

EF2452: Mathematics for Economics and Finance

Lecture 6

Part II of II

Department of Economics and Finance
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Main Topics

- Finding critical points of two-variable functions
- Classification of critical points: maxima, minima, and saddle points
- Profit maximization for a firm producing two goods

Critical Points

- When we say that the point (a, b) is a local maximum of a function $f(x, y)$ we mean that

$$f(x, y) \leq f(a, b)$$

for all points (x, y) in the neighborhood of (a, b) .

- When we say that the point (a, b) is a local minimum of a function $f(x, y)$ we mean that

$$f(a, b) \leq f(x, y)$$

for all points (x, y) in the neighborhood of (a, b) .

Critical Points

- A point (x, y) is a critical point if the following first-order conditions hold at the point:

$$\frac{\partial f}{\partial x} = f_1(x, y) = 0$$

$$\frac{\partial f}{\partial y} = f_2(x, y) = 0$$

- These are the necessary conditions for a critical point.

Critical Points

- As an example, consider the function

$$f(x, y) = x^4 + 2x^2y + 2y^2 + y$$

- The partial derivatives of f are:

$$\frac{\partial f}{\partial x} = f_1(x, y) = 4x^3 + 4xy$$

$$\frac{\partial f}{\partial y} = f_2(x, y) = 2x^2 + 4y + 1$$

Critical Points

- The first-order conditions are:

$$\boxed{\frac{\partial f}{\partial x} = 4x^3 + 4xy = 0} \implies x(x^2 + y) = 0$$
$$\implies x = 0 \quad \text{or} \quad y = -x^2$$

$$\boxed{\frac{\partial f}{\partial y} = 2x^2 + 4y + 1 = 0}$$

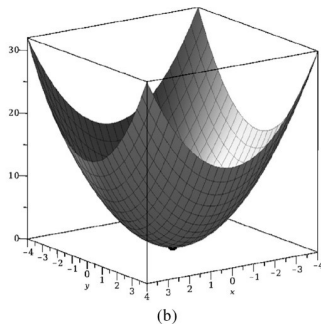
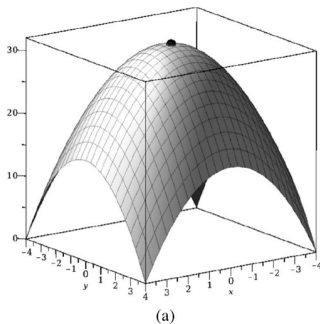
- When $x = 0$, the second equation gives $y = -1/4$. Thus, one critical point is $(0, -1/4)$.
- When $y = -x^2$, the second equation becomes

$$2x^2 - 4x^2 + 1 = 0 \implies 2x^2 = 1 \implies x = 1/\sqrt{2} \quad \text{or} \quad -1/\sqrt{2}$$

In this case, $y = -1/2$. Thus, the other two critical points are $(1/\sqrt{2}, -1/2)$ and $(-1/\sqrt{2}, -1/2)$.

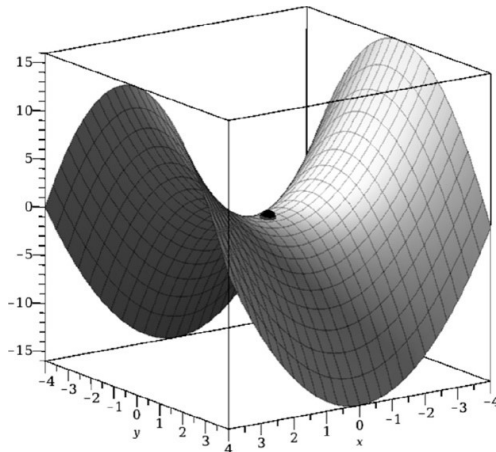
Critical Points

- You may have a local maximum or local minimum:



Critical Points

- You may have neither a local maximum nor a local minimum but a **saddle point**:



Classification of Critical Points

- For any function of two variables, the critical points of f can be classified using f_{11} and $f_{11}f_{22} - f_{12}^2$, according to the following second-order conditions.
- If $f_{11}f_{22} - f_{12}^2 > 0$ and $f_{11} < 0$, it is a maximum.
- If $f_{11}f_{22} - f_{12}^2 > 0$ and $f_{11} > 0$, it is a minimum.
- If $f_{11}f_{22} - f_{12}^2 < 0$, it is a saddle point.

