

MA1201 Calculus and Basic Linear Algebra II

Solution of Problem Set 2

Problem 1

(a) Let $y = 1 + \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = -\frac{2}{x^3} \Rightarrow dx = -\frac{x^3}{2} dy$.

Then the integral becomes

$$\int \frac{e^{1+\frac{1}{x^2}}}{x^3} dx = \int \frac{e^{1+\frac{1}{x^2}}}{x^3} \left(-\frac{x^3}{2} dy \right) = -\frac{1}{2} \int e^y dy = -\frac{1}{2} e^y + C = -\frac{1}{2} e^{1+\frac{1}{x^2}} + C.$$

(b) Let $y = 1 - 2x^3 \Rightarrow \frac{dy}{dx} = -6x^2 \Rightarrow dx = -\frac{1}{6x^2} dy$.

Then the integral becomes

$$\begin{aligned} \int x^2 \sec(1 - 2x^3) dx &= \int x^2 \sec(1 - 2x^3) \left(-\frac{1}{6x^2} dy \right) = -\frac{1}{6} \int \sec y dy \\ &= -\frac{1}{6} \ln|\sec y + \tan y| + C = -\frac{1}{6} \ln|\sec(1 - 2x^3) + \tan(1 - 2x^3)| + C. \end{aligned}$$

(c) Let $y = 1 + x^4 \Rightarrow \frac{dy}{dx} = 4x^3 \Rightarrow dx = \frac{1}{4x^3} dy$.

Then the integral becomes

$$\begin{aligned} \int x^{11} \sqrt{1 + x^4} dx &= \int x^{11} \sqrt{1 + x^4} \left(\frac{1}{4x^3} dy \right) = \frac{1}{4} \int x^8 \sqrt{1 + x^4} dy = \frac{1}{4} \int (y - 1)^2 \sqrt{y} dy \\ &= \frac{1}{4} \int \left(y^{\frac{5}{2}} - 2y^{\frac{3}{2}} + y^{\frac{1}{2}} \right) dy = \frac{1}{4} \left(\frac{y^{\frac{7}{2}}}{\frac{7}{2}} - 2 \frac{y^{\frac{5}{2}}}{\frac{5}{2}} + \frac{y^{\frac{3}{2}}}{\frac{3}{2}} \right) + C \\ &= \frac{1}{14} (1 + x^4)^{\frac{7}{2}} - \frac{1}{5} (1 + x^4)^{\frac{5}{2}} + \frac{1}{6} (1 + x^4)^{\frac{3}{2}} + C. \end{aligned}$$

(d) Let $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x} dy$.

Then the integral becomes

$$\begin{aligned} \int x \cos^2(x^2) dx &= \int x \cos^2(x^2) \left(\frac{1}{2x} dy \right) = \frac{1}{2} \int \cos^2 y dy \\ &= \frac{1}{2} \int \frac{1}{2} [\cos(y + y) + \cos(y - y)] dy = \frac{1}{4} \int (\cos 2y + 1) dy \\ &= \frac{1}{4} \left(\frac{1}{2} \sin 2y + y \right) + C = \frac{1}{8} \sin(2x^2) + \frac{x^2}{4} + C. \end{aligned}$$

(e) Let $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow dx = -\frac{1}{\sin x} dy$.

Together with the fact that $\sin 2x = 2 \sin x \cos x$, the integral then becomes

$$\begin{aligned} \int \sin 2x \sqrt{\cos x} dx &= \int 2 \sin x \cos x \sqrt{\cos x} \left(-\frac{1}{\sin x} dy \right) = -2 \int y \sqrt{y} dy = -2 \int y^{\frac{3}{2}} dy \\ &= -2 \frac{y^{\frac{5}{2}}}{\frac{5}{2}} + C = -\frac{4}{5} \cos^{\frac{5}{2}} x + C. \end{aligned}$$

(f) Let $y = 1 + e^x \Rightarrow \frac{dy}{dx} = e^x \Rightarrow dx = \frac{1}{e^x} dy$.

Then the integral becomes

$$\begin{aligned} \int \frac{e^{2x}}{(1 + e^x)^3} dx &= \int \frac{e^{2x}}{(1 + e^x)^3} \left(\frac{1}{e^x} dy \right) = \int \frac{e^x}{(1 + e^x)^3} dy = \int \frac{y - 1}{y^3} dy = \int (y^{-2} - y^{-3}) dy \\ &= -y^{-1} + \frac{1}{2} y^{-2} + C = -\frac{1}{1 + e^x} + \frac{1}{2(1 + e^x)^2} + C. \end{aligned}$$

(g)

$$\text{Let } y = x^2 - 1 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x} dy.$$

When $x = 1$, $y = 1^2 - 1 = 0$. When $x = 2$, $y = 2^2 - 1 = 3$.

Then the integral becomes

$$\int_1^2 x e^{x^2-1} dx = \int_0^3 x e^{x^2-1} \left(\frac{1}{2x} dy \right) = \frac{1}{2} \int_0^3 e^y dy = \frac{1}{2} e^y \Big|_0^3 = \frac{e^3}{2} - \frac{1}{2}.$$

(h)

$$\text{Let } y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow dx = x dy.$$

When $x = 5$, $y = \ln 5$. When $x = 1$, $y = \ln 1 = 0$.

Then the integral becomes

$$\begin{aligned} \int_1^5 \frac{\sin^2(\ln x)}{x} dx &= \int_0^{\ln 5} \frac{\sin^2(\ln x)}{x} (x dy) = \int_0^{\ln 5} \sin^2 y dy \\ &= \int_0^{\ln 5} -\frac{1}{2} (\cos(y+y) - \cos(y-y)) dy = -\frac{1}{2} \int_0^{\ln 5} (\cos 2y - 1) dy \\ &= -\frac{1}{2} \left(\frac{1}{2} \sin 2y - y \right) \Big|_0^{\ln 5} = -\frac{1}{4} \sin(2 \ln 5) + \frac{1}{2} \ln 5. \end{aligned}$$

(i)

$$\text{Following the idea of Example 7, let } y = x^2 + 4 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x} dy.$$

We compute the integral by decomposing it as follows:

$$\begin{aligned} \int \frac{3x+2}{x^2+4} dx &= \int \frac{3x}{x^2+4} dx + \int \frac{2}{x^2+4} dx = \int \frac{3x}{x^2+4} \left(\frac{1}{2x} dy \right) + 2 \int \frac{1}{x^2+4} dx \\ &= \frac{3}{2} \int \frac{1}{y} dy + \frac{2}{4} \int \frac{1}{\left(\frac{x}{2}\right)^2 + 1} dx = \frac{3}{2} \ln y + \frac{1}{2} \left(\frac{1}{\frac{1}{2}} \tan^{-1} \frac{x}{2} \right) + C = \frac{3}{2} \ln|x^2+4| + \tan^{-1} \frac{x}{2} + C. \end{aligned}$$

(j)

$$\text{Let } y = x^2 - 2x + 5 \Rightarrow \frac{dy}{dx} = 2x - 2 \Rightarrow dx = \frac{1}{2x-2} dy.$$

Similar to (i), the integral can be computed as

$$\begin{aligned} \int \frac{2x+1}{x^2-2x+5} dx &= \int \frac{2x-2}{x^2-2x+5} dx + \int \frac{3}{x^2-2x+5} dx \\ &= \int \frac{2x-2}{x^2-2x+5} \left(\frac{1}{2x-2} dy \right) + 3 \int \frac{1}{x^2-2x+5} dx \\ &= \int \frac{1}{y} dy + 3 \int \frac{1}{(x-1)^2+4} dx = \int \frac{1}{y} dy + \frac{3}{4} \int \frac{1}{\left(\frac{x-1}{2}\right)^2 + 1} dx \\ &= \ln|y| + \frac{3}{4} \left(\frac{1}{\frac{1}{2}} \tan^{-1} \frac{x-1}{2} \right) + C = \ln|x^2-2x+5| + \frac{3}{2} \tan^{-1} \frac{x-1}{2} + C. \end{aligned}$$

(k) Let $y = 3x^2 + 6x + 19 \Rightarrow \frac{dy}{dx} = 6x + 6 \Rightarrow dx = \frac{1}{6x + 6} dy$.

Then the integral can be computed as

$$\begin{aligned} \int \frac{4x}{3x^2 + 6x + 19} dx &= \int \frac{4x + 4}{3x^2 + 6x + 19} dx - \int \frac{4}{3x^2 + 6x + 19} dx \\ &= \frac{2}{3} \int \frac{1}{y} dy - 4 \int \frac{1}{3x^2 + 6x + 19} dx = \frac{2}{3} \int \frac{1}{y} dy - 4 \int \frac{1}{3(x+1)^2 + 16} dx \\ &= \frac{2}{3} \ln|y| - \frac{4}{16} \int \frac{1}{\left(\frac{\sqrt{3}}{4}x + \frac{\sqrt{3}}{4}\right)^2 + 1} dx \\ &= \frac{2}{3} \ln|3x^2 + 6x + 19| - \frac{1}{4} \left(\frac{1}{\frac{\sqrt{3}}{4}} \tan^{-1} \left(\frac{\sqrt{3}}{4}x + \frac{\sqrt{3}}{4} \right) \right) + C \\ &= \frac{2}{3} \ln|3x^2 + 6x + 19| - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{\sqrt{3}}{4}x + \frac{\sqrt{3}}{4} \right) + C. \end{aligned}$$

(l) Let $x = \sin \theta \Rightarrow \frac{dx}{d\theta} = \cos \theta \Rightarrow dx = \cos \theta d\theta$.

The integral then becomes

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{1-x^2}} dx &= \int \frac{1}{\sin^2 \theta \sqrt{1-\sin^2 \theta}} \cos \theta d\theta = \int \frac{1}{\sin^2 \theta} d\theta = \int \csc^2 \theta d\theta \\ &= -\cot \theta + C \stackrel{\substack{\sin \theta = x \\ \cos \theta = \sqrt{1-x^2}}}{=} -\frac{\sqrt{1-x^2}}{x} + C. \end{aligned}$$

(m) Let $x = \frac{1}{\sqrt{3}} \tan \theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{\sqrt{3}} \sec^2 \theta \Rightarrow dx = \frac{1}{\sqrt{3}} \sec^2 \theta d\theta$

The integral becomes

$$\begin{aligned} \int \frac{1}{(1+3x^2)^{\frac{3}{2}}} dx &= \int \frac{1}{\left(1+3\left(\frac{1}{3}\tan^2 \theta\right)\right)^{\frac{3}{2}}} \left(\frac{1}{\sqrt{3}} \sec^2 \theta d\theta\right) = \frac{1}{\sqrt{3}} \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta = \frac{1}{\sqrt{3}} \int \frac{1}{\sec \theta} d\theta \\ &= \frac{1}{\sqrt{3}} \int \cos \theta d\theta = \frac{1}{\sqrt{3}} \sin \theta + C = \frac{1}{\sqrt{3}} \frac{\sqrt{3}x}{\sqrt{1+3x^2}} + C = \frac{x}{\sqrt{1+3x^2}} + C. \end{aligned}$$

(n) Let $y = 4x^2 + 1 \Rightarrow \frac{dy}{dx} = 8x \Rightarrow dx = \frac{1}{8x} dy$.

Then the integral becomes

$$\int \frac{3x}{\sqrt{4x^2 + 1}} dx = \int \frac{3x}{\sqrt{4x^2 + 1}} \left(\frac{1}{8x} dy \right) = \frac{3}{8} \int \frac{1}{\sqrt{y}} dy = \frac{3}{8} \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{3}{4} \sqrt{4x^2 + 1} + C.$$

(o) Let $x = \frac{3}{4} \sin \theta \Rightarrow \frac{dx}{d\theta} = \frac{3}{4} \cos \theta \Rightarrow dx = \frac{3}{4} \cos \theta d\theta$.

