} \int_0^{\pi} t^3 \frac{e^{-j2kt}}{j} dt &= \frac{1}{\pi} \left[\frac{t^3 e^{-j2kt}}{(-j2k)} \Big|_0^{\pi} - \frac{3}{(-j2k)} \int_0^{\pi} t^2 e^{-j2kt} dt \right] \\ &= \frac{1}{\pi} \left[\frac{\pi^3}{(-j2k)} - \frac{3}{(-j2k)} \left\{ \frac{t^2 e^{-j2kt}}{(-j2k)} \Big|_0^{\pi} - \frac{2}{(-j2k)} \int_0^{\pi} t e^{-j2kt} dt \right\} \right] \\ &= \frac{1}{\pi} \left[\frac{\pi^3}{(-j2k)} - \frac{3}{(-j2k)} \left(\frac{\pi^2}{(-j2k)} \right) + \frac{6}{(-j2k)^2} \left\{ \frac{t e^{-j2kt}}{(-j2k)} \Big|_0^{\pi} - \frac{1}{(-j2k)} \int_0^{\pi} e^{-j2kt} dt \right\} \right] \\ &= \frac{1}{\pi} \left[\frac{\pi^3}{(-j2k)} - \frac{3}{(-j2k)^2} \pi^2 + \frac{6\pi}{(-j2k)^3} \right] \\ \frac{1}{\pi} \int_0^{\pi} t e^{-j2kt} dt &= \frac{1}{\pi} \left\{ \frac{t e^{-j2kt}}{(-j2k)} \Big|_0^{\pi} - \frac{1}{(-j2k)} \int_0^{\pi} e^{-j2kt} dt \right\} = \frac{1}{(-j2k)} \end{aligned}$$

(5 pts)

- (b) Derive the trigonometric FS coefficients a_0, a_k , and b_k of the signal $x(t)$

a) (continue)

$$\Rightarrow c_k = \frac{1}{\pi} \int_0^{\pi} \left(t - \frac{1}{\pi^2} t^3 \right) e^{-j2kt} dt$$

$$= \frac{1}{(-j2k)} - \left(\frac{1}{(-j2k)} - \frac{3}{\pi} \frac{1}{(-j2k)^2} + \frac{6}{\pi^2} \frac{1}{(-j2k)^3} \right)$$

$$= \frac{3}{\pi} \frac{1}{(-j2k)^2} - \frac{6}{\pi^2} \frac{1}{(-j2k)^3}$$

$$= -\frac{3}{4k^2\pi} + \frac{3}{4k^3\pi^2} j$$

$$c_0 = \frac{1}{\pi} \int_0^{\pi} \left(t - \frac{1}{\pi^2} t^3 \right) dt = \frac{1}{\pi} \left[\frac{1}{2} t^2 - \frac{1}{4\pi^2} t^4 \right]_0^{\pi} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

(Answer Page for Question 4)

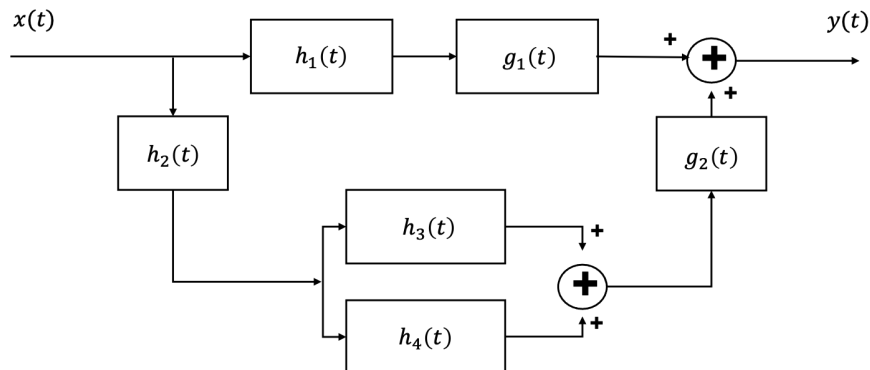
Sol) Q4 - b)

$$a_0 = 2C_0 = \frac{\pi}{2}$$

$$a_k = 2\operatorname{Re}(C_k) = -\frac{3}{2k^2\pi}$$

$$b_k = -2\operatorname{Im}(C_k) = \frac{-3}{2k^3\pi^2}$$

5. (25 points) Consider the interconnection of six LTI systems, as depicted in the following figure.