Classification of Critical Points

- As an example, consider again the function

$$f(x, y) = x^4 + 2x^2y + 2y^2 + y$$

- We showed that the critical points are

$$(0, -1/4), \quad (1/\sqrt{2}, -1/2), \quad (-1/\sqrt{2}, -1/2)$$

- To classify them, we need the second partial derivatives of f :

$$f_{11}(x, y) = 12x^2 + 4y \quad f_{12}(x, y) = 4x, \quad f_{22}(x, y) = 4$$

Classification of Critical Points

- At $(0, -1/4)$, we have

$$f_{11}(0, -1/4) = 12(0)^2 + 4(-1/4) = -1$$

$$f_{12}(0, -1/4) = 0$$

$$f_{22}(0, -1/4) = 4$$

$$f_{11}f_{22} - f_{12}^2 = (-1)(4) - 0 = -4 < 0$$

- Therefore, $(0, -1/4)$ is a saddle point.

Classification of Critical Points

- At $(1/\sqrt{2}, -1/2)$, we have

$$f_{11}(1/\sqrt{2}, -1/2) = 12(1/\sqrt{2})^2 + 4(-1/2) = 4$$

$$f_{12}(1/\sqrt{2}, -1/2) = (4)(1/\sqrt{2}) = 2\sqrt{2}$$

$$f_{22}(1/\sqrt{2}, -1/2) = 4$$

$$f_{11}f_{22} - f_{12}^2 = (4)(4) - (2\sqrt{2})^2 = 16 - 8 = 8 > 0$$

- Since, $f_{11}f_{22} - f_{12}^2 > 0$ and $f_{11} > 0$, then $(1/\sqrt{2}, -1/2)$ is a minimum.

Classification of Critical Points

- At $(-1/\sqrt{2}, -1/2)$, we have

$$f_{11}(-1/\sqrt{2}, -1/2) = 12(-1/\sqrt{2})^2 + 4(-1/2) = 4$$

$$f_{12}(-1/\sqrt{2}, -1/2) = (4)(-1/\sqrt{2}) = -2\sqrt{2}$$

$$f_{22}(-1/\sqrt{2}, -1/2) = 4$$

$$f_{11}f_{22} - f_{12}^2 = (4)(4) - (-2\sqrt{2})^2 = 16 - 8 = 8 > 0$$

- Since, $f_{11}f_{22} - f_{12}^2 > 0$ and $f_{11} > 0$, then $(-1/\sqrt{2}, -1/2)$ is also a minimum.

Profit Maximization

- Problems in which we need to optimize a function of several variables occur frequently in economics.
- A simple example occurs when we consider the profit maximization problem for a firm that makes two goods.
- Suppose a firm makes two goods, X and Y . Let x and y denote the quantities of X and Y produced, respectively.
- We need to first write the expressions for revenue and profit as a function of x and y . In order to do so, we have to understand how prices are related to quantities.

Revenue

- The revenue obtained by producing amounts x and y is

$$R(x, y) = xp^X + yp^Y$$

- But how are prices p^X and p^Y affected by x and y ? There are three cases:

Case 1: p^X and p^Y are constants

Case 2: p^X depends only on x and p^Y depends only on y

Case 3: each of p^X and p^Y depends on both x and y
(markets interact)

Case 1: p^X and p^Y are constants

- This happens when there is perfect competition in both markets, X and Y so that the firm is “small” and is a price taker in both markets.
- For example, suppose the selling price of each unit of X is fixed at $p^X = 4$, and the price of each unit of Y is fixed at $p^Y = 1$.
- Then the revenue when the firm produces x units of X and y units of Y is given by

$$R(x, y) = 4x + y$$

- Now, suppose the (joint) cost function of producing x and y is given by

$$C(x, y) = 5 + x^2 - xy + y^2$$

Case 1: p^X and p^Y are constants

- Then, the profit function of producing x and y is given by

$$\Pi(x, y) = R(x, y) - C(x, y)$$

$$\implies \Pi(x, y) = 4x + y - (5 + x^2 - xy + y^2)$$

$$\implies \Pi(x, y) = 4x + y - 5 - x^2 + xy - y^2$$

- To determine the levels of production of each good that will maximize the profit, we shall, therefore, have to find the point (x, y) , which gives the maximum value of the function $\Pi(x, y)$.

Case 2: p^X depends only on x and p^Y depends only on y

- This is the case when the firm has a monopoly in both markets, X and Y , and there is no interaction between the markets for the two goods.
- For example, suppose the demand sets for X and Y are

$$D_X = \{(x, p) | x + 2p = 10\}$$

$$D_Y = \{(y, p) | y + p = 24\}$$

- From the above demand sets, we can write the inverse demand functions for X and Y :

$$p^X(x) = 5 - 0.5x$$

$$p^Y(y) = 24 - y$$

Case 2: p^X depends only on x and p^Y depends only on y

- Then, the profit function of producing x and y is given by

$$\Pi(x, y) = R(x, y) - C(x, y)$$

$$\implies \Pi(x, y) = xp^X + yp^Y - (5 + x^2 - xy + y^2)$$

$$\implies \Pi(x, y) = x(5 - 0.5x) + y(24 - y) - (5 + x^2 - xy + y^2)$$

$$\implies \Pi(x, y) = 5x - 0.5x^2 + 24y - y^2 - 5 - x^2 + xy - y^2$$

$$\implies \Pi(x, y) = 5x + 24y - 1.5x^2 - 2y^2 + xy - 5$$

- To determine the levels of production of each good that will maximize the profit, we shall, therefore, have to find the point (x, y) , which gives the maximum value of the function $\Pi(x, y)$.