Then the integral becomes

$$\begin{aligned}
\int \sqrt{9-16x^2} dx &= \int \sqrt{9-16\left(\frac{3}{4}\sin\theta\right)^2} \left(\frac{3}{4}\cos\theta d\theta\right) = \frac{9}{4} \int \cos^2\theta d\theta \\
&= \frac{9}{4} \int \frac{1}{2} [\cos(\theta+\theta) + \cos(\theta-\theta)] d\theta = \frac{9}{8} \int \cos 2\theta d\theta + \frac{9}{8} \int 1 d\theta \\
&= \frac{9}{16} \sin 2\theta + \frac{9}{8} \theta + C = \frac{9}{8} \sin\theta \cos\theta + \frac{9}{8} \theta + C \\
&= \frac{9}{8} \left(\frac{4x}{3}\right) \frac{\sqrt{9-16x^2}}{3} + \frac{9}{8} \sin^{-1} \frac{4}{3}x + C = \frac{1}{2}x\sqrt{9-16x^2} + \frac{9}{8} \sin^{-1} \frac{4}{3}x + C.
\end{aligned}$$

(p) Using completing square technique, we first rewrite the integral as

$$\int \frac{1}{(x^2+6x+10)^{\frac{3}{2}}} dx = \int \frac{1}{[(x+3)^2+1]^{\frac{3}{2}}} dx.$$

$$\text{Let } x+3 = \tan\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta d\theta.$$

Then the integral becomes

$$\begin{aligned}
\int \frac{1}{[(x+3)^2+1]^{\frac{3}{2}}} dx &= \int \frac{1}{(\tan^2\theta+1)^{\frac{3}{2}}} \sec^2\theta d\theta = \int \frac{1}{\sec\theta} d\theta = \int \cos\theta d\theta = \sin\theta + C \\
&= \frac{x+3}{\sqrt{x^2+6x+10}} + C.
\end{aligned}$$

(q) Let $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow dx = -\frac{1}{\sin x} dy$.

The integral then becomes

$$\begin{aligned}
\int \sin^7 x dx &= \int \sin^6 x \left(-\frac{1}{\sin x} dy\right) = -\int \sin^6 x dy = -\int (1-\cos^2 x)^3 dy \\
&= -\int (1-y^2)^3 dy = -\int (1-3y^2+3y^4-y^6) dy \\
&= -y + y^3 - \frac{3}{5}y^5 + \frac{y^7}{7} + C = -\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{\cos^7 x}{7} + C.
\end{aligned}$$

(r) Let $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow dx = \frac{1}{\cos x} dy$.

The integral can be rewritten as

$$\begin{aligned}
\int \sin^3 x \cos^5 x dx &= \int \sin^3 x \cos^5 x \left(\frac{1}{\cos x} dy\right) = \int \sin^3 x \cos^4 x dy = \int \sin^3 x (1-\sin^2 x)^2 dy \\
&= \int y^3(1-y^2)^2 dy = \int (y^3-2y^5+y^7) dy = \frac{y^4}{4} - \frac{y^6}{3} + \frac{y^8}{8} + C \\
&= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8} + C.
\end{aligned}$$

Problem 2

(a) Take $u = x$ and $dv = e^{-3x} dx \Rightarrow v = \int e^{-3x} dx = -\frac{1}{3}e^{-3x}$.

Using integration by parts, we get

$$\begin{aligned}
\int \underbrace{x}_{u} \underbrace{e^{-3x} dx}_{dv} &= \underbrace{-\frac{1}{3}xe^{-3x}}_{uv} - \int \underbrace{-\frac{1}{3}e^{-3x}}_v \underbrace{dx}_{du} = -\frac{1}{3}xe^{-3x} + \frac{1}{3} \int e^{-3x} dx \\
&= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C.
\end{aligned}$$

(b) Take $u = \ln x$ and $dv = \sqrt{x} dx \Rightarrow v = \int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{2}{3}x^{\frac{3}{2}}$.

Using integration by parts, we get

$$\begin{aligned}
\int_1^e \sqrt{x} \ln x \, dx &= \int_1^e \underbrace{\ln x}_u \underbrace{\sqrt{x} dx}_{dv} = \underbrace{\frac{2}{3} x^{\frac{3}{2}} \ln x}_{uv} \Big|_1^e - \int_1^e \underbrace{\frac{2}{3} x^{\frac{3}{2}}}_v \underbrace{d(\ln x)}_{du} \\
&= \frac{2}{3} e^{\frac{3}{2}} \ln e - \frac{2}{3} x^{\frac{3}{2}} \ln 1 - \frac{2}{3} \int_1^e x^{\frac{3}{2}} \left(\frac{1}{x} dx \right) \\
&= \frac{2}{3} e^{\frac{3}{2}} - \frac{2}{3} \int_1^e x^{\frac{1}{2}} dx = \frac{2}{3} e^{\frac{3}{2}} - \frac{2}{3} \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \Big|_1^e \\
&= \frac{2}{3} e^{\frac{3}{2}} - \frac{4}{9} e^{\frac{3}{2}} + \frac{4}{9} = \frac{2}{9} e^{\frac{3}{2}} + \frac{4}{9}.
\end{aligned}$$

(c) Take $u = x^2$ and $dv = \sin x \, dx \Rightarrow v = \int \sin x \, dx = -\cos x$.

Using integration by parts, we get

$$\int \underbrace{x^2}_u \underbrace{\sin x \, dx}_{dv} = \underbrace{-x^2 \cos x}_{uv} - \int \underbrace{-\cos x}_v \underbrace{d(x^2)}_{du} = -x^2 \cos x + 2 \int x \cos x \, dx.$$

To compute the second integral, we take $u = x$ and $dv = \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x$. Using integration by parts, we have

$$\int \underbrace{x}_u \underbrace{\cos x \, dx}_{dv} = \underbrace{x \sin x}_{uv} - \int \underbrace{\sin x}_v \underbrace{dx}_{du} = x \sin x + \cos x + C.$$

Therefore, we conclude that

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

(d) We first rewrite the integral by using product-to-sum formula:

$$\begin{aligned}
\int x \sin^2 x \, dx &= \int x \left[-\frac{1}{2} (\cos(x+x) - \cos(x-x)) \right] dx = -\frac{1}{2} \int \underbrace{x}_{u} \underbrace{\cos 2x \, dx}_{dv} + \frac{1}{2} \int x \, dx \\
&\stackrel{u=x}{\Rightarrow v=\int \cos 2x \, dx = \frac{1}{2} \sin 2x} \\
&\cong -\frac{1}{2} \left(\underbrace{\frac{1}{2} x \sin 2x}_{uv} - \int \underbrace{\frac{1}{2} \sin 2x}_v \underbrace{dx}_{du} \right) + \frac{1}{2} \left(\frac{x^2}{2} \right) \\
&= -\frac{1}{4} x \sin 2x + \frac{1}{4} \left(-\frac{1}{2} \cos 2x \right) + \frac{x^2}{4} + C = -\frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + \frac{x^2}{4} + C.
\end{aligned}$$

(e) Take $u = \cos^{-1} x$ and $dv = x^2 \, dx \Rightarrow v = \int x^2 \, dx = \frac{x^3}{3}$.

Using integration by parts, we have

$$\begin{aligned}
\int x^2 \cos^{-1} x \, dx &= \int \underbrace{\cos^{-1} x}_u \underbrace{(x^2 \, dx)}_{dv} = \underbrace{\frac{x^3}{3} \cos^{-1} x}_{uv} - \int \underbrace{\frac{x^3}{3}}_v \underbrace{d(\cos^{-1} x)}_{du} \\
&= \frac{x^3}{3} \cos^{-1} x + \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} \, dx \\
&\stackrel{y=1-x^2}{\frac{dy}{dx}=-2x}{\cong} \frac{x^3}{3} \cos^{-1} x + \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} \left(-\frac{1}{2x} dy \right) = \frac{x^3}{3} \cos^{-1} x - \frac{1}{6} \int \frac{1-y}{\sqrt{y}} \, dy \\
&= \frac{x^3}{3} \cos^{-1} x - \frac{1}{6} \int \left(y^{-\frac{1}{2}} - y^{\frac{1}{2}} \right) dy = \frac{x^3}{3} \cos^{-1} x - \frac{1}{6} \left(\frac{y^{\frac{1}{2}}}{\frac{1}{2}} - \frac{y^{\frac{3}{2}}}{\frac{3}{2}} \right) + C \\
&= \frac{x^3}{3} \cos^{-1} x - \frac{1}{3} \sqrt{1-x^2} + \frac{1}{9} (1-x^2)^{\frac{3}{2}} + C.
\end{aligned}$$

- (f) Take $u = \tan^{-1} x$ and $dv = dx \Rightarrow v = \int dx = x$.

Using integration by parts, we have

$$\begin{aligned} \int \underbrace{\tan^{-1} x}_u \underbrace{dx}_{dv} &= \underbrace{x \tan^{-1} x}_{uv} - \int \underbrace{x d(\tan^{-1} x)}_{vdu} = x \tan^{-1} x - \int \frac{x}{1+x^2} dx \\ &\stackrel{y=1+x^2}{\Rightarrow \frac{dy}{dx}=2x} \equiv x \tan^{-1} x - \int \frac{x}{1+x^2} \left(\frac{1}{2x} dy \right) = x \tan^{-1} x - \frac{1}{2} \int \frac{1}{y} dy \\ &= x \tan^{-1} x - \frac{1}{2} \ln|y| + C = x \tan^{-1} x - \frac{1}{2} \ln|1+x^2| + C. \end{aligned}$$

- (g) Take $u = \csc x$, $dv = \csc^2 x dx \Rightarrow v = \int \csc^2 x dx = -\cot x$.

Using integration by parts, we have

$$\begin{aligned} \int \csc^3 x dx &= \int \underbrace{\csc x}_u \underbrace{(\csc^2 x dx)}_{dv} = \underbrace{-\csc x \cot x}_{uv} - \int \underbrace{(-\cot x)}_v \underbrace{d(\csc x)}_{du} \\ &= -\csc x \cot x - \int \csc x \cot^2 x dx = -\csc x \cot x - \int \csc x (\csc^2 x - 1) dx \\ &= -\csc x \cot x - \int \csc^3 x dx + \int \csc x dx \\ &= -\csc x \cot x - \ln|\cot x + \csc x| - \int \csc^3 x dx. \end{aligned}$$

Summing up, we get

$$\begin{aligned} \int \csc^3 x dx &= -\csc x \cot x - \ln|\cot x + \csc x| - \int \csc^3 x dx \\ \Rightarrow 2 \int \csc^3 x dx &= -\csc x \cot x - \ln|\cot x + \csc x| \\ \Rightarrow \int \csc^3 x dx &= -\frac{1}{2} (\csc x \cot x + \ln|\cot x + \csc x|) + C. \end{aligned}$$

- (h) Take $u = \cos^2 x$, $dv = \cos x dx \Rightarrow v = \int \cos x dx = \sin x$.

Using integration by parts, we have

$$\begin{aligned} \int \cos^3 x dx &= \int \underbrace{\cos^2 x}_u \underbrace{(\cos x dx)}_{dv} = \underbrace{\cos^2 x \sin x}_{uv} - \int \underbrace{(\sin x)}_v \underbrace{d(\cos^2 x)}_{du} \\ &= \cos^2 x \sin x - \int \sin x (-2 \cos x \sin x) dx = \cos^2 x \sin x + 2 \int \sin^2 x \cos x dx \\ &= \cos^2 x \sin x + 2 \int (1 - \cos^2 x) \cos x dx \\ &= \cos^2 x \sin x + 2 \int \cos x dx - 2 \int \cos^3 x dx \\ &= \cos^2 x \sin x + 2 \sin x - 2 \int \cos^3 x dx. \end{aligned}$$

Summing up, we get

$$\begin{aligned} \int \cos^3 x dx &= \cos^2 x \sin x + 2 \sin x - 2 \int \cos^3 x dx \\ \Rightarrow 3 \int \cos^3 x dx &= \cos^2 x \sin x + 2 \sin x \\ \Rightarrow \int \cos^3 x dx &= \frac{1}{3} (\cos^2 x \sin x + 2 \sin x) + C. \end{aligned}$$

- (i) Take $u = (\ln x)^2$ and $dv = \frac{1}{x^2} dx \Rightarrow v = \int \frac{1}{x^2} dx = -x^{-1} = -\frac{1}{x}$.