The impulse responses of each component are given by

$$h_1(t) = \text{tri}(t), \quad h_2(t) = \text{rect}(t), \quad h_3(t) = U(t + 0.5), \\ h_4(t) = -U(t - 0.5), \quad g_1(t) = \delta(t + 0.5), \quad g_2(t) = \delta(t - 0.5),$$

where $\text{tri}(t) = 1 - |t|$ for $|t| < 1$ and 0 otherwise, $\text{rect}(t)$ is the rectangular pulse signal, $U(t)$ is the unit step function, and $\delta(t)$ is the unit impulse function.

- (15 pts) (a) Find the impulse response $h(t)$ of the overall system.

Sol) a) $h(t) = h_1(t) * g_1(t) + h_2(t) * (h_3(t) + h_4(t)) * g_2(t)$

where $h_1(t) * g_1(t) = \text{tri}(t) * \delta(t + 0.5) = \text{tri}(t + 0.5)$

$h_3(t) + h_4(t) = U(t + 0.5) - U(t - 0.5) = \text{rect}(t)$

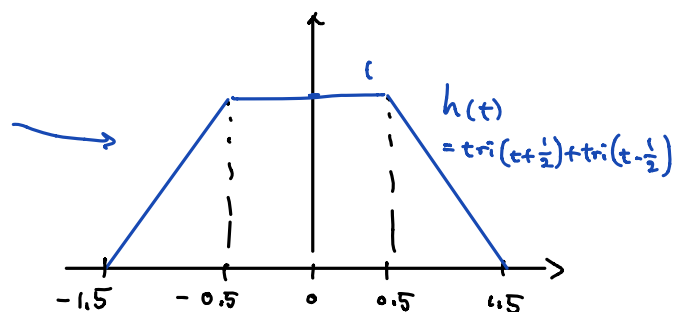
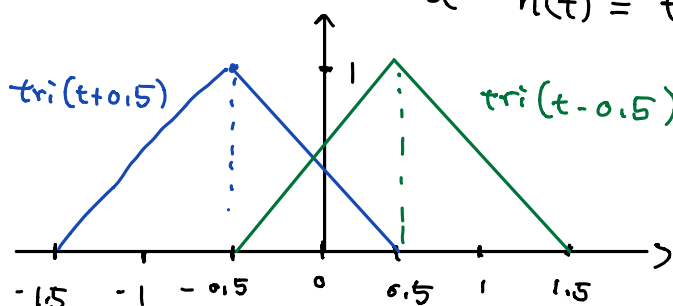
$h_2(t) * (h_3(t) + h_4(t)) = \text{rect}(t) * \text{rect}(t) = \text{tri}(t)$

- (10 pts) (b) Check if the given system $h(t)$ is memoryless, causal and stable.



Therefore $h_2(t) * (h_3(t) + h_4(t)) * g_2(t) = \text{tri}(t - 0.5)$

and $h(t) = \text{tri}(t + 0.5) + \text{tri}(t - 0.5)$



(Answer Page for Question 5)

Solution) (continue)

a)

$$h(t) = \begin{cases} t+1.5 & , \text{ for } -1.5 \leq t < -0.5 \\ 1 & , \text{ for } -0.5 \leq t < 0.5 \\ 1.5-t & , \text{ for } 0.5 \leq t < 1.5 \\ 0 & , \text{ otherwise} \end{cases}$$

b) memory, non-causal, stable system