Case 3: each of p^X and p^Y depends on both x and y (markets interact)

- Here, we are assuming that the price of X not only affects the quantity demanded in market X but also affects the demand for Y , and vice versa.
- For example, if X and Y are e-books and printed books sold by a publisher for a specific novel, these goods can be viewed as **substitutes**. As the price of e-books falls, the demand for e-books increases, while the demand for printed books decreases.
- On the other hand, if X is a necklace and Y is a matching bracelet, then these goods are **complements**: the increase in the price of one good decreases the demand for its own as well as the other.

Case 3: each of p^X and p^Y depends on both x and y (markets interact)

- For example, consider the following demand for X :

$$x = 2 - 2p^X + p^Y$$

and the following demand for Y :

$$y = 13 + p^X - 2p^Y$$

- See that the law of demand holds: an increase in own price results in a reduction in the demand for its own product; i.e., there is an inverse relationship between x and p^X , and between y and p^Y .
- The goods are substitutes: if the price of X is fixed, an increase in the price of Y results in an increase in the demand for X . Similarly, if p^Y is fixed and p^X rises, then there is an increase in demand for Y .

Case 3: each of p^X and p^Y depends on both x and y (markets interact)

- Rearrange the equations to find expressions for p^X and p^Y in terms of x and y .
- One way is to multiply the first equation (the demand for X) by 2:

$$2x = 4 - 4p^X + 2p^Y$$

Then add the above to the second equation (the demand for Y) to get

$$\begin{aligned} 2x + y &= 17 - 3p^X \\ \implies p^X &= \frac{17 - 2x - y}{3} \end{aligned}$$

Case 3: each of p^X and p^Y depends on both x and y (markets interact)

- In a similar way, multiply the second equation (the demand for Y) by 2:

$$2y = 26 + 2p^X - 4p^Y$$

Then add the above to the first equation (the demand for X) to get

$$\begin{aligned}x + 2y &= 28 - 3p^Y \\ \implies p^Y &= \frac{28 - x - 2y}{3}\end{aligned}$$

Case 3: each of p^X and p^Y depends on both x and y (markets interact)

- Then, the profit function of producing x and y is given by

$$\Pi(x, y) = R(x, y) - C(x, y)$$

$$\implies \Pi(x, y) = xp^X + yp^Y - (5 + x^2 - xy + y^2)$$

$$\implies \Pi(x, y) = x\left(\frac{17 - 2x - y}{3}\right) + y\left(\frac{28 - x - 2y}{3}\right) - (5 + x^2 - xy + y^2)$$

$$\implies \Pi(x, y) = -5 + \frac{17}{3}x + \frac{28}{3}y - \frac{5}{3}x^2 - \frac{5}{3}y^2 + \frac{1}{3}xy$$

- To determine the levels of production of each good that will maximize the profit, we shall, therefore, have to find the point (x, y) , which gives the maximum value of the function $\Pi(x, y)$.

Profit Maximization: An Example

- Consider the profit function of producing x and y from the example in Case 2:

$$\Pi(x, y) = 5x + 24y - 1.5x^2 - 2y^2 + xy - 5$$

- The first-order conditions are (find the partial derivatives w.r.t x and y , respectively, and set them equal to zero)

$$\Pi_1(x, y) = 5 - 3x + y = 0$$

$$\Pi_2(x, y) = 24 - 4y + x = 0$$

$$\implies 3x - y = 5$$

$$\implies -x + 4y = 24$$

- We need to find the x and y that satisfies the above two linear equations. You may use the method of substitution or multiply the first equation by 4 and add to the second equation to obtain

$$12x - 4y - x + 4y = 20 + 24 \implies 11x = 44 \implies x = 4$$

Profit Maximization: An Example

- Then $y = (3)(4) - 5 = 7$. Therefore, the profit function has a critical point at $(4, 7)$.
- To classify the critical point, we use the second-order conditions. We have

$$\Pi_{11} = -3, \quad \Pi_{12} = 1, \quad \Pi_{22} = -4$$

$$\Pi_{11}\Pi_{22} - \Pi_{12}^2 = (-3)(-4) - 1 = 11$$

- Since $\Pi_{11} < 0$ and $\Pi_{11}\Pi_{22} - \Pi_{12}^2 > 0$, the critical point is a maximum, and the maximum profit is $\Pi(4, 7) = 89$.

References

Following chapters of the textbook:

13 Optimisation in Two Variables