Using integration by parts, we have

$$\begin{aligned}
\int_1^e \left(\frac{\ln x}{x}\right)^2 dx &= \int_1^e \underbrace{(\ln x)^2}_u \underbrace{\left(\frac{1}{x^2} dx\right)}_{dv} = -\underbrace{\frac{(\ln x)^2}{x}}_{uv} \Big|_1^e - \int_1^e \underbrace{-\frac{1}{x}}_v \underbrace{d(\ln x)}_{du} \\
&\stackrel{\frac{d}{dx}(\ln x)^2 = 2(\ln x)\left(\frac{1}{x}\right)}{\cong} -\frac{(\ln e)^2}{e} + \frac{(\ln 1)^2}{1} + 2 \int_1^e \frac{\ln x}{x^2} dx = -\frac{1}{e} + 2 \int_1^e \underbrace{\frac{(\ln x)}{u}}_u \underbrace{\left(\frac{1}{x^2} dx\right)}_{dv} \\
&\stackrel{\substack{u=\ln x \\ dv=\frac{1}{x^2} dx}}{\Rightarrow v \int \frac{1}{x^2} dx = -\frac{1}{x}}{\cong} -\frac{1}{e} + 2 \left(-\underbrace{\frac{\ln x}{x}}_{uv} \Big|_1^e - \int_1^e \underbrace{-\frac{1}{x}}_v \underbrace{d(\ln x)}_{du} \right) \\
&= -\frac{1}{e} + 2 \left(-\frac{\ln e}{e} + \frac{\ln 1}{1} + \int_1^e \frac{1}{x^2} dx \right) = -\frac{1}{e} - \frac{2}{e} + 2 \left(-\frac{1}{x} \right) \Big|_1^e = 2 - \frac{5}{e}.
\end{aligned}$$

(j) We take $u = \sin 3x$ and $dv = e^x dx \Rightarrow v = \int e^x dx = e^x$. Using integration by parts, we get

$$\begin{aligned}
\int e^x \sin 3x dx &= e^x \sin 3x - \int e^x d(\sin 3x) = e^x \sin 3x - 3 \int e^x \cos 3x dx \\
&\stackrel{\substack{u=\cos 3x \\ dv=e^x dx \Rightarrow v=\int e^x dx=e^x}}{\cong} e^x \sin 3x - 3 \left(e^x \cos 3x - \int e^x d(\cos 3x) \right) \\
&= e^x \sin 3x - 3e^x \cos 3x - 9 \int e^x \sin 3x dx.
\end{aligned}$$

Summing up, we get

$$\begin{aligned}
\int e^x \sin 3x dx &= e^x \sin 3x - 3e^x \cos 3x - 9 \int e^x \sin 3x dx \\
\Rightarrow 10 \int e^x \sin 3x dx &= e^x \sin 3x - 3e^x \cos 3x \\
\Rightarrow \int e^x \sin 3x dx &= \frac{1}{10} e^x \sin 3x - \frac{3}{10} e^x \cos 3x + C.
\end{aligned}$$

Problem 3

(a) Let $y = 2e^x + 1 \Rightarrow \frac{dy}{dx} = 2e^x \Rightarrow dx = \frac{1}{2e^x} dy$

The integral then becomes

$$\begin{aligned}
\int e^{2x} \sin(2e^x + 1) dx &= \int e^{2x} \sin(2e^x + 1) \left(\frac{1}{2e^x}\right) dy = \frac{1}{2} \int e^x \sin(2e^x + 1) dy \\
&= \frac{1}{2} \int \frac{y-1}{2} \sin y dy = \frac{1}{4} \int y \sin y dy - \frac{1}{4} \int \sin y dy = \frac{1}{4} \int \underbrace{y \sin y}_{u \frac{dv}{dv}} dy + \frac{1}{4} \cos y \\
&\stackrel{\substack{u=y \\ dv=\sin y dy \\ \Rightarrow v=\int \sin y dy = -\cos y}}{\cong} \frac{1}{4} \left(-y \cos y - \int (-\cos y) dy + \frac{1}{4} \cos y \right) \\
&= -\frac{1}{4} y \cos y + \frac{1}{4} \sin y + \frac{1}{4} \cos y + \\
&= -\frac{1}{4} (2e^x + 1) \cos(2e^x + 1) + \frac{1}{4} \sin(2e^x + 1) + \frac{1}{4} \cos(2e^x + 1) + C.
\end{aligned}$$

(b) Let $y = 2\sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x}} \Rightarrow dx = \sqrt{x} dy$

When $x = 1$, $y = 2$. When $x = 0$, $y = 0$.

The integral then becomes

$$\begin{aligned}
\int_0^1 \sin(2\sqrt{x}) dx &= \int_0^2 \sin(2\sqrt{x}) (\sqrt{x} dy) = \int_0^2 \frac{y}{2} \sin y dy = \frac{1}{2} \int_0^2 \underbrace{y}_{u} \underbrace{\sin y dy}_{dv} \\
&\stackrel{u=y}{\Rightarrow v=\int \sin y dy = -\cos y} \stackrel{dv=\sin y dy}{=} \frac{1}{2} \left(-y \cos y \Big|_0^2 - \int_0^2 (-\cos y) dy \right) = -\cos 2 + \frac{1}{2} \sin y \Big|_0^2 \\
&= -\cos 2 + \frac{1}{2} \sin 2.
\end{aligned}$$

(c) Let $y = 1 + \sqrt[3]{x} \Rightarrow \frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}} \Rightarrow dx = 3x^{\frac{2}{3}} dy$.

When $x = 1$, $y = 2$. When $x = 0$, $y = 1$.

$$\int_0^1 \ln(1 + \sqrt[3]{x}) dx = \int_1^2 3x^{\frac{2}{3}} \ln(1 + \sqrt[3]{x}) dy = 3 \int_1^2 (y-1)^2 \ln y dy.$$

Using integration by parts with $u(y) = \ln y$ and $dv = (y-1)^2 dy \Rightarrow v = \int (y-1)^2 dy = \frac{(y-1)^3}{3}$. We get

$$\begin{aligned}
\int_1^2 (y-1)^2 \ln y dy &= \frac{(y-1)^3}{3} \ln y \Big|_1^2 - \int_1^2 \frac{(y-1)^3}{3} d(\ln y) \\
&= \frac{1}{3} \ln 2 - \frac{1}{3} \int_1^2 \frac{y^3 - 3y^2 + 3y - 1}{y} dy = \frac{1}{3} \ln 2 - \frac{1}{3} \int_1^2 \left(y^2 - 3y + 3 - \frac{1}{y} \right) dy \\
&= \frac{1}{3} \ln 2 - \frac{1}{3} \left(\frac{y^3}{3} - \frac{3y^2}{2} + 3y - \ln y \right) \Big|_1^2 = -\frac{5}{18}.
\end{aligned}$$

(d) Let $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow dx = x dy$.

Then the integral becomes

$$\int \cos(\ln x) dx = \int x \cos(\ln x) dy = \int e^y \cos y dy.$$

Using similar techniques as in Problem 2(j) (integration by parts twice), we get

$$\int e^y \cos y dy = \frac{1}{2} (e^y \cos y + e^y \sin y).$$

Therefore, we conclude that

$$\begin{aligned}
\int \cos(\ln x) dx &= \frac{1}{2} (e^y \cos y + e^y \sin y) = \frac{1}{2} (e^{\ln x} \cos(\ln x) + e^{\ln x} \sin(\ln x)) \\
&= \frac{1}{2} x \cos(\ln x) + \frac{1}{2} x \sin(\ln x).
\end{aligned}$$

(e) Let $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow dx = \frac{1}{\cos x} dy$

Together with the fact that $\sin 2x = 2 \sin x \cos x$, the integral can be rewritten as

$$\int \sin 2x \ln(\sin x) dx = 2 \int \sin x \cos x \ln(\sin x) \left(\frac{1}{\cos x} dy \right) = 2 \int y \ln y dy$$

Using integration by parts with $u = \ln y$ and $dv = y dy \Rightarrow v = \int y dy = \frac{y^2}{2}$, the latter integral can be computed as

$$\begin{aligned}
\int y \ln y dy &= \frac{y^2}{2} \ln y - \int \frac{y^2}{2} d(\ln y) = \frac{y^2}{2} \ln y - \int \frac{y^2}{2} \left(\frac{1}{y} dy \right) = \frac{y^2}{2} \ln y - \frac{1}{2} \int y dy \\
&= \frac{y^2}{2} \ln|y| - \frac{y^2}{4} + C.
\end{aligned}$$

Therefore, we conclude that

$$\int \sin 2x \ln(\sin x) dx = 2 \left(\frac{y^2}{2} \ln y - \frac{y^2}{4} \right) + C = \sin^2 x \ln|\sin x| - \frac{\sin^2 x}{2} + C.$$

(f) Let $y = x + 3 \Rightarrow \frac{dy}{dx} = 1 \Rightarrow dx = dy$.

The integral becomes

$$\int (x + 1) \ln(x + 3) dx = \int (y - 2) \ln y dy.$$

Using integration by parts with $u(y) = \ln y$ and $dv = (y - 2)dy \Rightarrow v = \int (y - 2)dy = \frac{(y-2)^2}{2}$.

The later integral can be computed as

$$\begin{aligned} \int (y - 2) \ln y dy &= \frac{(y - 2)^2}{2} \ln y - \int \frac{(y - 2)^2}{2} d(\ln y) = \frac{(y - 2)^2}{2} \ln y - \frac{1}{2} \int \frac{y^2 - 4y + 4}{y} dy \\ &= \frac{(y - 2)^2}{2} \ln y - \frac{1}{2} \int \left(y - 4 + \frac{4}{y} \right) dy = \frac{(y - 2)^2}{2} \ln y - \frac{y^2}{4} + 2y - 2 \ln y + C. \end{aligned}$$

Thus we conclude that

$$\begin{aligned} \int (x + 1) \ln(x + 3) dx &= \frac{(y - 2)^2}{2} \ln y - \frac{y^2}{4} + 2y - 2 \ln y + C \\ &= \frac{(x - 1)^2}{2} \ln|x + 3| - \frac{(x + 3)^2}{4} + 2(x + 3) - 2 \ln|x + 3| + C. \end{aligned}$$

(g) Let $y = e^x - 1 \Rightarrow \frac{dy}{dx} = e^x \Rightarrow dx = \frac{1}{e^x} dy$

When $x = 2$, $y = e^2 - 1$. When $x = 1$, $y = e - 1$.

The integral then becomes

$$\begin{aligned} \int_1^2 \frac{e^{2x}}{e^x - 1} dx &= \int_{e-1}^{e^2-1} \frac{e^{2x}}{e^x - 1} \left(\frac{1}{e^x} dy \right) = \int_{e-1}^{e^2-1} \frac{e^x}{e^x - 1} dy = \int_{e-1}^{e^2-1} \frac{y + 1}{y} dy \\ &= \int_{e-1}^{e^2-1} \left(1 + \frac{1}{y} \right) dy = (y + \ln y) \Big|_{e-1}^{e^2-1} = e^2 - e + \ln|e^2 - 1| - \ln|e - 1|. \end{aligned}$$

(h) Let $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x} dy$

The integral then becomes

$$\begin{aligned} \int x^3 \cos(3x^2) \sin(x^2) dx &= \int x^3 \cos(3x^2) \sin(x^2) \left(\frac{1}{2x} dy \right) = \frac{1}{2} \int y \cos 3y \sin y dy \\ &= \frac{1}{2} \int y \left[\frac{1}{2} (\sin(y + 3y) + \sin(y - 3y)) \right] dy \\ &= \frac{1}{4} \int \underbrace{y}_{u} \underbrace{\sin 4y}_{dv} dy - \frac{1}{4} \int \underbrace{y}_{u} \underbrace{\sin 2y}_{dv} dy \\ &= \frac{1}{4} \left(-\frac{y \cos 4y}{4} - \int \frac{-\cos 4y}{4} dy \right) - \frac{1}{4} \left(-\frac{y \cos 2y}{2} - \int \frac{-\cos 2y}{2} dy \right) \\ &= -\frac{1}{16} y \cos 4y + \frac{1}{64} \sin 4y + \frac{1}{8} y \cos 2y - \frac{1}{16} \sin 2y + C \\ &= -\frac{1}{16} x^2 \cos(4x^2) + \frac{1}{64} \sin(4x^2) + \frac{1}{8} x^2 \cos(2x^2) - \frac{1}{16} \sin(2x^2) + C. \end{aligned}$$

(i) Let $x = 2 \sin \theta \Rightarrow \frac{dx}{d\theta} = 2 \cos \theta \Rightarrow dx = 2 \cos \theta d\theta$

Then the integral becomes

$$\begin{aligned} \int x^2 \sqrt{4-x^2} dx &= \int 4 \sin^2 \theta \sqrt{4-4 \sin^2 \theta} (2 \cos \theta d\theta) = 16 \int \sin^2 \theta \cos^2 \theta d\theta \\ &= 16 \int (\sin \theta \cos \theta)^2 d\theta = 16 \int \left(\frac{1}{2} \sin 2\theta\right)^2 d\theta = 4 \int \sin^2 2\theta d\theta \\ &= 4 \int -\frac{1}{2} [\cos(2\theta + 2\theta) - \cos(2\theta - 2\theta)] d\theta = -2 \int \cos 4\theta d\theta + 2 \int 1 d\theta \\ &= -\frac{1}{2} \sin 4\theta + 2\theta + C = -\sin 2\theta \cos 2\theta + 2\theta + C \\ &= -2 \sin \theta \cos \theta [\cos^2 \theta - \sin^2 \theta] + 2\theta + C \\ &= -2 \frac{x \sqrt{4-x^2}}{2} \left(\frac{4-x^2}{4} - \frac{x^2}{4}\right) + 2 \sin^{-1} \frac{x}{2} + C \\ &= -\frac{1}{2} x \sqrt{4-x^2} \left(1 - \frac{x^2}{2}\right) + 2 \sin^{-1} \frac{x}{2} + C. \end{aligned}$$

(j) Let $y = 4 + x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x} dy$.

Then the integral becomes

$$\int x^3 \sin(4+x^2) dx = \int x^3 \sin(4+x^2) \left(\frac{1}{2x} dy\right) = \frac{1}{2} \int (y-4) \sin y dy.$$

Using integration by parts with $u = y-4$ and $dv = \sin y dy \Rightarrow v = \int \sin y dy = -\cos y$. The later integral can be computed as

$$\begin{aligned} \int (y-4) \sin y dy &= -(y-4) \cos y - \int (-\cos y) d(y-4) = (4-y) \cos y + \int \cos y dy \\ &= (4-y) \cos y + \sin y + C. \end{aligned}$$

Hence, we can conclude that

$$\int x^3 \sin(4+x^2) dx = \frac{1}{2} [(4-y) \cos y + \sin y] + C = -\frac{x^2}{2} \cos(4+x^2) + \frac{1}{2} \sin(4+x^2) + C.$$

Problem 4

(a) We shall compute $\int e^{2x} \sin 3x dx$ using integration by parts.

We take $u = \sin 3x$ and $dv = e^{2x} dx \Rightarrow v = \int e^{2x} dx = \frac{1}{2} e^{2x}$. Using integration by parts, we get

$$\int e^{2x} \sin 3x dx = \frac{1}{2} e^{2x} \sin 3x - \int \frac{1}{2} e^{2x} d(\sin 3x) = \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \int e^{2x} \cos 3x dx$$

$$\begin{aligned} &\stackrel{u=\cos 3x}{dv=e^{2x} dx \Rightarrow v=\frac{1}{2} e^{2x}} \int e^{2x} \cos 3x dx \\ &\stackrel{=}{=} \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \left(\frac{1}{2} e^{2x} \cos 3x - \int \frac{1}{2} e^{2x} d(\cos 3x) \right) \\ &= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{4} e^{2x} \cos 3x - \frac{9}{4} \int e^{2x} \sin 3x dx. \end{aligned}$$

Summing up, we get

$$\begin{aligned} \int e^{2x} \sin 3x dx &= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{4} e^{2x} \cos 3x - \frac{9}{4} \int e^{2x} \sin 3x dx \\ \Rightarrow \frac{13}{4} \int e^{2x} \sin 3x dx &= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{4} e^{2x} \cos 3x \\ \Rightarrow \int e^{2x} \sin 3x dx &= \frac{2}{13} e^{2x} \sin 3x - \frac{3}{13} e^{2x} \cos 3x + C. \end{aligned}$$

In a similar fashion, we obtain

$$\int e^{2x} \cos 3x dx = \frac{3}{13} e^{2x} \sin 3x + \frac{2}{13} e^{2x} \cos 3x + C.$$

- (b) Comparing with the similar integral in (a), there is an extra term " x " in the integrand. In order to apply the previous result, one has to eliminate " x " from the integrand. This can be done by using integration by parts.

$$\text{We take } u = x \text{ and } dv = e^{2x} \cos 3x \, dx \Rightarrow v = \int e^{2x} \cos 3x \, dx = \frac{3}{13} e^{2x} \sin 3x + \frac{2}{13} e^{2x} \cos 3x.$$

Using integration by parts, we get

$$\begin{aligned} & \int x e^{2x} \cos 3x \, dx \\ &= x \left(\frac{3}{13} e^{2x} \sin 3x + \frac{2}{13} e^{2x} \cos 3x \right) - \int \left(\frac{3}{13} e^{2x} \sin 3x + \frac{2}{13} e^{2x} \cos 3x \right) dx \\ &= \frac{3}{13} x e^{2x} \sin 3x + \frac{2}{13} x e^{2x} \cos 3x - \frac{3}{13} \int e^{2x} \sin 3x \, dx - \frac{2}{13} \int e^{2x} \cos 3x \, dx. \\ &= \frac{3}{13} x e^{2x} \sin 3x + \frac{2}{13} x e^{2x} \cos 3x - \frac{3}{13} \left(\frac{2}{13} e^{2x} \sin 3x - \frac{3}{13} e^{2x} \cos 3x \right) \\ &\quad - \frac{2}{13} \left(\frac{3}{13} e^{2x} \sin 3x + \frac{2}{13} e^{2x} \cos 3x \right) + C \\ &= \frac{3}{13} x e^{2x} \sin 3x + \frac{2}{13} x e^{2x} \cos 3x - \frac{12}{169} e^{2x} \sin 3x + \frac{5}{13} e^{2x} \cos 3x + C. \end{aligned}$$

Problem 5

The first objective is to eliminate the term " x " in the integrand. This can be done by using integration by parts.

Using integration by parts with $u = x$ and $dv = f'(x)dx \Rightarrow v = \int f'(x)dx = f(x)$, we have

$$\int_a^b \underbrace{x}_{u} \underbrace{f'(x)dx}_{dv} = \underbrace{xf(x)}_{uv} \Big|_a^b - \int_a^b \underbrace{f(x)}_v \underbrace{dx}_{du} = b \underbrace{f(b)}_{=1} - a \underbrace{f(a)}_{=1} - \underbrace{\int_a^b f(x)dx}_{=0} = b - a.$$

Problem 6

We first simplify the first integral $\int_0^a x^3 f(x^2)dx$ on the left hand side.

$$\text{Let } y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x} dy.$$

When $x = a$, $y = a^2$. When $x = 0$, $y = 0$.

Then the integral $\int_0^a x^3 f(x^2)dx$ can be rewritten as

$$\int_0^a x^3 f(x^2)dx = \int_0^{a^2} x^3 f(x^2) \left(\frac{1}{2x} dy \right) = \frac{1}{2} \int_0^{a^2} y f(y) dy.$$

Since the definite integrals $\int_0^{a^2} y f(y) dy$ and $\int_0^{a^2} x f(x) dx$ give the same values. Therefore we conclude that

$$\int_0^a x^3 f(x^2)dx - \frac{1}{2} \int_0^{a^2} x f(x) dx = \frac{1}{2} \int_0^{a^2} y f(y) dy - \frac{1}{2} \int_0^{a^2} x f(x) dx = 0.$$

Problem 7

Using integration by parts with $u = x(1-x)$ and $dv = f''(x)dx \Rightarrow v = \int f''(x)dx = f'(x)$, we have

$$\begin{aligned} & \int_0^1 x(1-x) f''(x) dx = x(1-x) f'(x) \Big|_0^1 - \int_0^1 f'(x) d[x(1-x)] \\ &= 0 - 0 - \int_0^1 f'(x) (1-2x) dx = \int_0^1 \underbrace{(2x-1)}_u \underbrace{f'(x) dx}_{dv} \end{aligned}$$

$$\begin{aligned}
& u=2x-1 \\
& dv=f'(x)dx \\
\Rightarrow v &= \int f'(x)dx = f(x) \\
& \cong (2x-1)f(x)|_0^1 - \int_0^1 f(x)d(2x-1) = \underbrace{f(1)}_{=1} - \left(-\underbrace{f(0)}_{=1} \right) - 2 \underbrace{\int_0^1 f(x)dx}_{=1} = 2 - 2 = 0.
\end{aligned}$$

Problem 8

(a) Using integration by parts with $u = x^n$ and $dv = e^{-3x}dx \Rightarrow v = \int e^{-3x}dx = -\frac{1}{3}e^{-3x}$, we have

$$\begin{aligned}
I_n &= \int_0^1 x^n e^{-3x} dx = -\frac{1}{3}x^n e^{-3x}|_0^1 - \int_0^1 -\frac{1}{3}e^{-3x}d(x^n) = -\frac{1}{3}e^{-3} + \frac{n}{3} \int_0^1 x^{n-1} e^{-3x} dx \\
&= -\frac{1}{3}e^{-3} + \frac{n}{3}I_{n-1}.
\end{aligned}$$

(b) Using the reduction formula in (a), the integral can be computed as

$$\begin{aligned}
\int_0^1 x^3 e^{-3x} dx &= I_3 = -\frac{1}{3}e^{-3} + \frac{3}{3}I_2 = -\frac{1}{3}e^{-3} + I_2 = -\frac{1}{3}e^{-3} + \left(-\frac{1}{3}e^{-3} + \frac{2}{3}I_1 \right) \\
&= -\frac{2}{3}e^{-3} + \frac{2}{3}I_1 = -\frac{2}{3}e^{-3} + \frac{2}{3} \left(-\frac{1}{3}e^{-3} + \frac{1}{3}I_0 \right) = -\frac{8}{9}e^{-3} + \frac{2}{9} \int_0^1 x^0 e^{-3x} dx \\
&= -\frac{8}{9}e^{-3} + \frac{2}{9} \int_0^1 e^{-3x} dx = -\frac{8}{9}e^{-3} - \frac{2}{27}e^{-3x}|_0^1 = \frac{2}{27} - \frac{26}{27}e^{-3}.
\end{aligned}$$

Problem 9

(a) Using integration by parts with $u = (\ln x)^n$ and $dv = x^a dx \Rightarrow v = \int x^a dx = \frac{x^{a+1}}{a+1}$ (since $a \neq -1$), we have

$$\begin{aligned}
I_n &= \int_1^e x^a (\ln x)^n dx = \frac{x^{a+1}}{a+1} (\ln x)^n \Big|_1^e - \int_1^e \frac{x^{a+1}}{a+1} d(\ln x)^n \\
&= \frac{e^{a+1}}{a+1} - \int_1^e \frac{x^{a+1}}{a+1} \left(n(\ln x)^{n-1} \frac{1}{x} dx \right) \\
&= \frac{e^{a+1}}{a+1} - \frac{n}{a+1} \int_1^e x^a (\ln x)^{n-1} dx = \frac{e^{a+1}}{a+1} - \frac{n}{a+1} I_{n-1}.
\end{aligned}$$

(b) Using the reduction formula in (a) (set $a = 2$), the integral can be compute as

$$\begin{aligned}
\int_1^e x^2 (\ln x)^3 dx &= I_3 \stackrel{n=3}{\cong} \frac{e^3}{3} - \frac{3}{3}I_2 = \frac{e^3}{3} - \left(\frac{e^3}{3} - \frac{2}{3}I_1 \right) = \frac{2}{3}I_1 = \frac{2}{3} \left(\frac{e^3}{3} - \frac{1}{3}I_0 \right) \\
&= \frac{2}{9}e^3 - \frac{2}{9} \int_1^e x^2 (\ln x)^0 dx = \frac{2}{9}e^3 - \frac{2}{9} \int_1^e x^2 dx = \frac{2}{9}e^3 - \frac{2}{27}x^3 \Big|_1^e = \frac{4}{27}e^3 + \frac{2}{27}.
\end{aligned}$$

(c) When $a = -1$, the integral becomes $I_n = \int_1^e \frac{(\ln x)^n}{x} dx$ and the above reduction formula cannot be applied. One can compute this integral using method of substitution.

$$\text{Let } y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow dx = x dy.$$

When $x = e$, $y = \ln e = 1$. When $x = 1$, $y = \ln 1 = 0$.

The integral then becomes

$$I_n = \int_1^e \frac{(\ln x)^n}{x} dx = \int_0^1 y^n dy = \frac{y^{n+1}}{n+1} \Big|_0^1 = \frac{1}{n+1}.$$

Problem 10

(a) Using integration by parts with $u = (x^2 + a^2)^n$ and $dv = dx \Rightarrow v = \int dx = x$, the integral can be expressed as

$$\begin{aligned}
I_n &= \int (x^2 + a^2)^n dx = x(x^2 + a^2)^n - \int x d(x^2 + a^2)^n \\
&= x(x^2 + a^2)^n - \int x(n(x^2 + a^2)^{n-1}(2x) dx) \\
&= x(x^2 + a^2)^n - 2n \int x^2 (x^2 + a^2)^{n-1} dx \\
&= x(x^2 + a^2)^n - 2n \int (x^2 + a^2 - a^2)(x^2 + a^2)^{n-1} dx \\
&= x(x^2 + a^2)^n - 2n \int (x^2 + a^2)^n dx + 2na^2 \int (x^2 + a^2)^{n-1} dx \\
&= x(x^2 + a^2)^n - 2nI_n + 2na^2I_{n-1}.
\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}
I_n &= x(x^2 + a^2)^n - 2nI_n + 2na^2I_{n-1} \\
\Rightarrow (2n + 1)I_n &= x(x^2 + a^2)^n + 2na^2I_{n-1} \\
\Rightarrow I_n &= \frac{1}{2n + 1} x(x^2 + a^2)^n + \frac{2n}{2n + 1} a^2I_{n-1}.
\end{aligned}$$

(b) Using the reduction formula in (a), we can compute the integral as

$$\begin{aligned}
\int (x^2 + a^2)^4 dx &= I_4 = \frac{1}{9} x(x^2 + a^2)^4 + \frac{8}{9} a^2 I_3 \\
&= \frac{1}{9} x(x^2 + a^2)^4 + \frac{8}{9} a^2 \left(\frac{1}{7} x(x^2 + a^2)^3 + \frac{6}{7} a^2 I_2 \right) \\
&= \frac{1}{9} x(x^2 + a^2)^4 + \frac{8}{63} a^2 x(x^2 + a^2)^3 + \frac{16}{21} a^4 I_2 \\
&= \frac{1}{9} x(x^2 + a^2)^4 + \frac{8}{63} a^2 x(x^2 + a^2)^3 + \frac{16}{21} a^4 \left(\frac{1}{5} x(x^2 + a^2)^2 + \frac{4}{5} a^2 I_1 \right) \\
&= \frac{1}{9} x(x^2 + a^2)^4 + \frac{8}{63} a^2 x(x^2 + a^2)^3 + \frac{16}{105} a^4 x(x^2 + a^2)^2 + \frac{64}{105} a^6 \left(\frac{1}{3} x(x^2 + a^2) + \frac{2}{3} a^2 I_0 \right) \\
&= \frac{1}{9} x(x^2 + a^2)^4 + \frac{8}{63} a^2 x(x^2 + a^2)^3 + \frac{16}{105} a^4 x(x^2 + a^2)^2 + \frac{64}{315} a^6 x(x^2 + a^2) \\
&\quad + \frac{128}{315} a^8 \int 1 dx \\
&= \frac{1}{9} x(x^2 + a^2)^4 + \frac{8}{63} a^2 x(x^2 + a^2)^3 + \frac{16}{105} a^4 x(x^2 + a^2)^2 + \frac{64}{315} a^6 x(x^2 + a^2) + \frac{128}{315} a^8 + C.
\end{aligned}$$

Problem 11

(a) Using integration by parts with $u = x^n$ and $dv = \cos 3x dx \Rightarrow v = \int \cos 3x dx = \frac{1}{3} \sin 3x$, we have

$$\begin{aligned}
I_n &= \int x^n \cos 3x dx = \frac{1}{3} x^n \sin 3x - \int \frac{1}{3} \sin 3x d(x^n) = \frac{1}{3} x^n \sin 3x - \frac{n}{3} \int x^{n-1} \sin 3x dx \\
&\stackrel{\substack{u=x^{n-1} \\ dv=\sin 3x dx \\ \Rightarrow v=\int \sin 3x dx = -\frac{1}{3} \cos 3x}}{\cong} \frac{1}{3} x^n \sin 3x - \frac{n}{3} \left(-\frac{1}{3} x^{n-1} \cos 3x - \int -\frac{1}{3} \cos 3x d(x^{n-1}) \right) \\
&= \frac{1}{3} x^n \sin 3x + \frac{n}{9} x^{n-1} \cos 3x - \frac{n(n-1)}{9} \int x^{n-2} \cos 3x dx \\
&= \frac{1}{3} x^n \sin 3x + \frac{n}{9} x^{n-1} \cos 3x - \frac{n(n-1)}{9} I_{n-2}.
\end{aligned}$$

- (b) Using the reduction formula in (a), the integral can be computed as

$$\begin{aligned}
 \int x^4 \cos 3x \, dx &= \frac{1}{3} x^4 \sin 3x + \frac{4}{9} x^3 \cos 3x - \frac{4}{3} I_2 \\
 &= \frac{1}{3} x^4 \sin 3x + \frac{4}{9} x^3 \cos 3x - \frac{4}{3} \left(\frac{1}{3} x^2 \sin 3x + \frac{2}{9} x \cos 3x - \frac{2}{9} I_0 \right) \\
 &= \frac{1}{3} x^4 \sin 3x + \frac{4}{9} x^3 \cos 3x - \frac{4}{9} x^2 \sin 3x - \frac{8}{27} x \cos 3x - \frac{8}{27} \int \cos 3x \, dx \\
 &= \frac{1}{3} x^4 \sin 3x + \frac{4}{9} x^3 \cos 3x - \frac{4}{9} x^2 \sin 3x - \frac{8}{27} x \cos 3x - \frac{8}{81} \sin 3x + C.
 \end{aligned}$$

Problem 12

- (a) Using similar method as in Example 28 of Lecture Note 2, we use integration by parts with $u = \cos^{n-1} x$ and $dv = \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x$ and obtain

$$\begin{aligned}
 I_n &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos^{n-1} x}_u \underbrace{(\cos x \, dx)}_{dv} = \cos^{n-1} x \sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, d(\cos^{n-1} x) \\
 &= 0 - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x ((n-1) \cos^{n-2} x (-\sin x \, dx)) = (n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x \, dx \\
 &= (n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^{n-2} x \, dx = (n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, dx \\
 &= (n-1) I_{n-2} - (n-1) I_n.
 \end{aligned}$$

Summing up, we have

$$I_n = (n-1) I_{n-2} - (n-1) I_n \Rightarrow n I_n = (n-1) I_{n-2} \Rightarrow I_n = \frac{n-1}{n} I_{n-2}.$$

- (b) Using the reduction formula obtained in (a), the first integral can be computed as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx = I_5 \stackrel{n=5}{=} \frac{4}{5} I_3 \stackrel{n=3}{=} \frac{4}{5} \left(\frac{2}{3} I_1 \right) = \frac{8}{15} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = \frac{8}{15} \sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{16}{15}.$$

Using the same reduction formula, the second integral can be computed as

$$\begin{aligned}
 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^4 x \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \, dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6 x \, dx = I_4 - I_6 \\
 &= \frac{3}{4} I_2 - \frac{5}{6} I_4 = \frac{3}{4} \left(\frac{1}{2} I_0 \right) - \frac{5}{6} \left(\frac{3}{4} I_2 \right) = \frac{5}{8} I_0 - \frac{15}{24} \left(\frac{1}{2} I_0 \right) = \frac{15}{48} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^0 \, dx \\
 &= \frac{15}{48} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, dx = \frac{15}{48} x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{15}{48} \pi = \frac{5}{16} \pi.
 \end{aligned}$$

- (c) Let $x = \sin \theta \Rightarrow \frac{dx}{d\theta} = \cos \theta \Rightarrow dx = \cos \theta \, d\theta$.

When $x = 1$, $\theta = \sin^{-1} 1 = \frac{\pi}{2}$. When $x = -1$, $\theta = \sin^{-1}(-1) = -\frac{\pi}{2}$.

The integral becomes

$$\begin{aligned}
\int_{-1}^1 (1-x^2)^{\frac{5}{2}} dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2 \theta)^{\frac{5}{2}} (\cos \theta d\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 \theta)^{\frac{5}{2}} (\cos \theta d\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6 \theta d\theta \\
&= I_6 \stackrel{\text{from (b)}}{=} \frac{15}{48} I_0 = \frac{15}{48} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^0 dx \stackrel{\text{from (b)}}{=} \frac{5}{16} \pi.
\end{aligned}$$

Problem 13

- (a) Using integration of parts with $u(x) = \sec^{n-2} x$ and $dv = \sec^2 x dx \Rightarrow v = \int \sec^2 x dx = \tan x$ and obtain

$$I_n = \int_0^{\frac{\pi}{4}} \sec^n x dx = \int_0^{\frac{\pi}{4}} \underbrace{\sec^{n-2} x}_u \underbrace{(\sec^2 x dx)}_{dv} = \sec^{n-2} x \tan x \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x d(\sec^{n-2} x)$$

$$\begin{aligned}
\sec \frac{\pi}{4} &= \frac{1}{\cos \frac{\pi}{4}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2} \\
&\stackrel{\text{from (b)}}{=} (\sqrt{2})^{n-2} - \int_0^{\frac{\pi}{4}} \tan x \underbrace{[(n-2) \sec^{n-3} x (\sec x \tan x) dx]}_{\frac{d}{dx}(\sec^{n-2} x) = \frac{d(\sec^{n-2} x)}{d(\sec x)} \frac{d(\sec x)}{dx}}
\end{aligned}$$

$$= (\sqrt{2})^{n-2} - (n-2) \int_0^{\frac{\pi}{4}} \sec^{n-2} x \tan^2 x dx$$

$$\stackrel{1+\tan^2 x = \sec^2 x}{=} (\sqrt{2})^{n-2} - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx$$

$$= (\sqrt{2})^{n-2} - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx$$

$$= (\sqrt{2})^{n-2} - (n-2)I_n + (n-2)I_{n-2}.$$

Summing up, we obtain

$$I_n = (\sqrt{2})^{n-2} - (n-2)I_n + (n-2)I_{n-2} \Rightarrow I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

- (b) First Integral

$$\text{Let } x = \tan \theta \Rightarrow \frac{dx}{d\theta} = \sec^2 \theta \Rightarrow dx = \sec^2 \theta d\theta.$$

$$\text{When } x = 1, \theta = \tan^{-1} 1 = \frac{\pi}{4}. \text{ When } x = 0, \theta = \tan^{-1} 0 = 0.$$

The integral then becomes

$$\int_0^1 (1+x^2)^{\frac{3}{2}} dx = \int_0^{\frac{\pi}{4}} (1+\tan^2 \theta)^{\frac{3}{2}} \sec^2 \theta d\theta = \int_0^{\frac{\pi}{4}} \sec^5 \theta d\theta = I_5.$$

Using the reduction formula in (a), we have

$$\begin{aligned}
I_5 &= \frac{(\sqrt{2})^3}{4} + \frac{3}{4} I_3 = \frac{\sqrt{2}}{2} + \frac{3}{4} \left(\frac{\sqrt{2}}{2} + \frac{1}{2} I_1 \right) = \frac{7\sqrt{2}}{8} + \frac{3}{8} \int_0^{\frac{\pi}{4}} \sec \theta d\theta = \frac{7\sqrt{2}}{8} + \frac{3}{8} \ln |\sec \theta + \tan \theta| \Big|_0^{\frac{\pi}{4}} \\
&= \frac{7\sqrt{2}}{8} + \frac{3}{8} \ln |\sqrt{2} + 1|.
\end{aligned}$$

Hence, we can conclude that

$$\int_0^1 (1+x^2)^{\frac{3}{2}} dx = I_5 = \frac{7\sqrt{2}}{8} + \frac{3}{8} \ln |\sqrt{2} + 1|.$$

Second Integral

Let $x = 2 \sec \theta \Rightarrow \frac{dx}{d\theta} = 2 \sec \theta \tan \theta \Rightarrow dx = 2 \sec \theta \tan \theta d\theta$.

When $x = 2\sqrt{2}$, $\sqrt{2} = \sec \theta \Rightarrow \theta = \cos^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$. When $x = 2$, $1 = \sec \theta \Rightarrow \theta = \cos^{-1} 1 = 0$.

The integral then becomes

$$\begin{aligned} \int_2^{2\sqrt{2}} x^2 \sqrt{x^2 - 4} dx &= \int_0^{\frac{\pi}{4}} 4 \sec^2 \theta \sqrt{4 \sec^2 \theta - 4} (2 \sec \theta \tan \theta d\theta) = 16 \int_0^{\frac{\pi}{4}} \sec^3 \theta \tan^2 \theta d\theta \\ &= 16 \int_0^{\frac{\pi}{4}} \sec^3 \theta (\sec^2 \theta - 1) d\theta = 16 \int_0^{\frac{\pi}{4}} \sec^5 \theta d\theta - 16 \int_0^{\frac{\pi}{4}} \sec^3 \theta d\theta = 16I_5 - 16I_3 \\ &= 16 \left(\frac{7\sqrt{2}}{8} + \frac{3}{8} \ln|\sqrt{2} + 1| \right) - 16 \left(\frac{\sqrt{2}}{2} + \frac{1}{2} \ln|\sqrt{2} + 1| \right) = 6\sqrt{2} - 2 \ln|\sqrt{2} + 1|. \end{aligned}$$

Problem 14

(a) Using the integration by parts with $u = x^{n-1}$ and $dv = x\sqrt{a^2 - x^2} dx \Rightarrow v =$

$$\int x\sqrt{a^2 - x^2} dx \stackrel{y=a^2-x^2}{=} -\frac{1}{2} \int \sqrt{y} dy = -\frac{1}{3} y^{\frac{3}{2}} = -\frac{1}{3} (a^2 - x^2)^{\frac{3}{2}}, \text{ we have}$$

$$\begin{aligned} I_n &= \int_0^a x^n \sqrt{a^2 - x^2} dx = -\frac{1}{3} x^{n-1} (a^2 - x^2)^{\frac{3}{2}} \Big|_0^a - \int_0^a -\frac{1}{3} (a^2 - x^2)^{\frac{3}{2}} d(x^{n-1}) \\ &= 0 - 0 + \frac{n-1}{3} \int_0^a x^{n-2} (a^2 - x^2)^{\frac{3}{2}} dx = \frac{n-1}{3} \int_0^a x^{n-2} (a^2 - x^2) \sqrt{a^2 - x^2} dx \\ &= \frac{n-1}{3} a^2 \int_0^a x^{n-2} \sqrt{a^2 - x^2} dx - \frac{n-1}{3} \int_0^a x^n \sqrt{a^2 - x^2} dx = \frac{n-1}{3} a^2 I_{n-2} - \frac{n-1}{3} I_n. \end{aligned}$$

Summing up, we have

$$I_n = \frac{n-1}{3} a^2 I_{n-2} - \frac{n-1}{3} I_n \Rightarrow I_n = \left(\frac{n-1}{n+2} \right) a^2 I_{n-2}.$$

Remark of (a)

It is not a good idea to use integration by parts with $u = x^n$ and $dv = \sqrt{a^2 - x^2} dx$. It is because the function v becomes $v = \int \sqrt{a^2 - x^2} dx = \dots = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$ and the integral will become

$$\begin{aligned} \int_0^a x^n \sqrt{a^2 - x^2} dx \\ &= x^n \left(\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right) \Big|_0^a - n \int_0^a x^{n-1} \left(\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right) dx. \end{aligned}$$

The computation will be tedious. Hence, one has to modify the choice of v (or dv) so that the integral will not be too complicated after using integration by parts.

(b) Using the reduction formula in (a), we have

$$\begin{aligned} \int_0^a x^2 (a^2 - x^2)^{\frac{3}{2}} dx &= \int_0^a x^2 (a^2 - x^2) \sqrt{a^2 - x^2} dx = a^2 \int_0^a x^2 \sqrt{a^2 - x^2} dx - \int_0^a x^4 \sqrt{a^2 - x^2} dx \\ &= a^2 I_2 - I_4 = a^2 \left(\frac{1}{4} a^2 I_0 \right) - \frac{3}{6} a^2 I_2 = \frac{1}{4} a^4 I_0 - \frac{1}{2} a^2 \left(\frac{1}{4} a^2 I_0 \right) = \frac{a^4}{8} I_0 = \frac{a^4}{8} \int_0^a \sqrt{a^2 - x^2} dx \\ &\stackrel{x=a \sin \theta}{\Rightarrow \frac{dx}{d\theta} = a \cos \theta} \stackrel{=}{=} \frac{a^4}{8} \int_0^{\frac{\pi}{2}} a^2 \cos^2 \theta d\theta = \frac{a^6}{8} \int_0^{\frac{\pi}{2}} \frac{1}{2} (\cos 2\theta + 1) d\theta = \frac{a^6}{16} \left(\frac{\sin 2\theta}{2} + \theta \right) \Big|_0^{\frac{\pi}{2}} = \frac{a^6}{16} \left(\frac{\pi}{2} \right) = \frac{a^6 \pi}{32}. \end{aligned}$$

Problem 15

(a) Using integration by parts with $u = (\sin^{-1} x)^n$ and $dv = dx \Rightarrow v = \int dx = x$, we have

$$\begin{aligned}
 I_n &= \int (\sin^{-1} x)^n dx = x (\sin^{-1} x)^n - \int x d(\sin^{-1} x)^n = x (\sin^{-1} x)^n - n \int \frac{x(\sin^{-1} x)^{n-1}}{\sqrt{1-x^2}} dx \\
 &= x (\sin^{-1} x)^n - n \int \underbrace{(\sin^{-1} x)^{n-1}}_u \underbrace{\frac{x}{\sqrt{1-x^2}} dx}_{dv} \\
 &\quad \begin{aligned} &u = (\sin^{-1} x)^{n-1} \\ &dv = \frac{x}{\sqrt{1-x^2}} dx \\ &\Rightarrow v = \int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \end{aligned} \\
 &\stackrel{\text{}}{=} x (\sin^{-1} x)^n - n \left(-\sqrt{1-x^2} (\sin^{-1} x)^{n-1} - \int -\sqrt{1-x^2} d(\sin^{-1} x)^{n-1} \right) \\
 &= x (\sin^{-1} x)^n + n\sqrt{1-x^2} (\sin^{-1} x)^{n-1} - n(n-1) \int \frac{\sqrt{1-x^2} (\sin^{-1} x)^{n-2}}{\sqrt{1-x^2}} dx \\
 &= x (\sin^{-1} x)^n + n\sqrt{1-x^2} (\sin^{-1} x)^{n-1} - n(n-1) \int (\sin^{-1} x)^{n-2} dx \\
 &= x (\sin^{-1} x)^n + n\sqrt{1-x^2} (\sin^{-1} x)^{n-1} - n(n-1) I_{n-2}.
 \end{aligned}$$

(b) Using the reduction formula in (a), the first integral can be computed as

$$\begin{aligned}
 \int (\sin^{-1} x)^3 dx &= I_3 = x (\sin^{-1} x)^3 + 3\sqrt{1-x^2} (\sin^{-1} x)^2 - 6I_1 \\
 &= x (\sin^{-1} x)^3 + 3\sqrt{1-x^2} (\sin^{-1} x)^2 - 6 \int \sin^{-1} x dx \dots (*)
 \end{aligned}$$

To compute the last integral, we use integration by parts with $u = \sin^{-1} x$ and $dv = dx \Rightarrow v = \int dx = x$ and obtain

$$\begin{aligned}
 \int \sin^{-1} x dx &= x \sin^{-1} x - \int x d(\sin^{-1} x) = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx \\
 &\quad \begin{aligned} &y = 1-x^2 \\ &\frac{dy}{dx} = -2x \end{aligned} \\
 &\stackrel{\text{}}{=} x \sin^{-1} x + \frac{1}{2} \int \frac{1}{\sqrt{y}} dy = x \sin^{-1} x + \sqrt{y} + C = x \sin^{-1} x + \sqrt{1-x^2} + C.
 \end{aligned}$$

We conclude from (*) that

$$\int (\sin^{-1} x)^3 dx = x (\sin^{-1} x)^3 + 3\sqrt{1-x^2} (\sin^{-1} x)^2 - 6x \sin^{-1} x - 6\sqrt{1-x^2} + C.$$

Consider the second integral, we use integration by parts with $u = (\sin^{-1} x)^5$ and $dv = \frac{x}{\sqrt{1-x^2}} dx \Rightarrow v = -\sqrt{1-x^2}$ (from (a)), we have

$$\begin{aligned}
 \int \frac{x(\sin^{-1} x)^5}{\sqrt{1-x^2}} dx &= \int \underbrace{(\sin^{-1} x)^5}_u \underbrace{\left(\frac{x}{\sqrt{1-x^2}} dx \right)}_{dv} \\
 &= -(\sin^{-1} x)^5 \sqrt{1-x^2} - \int -\sqrt{1-x^2} d(\sin^{-1} x)^5 \\
 &= -(\sin^{-1} x)^5 \sqrt{1-x^2} + \int 5 \frac{\sqrt{1-x^2} (\sin^{-1} x)^4}{\sqrt{1-x^2}} dx \\
 &= -(\sin^{-1} x)^5 \sqrt{1-x^2} + 5 \int (\sin^{-1} x)^4 dx = -(\sin^{-1} x)^5 \sqrt{1-x^2} + 5I_4 \\
 &= -(\sin^{-1} x)^5 \sqrt{1-x^2} + 5I_4 \dots \dots (**)
 \end{aligned}$$

Note that I_4 can be computed as

$$\begin{aligned}
I_4 &= x(\sin^{-1} x)^4 + 4(\sin^{-1} x)^3 \sqrt{1-x^2} - 12I_2 \\
&= x(\sin^{-1} x)^4 + 4(\sin^{-1} x)^3 \sqrt{1-x^2} - 12 \left(x(\sin^{-1} x)^2 + 2(\sin^{-1} x)^3 \sqrt{1-x^2} - 2I_0 \right) \\
&= x(\sin^{-1} x)^4 + 4(\sin^{-1} x)^3 \sqrt{1-x^2} - 12x(\sin^{-1} x)^2 - 24(\sin^{-1} x)^3 \sqrt{1-x^2} + 24 \int 1 dx \\
&= x(\sin^{-1} x)^4 + 4(\sin^{-1} x)^3 \sqrt{1-x^2} - 12x(\sin^{-1} x)^2 - 24(\sin^{-1} x)^3 \sqrt{1-x^2} + 24x + C.
\end{aligned}$$

Substitute this into (**), we obtain the value of $\int \frac{x(\sin^{-1} x)^5}{\sqrt{1-x^2}} dx$. (We skip the detail to save space).

Problem 16

(a) Note that $3x - x^2 = x(3 - x)$, we may try the following decomposition:

$$\begin{aligned}
\frac{1}{3x - x^2} &= \frac{1}{x(3 - x)} = \frac{A}{x} + \frac{B}{3 - x} \\
&\Rightarrow 1 = A(3 - x) + Bx.
\end{aligned}$$

- Substitute $x = 3$, we have $1 = 3B \Rightarrow B = \frac{1}{3}$.
- Substitute $x = 0$, we have $1 = 3A \Rightarrow A = \frac{1}{3}$.

The integral can be computed as

$$\int \frac{1}{3x - x^2} dx = \frac{1}{3} \int \frac{1}{x} dx + \frac{1}{3} \int \frac{1}{3 - x} dx = \frac{1}{3} \ln|x| - \frac{1}{3} \ln|3 - x| + C.$$

(b) Note that the denominator consists of a repeated factor $(x - 1)^2$, we should try the following decomposition:

$$\begin{aligned}
\frac{2x^2 - 5x + 5}{(x - 1)^2(x - 2)} &= \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x - 2} \\
&\Rightarrow 2x^2 - 5x + 5 = A(x - 1)(x - 2) + B(x - 2) + C(x - 1)^2.
\end{aligned}$$

- Put $x = 1$, we have $2 = -B \Rightarrow B = -2$.
- Put $x = 2$, we have $3 = C$.
- Compare the coefficient of x^2 : $2 = A + C \Rightarrow A = -1$.

The integral can be computed as

$$\begin{aligned}
\int \frac{2x^2 - 5x + 5}{(x - 1)^2(x - 2)} dx &= - \int \frac{1}{x - 1} dx - \int \frac{2}{(x - 1)^2} dx + 3 \int \frac{1}{x - 2} dx \\
&= -\ln|x - 1| + \frac{2}{x - 1} + 3 \ln|x - 2| + C.
\end{aligned}$$

(c) Note that the rational function is the improper one, we first use *long division* (numerator divided by denominator) to decompose the integrand into the sum of polynomial and proper rational function.

$$\begin{aligned}
\int \frac{3x^4 - 5x^3 + x^2 + 2x + 1}{3x^3 - 2x^2 - x} dx &= \int \frac{(3x^3 - 2x^2 - x)(x - 1) + x + 1}{3x^3 - 2x^2 - x} dx \\
&= \int (x - 1) dx + \int \frac{x + 1}{3x^3 - 2x^2 - x} dx = \frac{x^2}{2} - x + \int \frac{x + 1}{3x^3 - 2x^2 - x} dx \dots (*)
\end{aligned}$$

To compute the second integral, we first note that $3x^3 - 2x^2 - x = x(3x + 1)(x - 1)$. We then decompose the integrand as follows:

$$\begin{aligned}
\frac{x + 1}{3x^3 - 2x^2 - x} &= \frac{x + 1}{x(3x + 1)(x - 1)} = \frac{A}{x} + \frac{B}{3x + 1} + \frac{C}{x - 1} \\
&\Rightarrow x + 1 = A(3x + 1)(x - 1) + Bx(x - 1) + Cx(3x + 1).
\end{aligned}$$

- Put $x = 0$, $1 = -A \Rightarrow A = -1$.
- Put $x = 1$, $2 = 4C \Rightarrow C = \frac{1}{2}$.
- Put $x = -\frac{1}{3}$, $\frac{2}{3} = -\frac{4B}{9} \Rightarrow B = -\frac{3}{2}$.

Then the second integral can be computed as

$$\begin{aligned}\int \frac{x+1}{3x^3-2x^2-x} dx &= -\int \frac{1}{x} dx - \frac{3}{2} \int \frac{1}{3x+1} dx + \frac{1}{2} \int \frac{1}{x-1} dx \\ &= -\ln|x| - \frac{1}{2} \ln|3x+1| + \frac{1}{2} \ln|x-1| + C.\end{aligned}$$

From (*), we conclude that

$$\int \frac{3x^4 - 5x^3 + x^2 + 2x + 1}{3x^3 - 2x^2 - x} dx = \frac{x^2}{2} - x - \ln|x| - \frac{1}{2} \ln|3x+1| + \frac{1}{2} \ln|x-1| + C.$$

- (d) Since the denominator consists of a quadratic factor, we shall decompose the integrand as follows:

$$\begin{aligned}\frac{3x^2+10}{(x+3)(x^2-6x+10)} &= \frac{Ax+B}{(x^2-6x+10)} + \frac{C}{x+3} \\ \Rightarrow 3x^2+10 &= (Ax+B)(x+3) + C(x^2-6x+10)\end{aligned}$$

- Put $x = -3$, $37 = 37C \Rightarrow C = 1$.
- Compare the coefficient of x^2 : $3 = A + C \Rightarrow A = 2$.
- Compare the constant term : $10 = 3B + 10C \Rightarrow B = 0$.

Thus the integral can be computed as

$$\begin{aligned}\int \frac{3x^2+10}{(x+3)(x^2-6x+10)} dx &= \int \frac{2x}{x^2-6x+10} dx + \int \frac{1}{x+3} dx \\ &= \int \frac{2x-6}{x^2-6x+10} dx + \int \frac{6}{x^2-6x+10} dx + \ln|x+3|\end{aligned}$$

Let $y = x^2 - 6x + 10$

$$\frac{dy}{dx} = 2x - 6$$

$$\cong \int \frac{1}{y} dy + 6 \int \frac{1}{(x-3)^2+1} dx + \ln|x+3|$$

$$= \ln|y| + 6 \tan^{-1}(x-3) + \ln|x+3| + C$$

$$= \ln|x^2-6x+10| + 6 \tan^{-1}(x-3) + \ln|x+3| + C.$$

- (e) Since the denominator contains a quadratic factor, we shall decompose the integrand as follows:

$$\begin{aligned}\frac{-7x+19}{(2x+1)(x^2-4x+9)} &= \frac{A}{2x+1} + \frac{Bx+C}{x^2-4x+9} \\ \Rightarrow -7x+19 &= A(x^2-4x+9) + (Bx+C)(2x+1).\end{aligned}$$

- Put $x = -\frac{1}{2}$, we have $\frac{45}{2} = \frac{45}{4}A \Rightarrow A = 2$.
- Compare the coefficient of x^2 : $0 = A + 2B \Rightarrow B = -1$.
- Compare the constant term : $19 = 9A + C \Rightarrow C = 1$.

Then the integral becomes

$$\begin{aligned}\int \frac{-7x+19}{(2x+1)(x^2-4x+9)} dx &= \int \frac{2}{2x+1} dx + \int \frac{-x+1}{x^2-4x+9} dx \\ &= 2 \int \frac{1}{2x+1} dx - \int \frac{x-1}{x^2-4x+9} dx \\ &= 2 \int \frac{1}{2x+1} dx - \int \frac{x-2}{x^2-4x+9} dx - \int \frac{1}{x^2-4x+9} dx\end{aligned}$$

Let $y = x^2 - 4x + 9$

$$\frac{dy}{dx} = 2x - 4$$

$$\begin{aligned} &\cong \ln|2x + 1| - \frac{1}{2} \int \frac{1}{y} dy - \int \frac{1}{(x-2)^2 + 5} dx \\ &= \ln|2x + 1| - \frac{1}{2} \ln|y| - \frac{1}{5} \int \frac{1}{\left(\frac{x-2}{\sqrt{5}}\right)^2 + 1} dx \\ &= \ln|2x + 1| - \frac{1}{2} \ln|x^2 - 4x + 9| - \frac{1}{5} \left(\frac{1}{\frac{1}{\sqrt{5}}} \tan^{-1} \frac{x-2}{\sqrt{5}} \right) + C \\ &= \ln|2x + 1| - \frac{1}{2} \ln|x^2 - 4x + 9| - \frac{\sqrt{5}}{5} \tan^{-1} \frac{x-2}{\sqrt{5}} + C. \end{aligned}$$

- (f) Note that by synthetic division $4x^3 - 3x + 1 = (x + 1)(2x - 1)^2$, we shall decompose the integrand as follows:

$$\begin{aligned} \frac{8x^2 - 3x - 2}{4x^3 - 3x + 1} &= \frac{8x^2 - 3x - 2}{(x + 1)(2x - 1)^2} = \frac{A}{x + 1} + \frac{B}{2x - 1} + \frac{C}{(2x - 1)^2} \\ \Rightarrow 8x^2 - 3x - 2 &= A(2x - 1)^2 + B(x + 1)(2x - 1) + C(x + 1) \end{aligned}$$

- Put $x = \frac{1}{2}$, $-\frac{3}{2} = \frac{3}{2}C \Rightarrow C = -1$.
- Put $x = -1$, $9 = 9A \Rightarrow A = 1$.
- Compare the coefficient of x^2 : $8 = 4A + 2B \Rightarrow B = 2$.

Then the integral can be computed as

$$\begin{aligned} \int \frac{8x^2 - 3x - 2}{4x^3 - 3x + 1} dx &= \int \frac{1}{x + 1} dx + 2 \int \frac{1}{2x - 1} dx - \int \frac{1}{(2x - 1)^2} dx \\ &= \ln|x + 1| + \ln|2x - 1| + \frac{1}{2(2x - 1)} + C. \end{aligned}$$

- (g) Note that the denominator contains a quadratic factor, we should decompose the integrand as follows:

$$\begin{aligned} \frac{x^2 - 5x - 5}{(x - 2)(x^2 + 2x + 3)} &= \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 2x + 3} \\ \Rightarrow x^2 - 5x - 5 &= A(x^2 + 2x + 3) + (Bx + C)(x - 2). \end{aligned}$$

- Put $x = 2$, $-11 = 11A \Rightarrow A = -1$.
- Compare the coefficient of x^2 : $1 = A + B \Rightarrow B = 2$.
- Compare the constant term: $-5 = 3A - 2C \Rightarrow C = 1$.

Thus the integral can be computed as

$$\begin{aligned} \int \frac{x^2 - 5x - 5}{(x - 2)(x^2 + 2x + 3)} dx &= - \int \frac{1}{x - 2} dx + \int \frac{2x + 1}{x^2 + 2x + 3} dx \\ &= - \ln|x - 2| + \int \frac{2x + 2}{x^2 + 2x + 3} dx - \int \frac{1}{x^2 + 2x + 3} dx \end{aligned}$$

$$\begin{aligned}
y &= x^2 + 2x + 3 \\
\frac{dy}{dx} &= 2x + 2 \\
&\cong -\ln|x - 2| + \int \frac{1}{y} dy - \int \frac{1}{x^2 + 2x + 3} dx \\
&= -\ln|x - 2| + \ln|y| - \int \frac{1}{(x + 1)^2 + 2} dx \\
&= -\ln|x - 2| + \ln|x^2 + 2x + 3| - \frac{1}{\sqrt{2}} \int \frac{1}{\left(\frac{x + 1}{\sqrt{2}}\right)^2 + 1} dx \\
&= -\ln|x - 2| + \ln|x^2 + 2x + 3| - \frac{\sqrt{2}}{2} \tan^{-1} \frac{x + 1}{\sqrt{2}} + C.
\end{aligned}$$

- (h) Similar to (c), the integrand is again an improper rational function, one has to use *long division* and rewrite the integrand as the sum of a polynomial and a proper rational function:

$$\begin{aligned}
\int \frac{x^5 + 2x^4 - x + 2}{x^3 + 2x^2 - x - 2} dx &= \int \frac{(x^2 + 1)(x^3 + 2x^2 - x - 2) + 4}{x^3 + 2x^2 - x - 2} dx \\
&= \int (x^2 + 1) dx + \int \frac{4}{x^3 + 2x^2 - x - 2} dx = \frac{x^3}{3} + x + \int \frac{4}{x^3 + 2x^2 - x - 2} dx \dots (*)
\end{aligned}$$

Note that by synthetic division $x^3 + 2x^2 - x - 2 = (x - 1)(x + 1)(x + 2)$, we can decompose the integrand as follows:

$$\begin{aligned}
\int \frac{4}{x^3 + 2x^2 - x - 2} dx &= \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 2} \\
\Rightarrow 4 &= A(x + 1)(x + 2) + B(x - 1)(x + 2) + C(x - 1)(x + 1).
\end{aligned}$$

- Put $x = -1$, $4 = B(-2)(1) \Rightarrow B = -2$.
- Put $x = -2$, $4 = C(-3)(-1) \Rightarrow C = \frac{4}{3}$.
- Put $x = 1$, $4 = 6A \Rightarrow A = \frac{2}{3}$.

Therefore the second integral of (*) can be computed as

$$\begin{aligned}
\int \frac{4}{x^3 + 2x^2 - x - 2} dx &= \frac{2}{3} \int \frac{1}{x - 1} dx - 2 \int \frac{1}{x + 1} dx + \frac{4}{3} \int \frac{1}{x + 2} dx \\
&= \frac{2}{3} \ln|x - 1| - 2 \ln|x + 1| + \frac{4}{3} \ln|x + 2| + C.
\end{aligned}$$

From (*), we conclude that

$$\int \frac{x^5 + 2x^4 - x + 2}{x^3 + 2x^2 - x - 2} dx = \frac{x^3}{3} + x + \frac{2}{3} \ln|x - 1| - 2 \ln|x + 1| + \frac{4}{3} \ln|x + 2| + C.$$

- (i) Since there is a repeated factor x^3 in the denominator, we shall decompose the integrand as follows:

$$\begin{aligned}
\frac{2x^2 - x + 1}{x^3(x - 1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x - 1} \\
\Rightarrow 2x^2 - x + 1 &= Ax^2(x - 1) + Bx(x - 1) + C(x - 1) + Dx^3.
\end{aligned}$$

- Put $x = 1$, $2 = D$.
 - Put $x = 0$, $1 = -C \Rightarrow C = -1$.
- $$\begin{aligned}
\Rightarrow Ax^2(x - 1) + Bx(x - 1) &= 2x^2 - x + 1 + (x - 1) - 2x^3 = -2x^3 + 2x^2 = -2x^2(x - 2) \\
\Rightarrow Ax + B &= -2x \Rightarrow A = -2, \quad B = 0
\end{aligned}$$

Hence, the integral can be computed as

$$\int \frac{2x - x + 1}{x^3(x - 1)} dx = -2 \int \frac{1}{x} dx - \int \frac{1}{x^3} dx + 2 \int \frac{1}{x - 1} dx = -2 \ln|x| + \frac{1}{2x^2} + 2 \ln|x - 1| + C.$$

(j) There are two quadratic factors in the denominator, we decompose the integrand as

$$\frac{3x^3 - 2x - 20}{(x^2 + 3)(2x^2 - 6x + 5)} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{2x^2 - 6x + 5}$$

Method I (Compare of coefficients):

$$\begin{aligned} \Rightarrow 3x^3 - 2x - 20 &= (Ax + B)(2x^2 - 6x + 5) + (Cx + D)(x^2 + 3) \quad (*) \\ &= (2A + C)x^3 + (-6A + 2B + D)x^2 + (5A - 6B + 3C)x + 5B + 3D \end{aligned}$$

- Compare the coefficient of x^3 : $2A + C = 3 \dots \dots \dots (1)$
- Compare the coefficient of x^2 : $-6A + 2B + D = 0 \dots \dots \dots (2)$
- Compare the coefficient of x : $5A - 6B + 3C = -2 \dots \dots \dots (3)$
- Compare the constant term: $5B + 3D = -20 \dots \dots \dots (4)$

$$3(2) - (4): -18A + B = 20 \dots \dots \dots (5)$$

$$3(1) - (3): A + 6B = 11 \dots \dots \dots (6)$$

$$(6) - 6(5): 109A = -109 \Rightarrow A = -1 \Rightarrow B = 2 \text{ from (5).}$$

From (1) $C = 5$ while from (4) $D = -10$.

Method II (Substitution using complex numbers):

- Put $x = \sqrt{3}i$ into (*), we have $-9\sqrt{3}i - 2\sqrt{3}i - 20 = (A\sqrt{3}i + B)(-6 - 6\sqrt{3}i + 5)$
 $\Rightarrow -20 - 11\sqrt{3}i = (A\sqrt{3}i + B)(-1 - 6\sqrt{3}i) = (18A - B) - (A + 6B)\sqrt{3}i$

By comparison, we have: $18A - B = -20 \dots \dots (7)$ and $A + 6B = 11 \dots \dots (8)$

$$6(7) + (8): 109A = -109 \Rightarrow A = -1 \Rightarrow B = 2 \text{ from (7)}$$

- Put $x = \frac{3}{2} + \frac{1}{2}i$ into (*), we have

$$\begin{aligned} 3\left(\frac{9}{4} + \frac{13}{4}i\right) - 2\left(\frac{3}{2} + \frac{1}{2}i\right) - 20 &= \left[\left(\frac{3}{2} + \frac{1}{2}i\right)C + D\right]\left[\left(2 + \frac{3}{2}i\right) + 3\right] = \left[\left(\frac{3}{2}C + D\right) + \frac{1}{2}Ci\right]\left(5 + \frac{3}{2}i\right) \\ \Rightarrow -\frac{65}{4} + \frac{35}{4}i &= \left(\frac{15}{2}C + 5D - \frac{3}{4}C\right) + \left(\frac{9}{4}C + \frac{3}{2}D + \frac{5}{2}C\right)i = \left(\frac{27}{4}C + 5D\right) + \left(\frac{19}{4}C + \frac{3}{2}D\right)i \end{aligned}$$

$$\text{By comparison, we have } \frac{27}{4}C + 5D = -\frac{65}{4} \dots \dots (9) \text{ and } \frac{19}{4}C + \frac{3}{2}D = \frac{35}{4} \dots \dots (10)$$

$$\frac{3}{2}(9) - 5(10): -\frac{109}{8}C = -\frac{545}{8} \Rightarrow C = 5. \text{ Thus from (9), } D = \frac{-\frac{65}{4} - \frac{135}{4}}{5} = -\frac{200}{20} = -10.$$

Then the integral can be computed as

$$\begin{aligned} \int \frac{3x^3 - 2x - 20}{(x^2 + 3)(2x^2 - 6x + 5)} dx &= \int \frac{-x + 2}{x^2 + 3} dx + \int \frac{5x - 10}{2x^2 - 6x + 5} dx \\ &= -\int \frac{x}{x^2 + 3} dx + 2 \int \frac{1}{x^2 + 3} dx + \int \frac{5x - \frac{15}{2}}{2x^2 - 6x + 5} dx - \frac{5}{2} \int \frac{1}{2x^2 - 6x + 5} dx \\ &\quad y=x^2+3 \Rightarrow \frac{dy}{dx}=2x \\ &\quad z=2x^2-6x+5 \Rightarrow \frac{dz}{dx}=4x-6 \\ &\quad \cong -\frac{1}{2} \int \frac{1}{y} dy + \frac{2}{3} \int \frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2 + 1} dx + \frac{5}{4} \int \frac{1}{z} dz - 5 \int \frac{1}{(2x-3)^2 + 1} dx \\ &= -\frac{1}{2} \ln|y| + \frac{2\sqrt{3}}{3} \tan^{-1} \frac{x}{\sqrt{3}} + \frac{5}{4} \ln|z| - \frac{5}{2} \tan^{-1}(2x-3) + C \\ &= -\frac{1}{2} \ln|x^2 + 3| + \frac{2\sqrt{3}}{3} \tan^{-1} \frac{x}{\sqrt{3}} + \frac{5}{4} \ln|2x^2 - 6x + 5| - \frac{5}{2} \tan^{-1}(2x-3) + C. \end{aligned}$$

(k) Note that there are one repeated factor and a quadratic factor in the denominator, we shall decompose the integrand as

$$\frac{6x^3 - 27x^2 + 5x - 1}{(x-2)^2(4x^2+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{4x^2+1}$$

$$\Rightarrow 6x^3 - 27x^2 + 5x - 1 = A(x-2)(4x^2+1) + B(4x^2+1) + (Cx+D)(x-2)^2.$$

- Put $x = 2$, $-51 = 17B \Rightarrow B = -3$.

$$\Rightarrow A(x-2)(4x^2+1) + (Cx+D)(x-2)^2 = 6x^3 - 27x^2 + 5x - 1 + 3(4x^2+1)$$

$$= 6x^3 - 15x^2 + 5x + 2 = (x-2)(6x^2 - 3x - 1) \text{ by synthetic division}$$

$$\Rightarrow A(4x^2+1) + (Cx+D)(x-2) = 6x^2 - 3x - 1$$

- Put $x = 2$, $17A = 24 - 6 - 1 = 17 \Rightarrow A = 1$.
- Compare the coefficient of x^2 : $4A + C = 6 \Rightarrow C = 2$.
- Compare the constant term: $A - 2D = -1 \Rightarrow D = 1$.

Then the integral can be computed as

$$\begin{aligned} \int \frac{6x^3 - 27x^2 + 5x - 1}{(x-2)^2(4x^2+1)} dx &= \int \frac{1}{x-2} dx - 3 \int \frac{1}{(x-2)^2} dx + \int \frac{2x+1}{4x^2+1} dx \\ &= \ln|x-2| + \frac{3}{x-2} + \int \frac{2x}{4x^2+1} dx + \int \frac{1}{4x^2+1} dx \\ &\quad \begin{matrix} y=4x^2+1 \\ \Rightarrow \frac{dy}{dx}=8x \end{matrix} \\ &\quad \cong \ln|x-2| + \frac{3}{x-2} + \frac{1}{4} \int \frac{1}{y} dy + \int \frac{1}{(2x)^2+1} dx \\ &= \ln|x-2| + \frac{3}{x-2} + \frac{1}{4} \ln|y| + \frac{1}{2} \tan^{-1} 2x + C \\ &= \ln|x-2| + \frac{3}{x-2} + \frac{1}{4} \ln|4x^2+1| + \frac{1}{2} \tan^{-1} 2x + C. \end{aligned}$$

- (I) Note that there is a repeated quadratic factor in the denominator, one shall decompose the integrand as

$$\frac{x^2 + 2x + 4}{x(x^2 + 2)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2 + 2)^2}$$

$$\Rightarrow x^2 + 2x + 4 = A(x^2 + 2)^2 + x(x^2 + 2)(Bx + C) + x(Dx + E).$$

- Put $x = 0$, $4A = 4 \Rightarrow A = 1$.

$$\Rightarrow x(x^2 + 2)(Bx + C) + x(Dx + E) = x^2 + 2x + 4 - (x^2 + 2)^2 = -x^4 - 3x^2 + 2x$$

$$= -x(x^3 + 3x - 2)$$

$$\Rightarrow (x^2 + 2)(Bx + C) + Dx + E = -(x^3 + 3x - 2) = -x(x^2 + 2) - (x - 2) \text{ by long division}$$

By comparing the coefficients, we get $B = -1$, $C = 0$, $D = -1$ and $E = 2$.

Then the integral can be computed as

$$\int \frac{x^2 + 2x + 4}{x(x^2 + 2)^2} dx = \int \frac{1}{x} dx - \int \frac{x}{x^2 + 2} dx - \int \frac{x - 2}{(x^2 + 2)^2} dx$$

$$= \ln|x| - \int \frac{x}{x^2 + 2} dx - \int \frac{x}{(x^2 + 2)^2} dx + 2 \int \frac{1}{(x^2 + 2)^2} dx$$

$y = x^2 + 2$ (for 1st and 2nd integrals)

$x = \sqrt{2} \tan \theta$ (for 3rd integral)

$$\begin{aligned} &\cong \ln|x| - \frac{1}{2} \int \frac{1}{y} dy - \frac{1}{2} \int \frac{1}{y^2} dy + \frac{\sqrt{2}}{2} \int \frac{1}{\sec^2 \theta} d\theta \\ &= \ln|x| - \frac{1}{2} \ln|y| + \frac{1}{2y} + \frac{\sqrt{2}}{2} \int \cos^2 \theta d\theta \\ &= \ln|x| - \frac{1}{2} \ln|x^2 + 2| + \frac{1}{2(x^2 + 2)} + \frac{\sqrt{2}}{4} \int (\cos 2\theta + 1) d\theta \\ &= \ln|x| - \frac{1}{2} \ln|x^2 + 2| + \frac{1}{2(x^2 + 2)} + \frac{\sqrt{2}}{8} \sin 2\theta + \frac{\sqrt{2}}{4} \theta + C \\ &= \ln|x| - \frac{1}{2} \ln|x^2 + 2| + \frac{1}{2(x^2 + 2)} + \frac{\sqrt{2}}{4} \sin \theta \cos \theta + \frac{\sqrt{2}}{4} \tan^{-1} \frac{x}{\sqrt{2}} + C \\ &= \ln|x| - \frac{1}{2} \ln|x^2 + 2| + \frac{1}{2(x^2 + 2)} + \frac{\sqrt{2}}{4} \frac{x}{\sqrt{x^2 + 2}} \frac{\sqrt{2}}{\sqrt{x^2 + 2}} + \frac{\sqrt{2}}{4} \tan^{-1} \frac{x}{\sqrt{2}} + C \\ &= \ln|x| - \frac{1}{2} \ln|x^2 + 2| + \frac{x + 1}{2(x^2 + 2)} + \frac{\sqrt{2}}{4} \tan^{-1} \frac{x}{\sqrt{2}} + C \end{aligned}$$