

Practice Problem Set 5 Solutions

Question 1: Use the Lagrangean method to find the positive values of x and y which maximize

$$\frac{2y}{y+2} + \frac{x}{x+1}$$

subject to the constraint $x + y = 120$.

The Lagrangean is

$$\mathcal{L}(x, y, \lambda) = \frac{2y}{y+2} + \frac{x}{x+1} - \lambda(x + y - 120),$$

where λ is a new variable known as the Lagrange multiplier. Then, the solution to the constrained optimization problem can be found by solving the equations

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{(x+1)^2} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{4}{(y+2)^2} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x + y - 120 = 0$$

From the first two equations, eliminating λ , we have

$$\frac{1}{(x+1)^2} = \frac{4}{(y+2)^2}$$

So, given that $x, y > 0$, $y + 2 = 2(x + 1)$ and hence

$$y = 2x$$

Then the third equation gives $3x = 120$, so we have

$$x = 40$$

$$y = 80$$

Question 2: For which value of c is the following system of equations consistent?

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 + x_2 + 2x_3 = 5$$

$$4x_1 + 3x_2 + 4x_3 = c$$

The augmented matrix is

$$A = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 5 \\ 4 & 3 & 4 & c \end{array} \right)$$

Performing row operations to reduce the above to echelon form, we have

$$\begin{aligned} &\xrightarrow{4R_1 - R_3 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 5 \\ 0 & 1 & 0 & 8 - c \end{array} \right) \\ &\xrightarrow{R_2 - 2R_1 \rightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 8 - c \end{array} \right) \\ &\xrightarrow{R_2 + R_3 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 9 - c \end{array} \right) \end{aligned}$$

Looking at the bottom row of the last matrix, we see that the system of equations is inconsistent unless $9 - c = 0$; that is, it is consistent only when $c = 9$.

Question 3: Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ -1 & 0 & 1 \end{pmatrix}$$

We form the matrix

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 6 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

Performing elementary row operations, we have

$$\begin{aligned} &(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 6 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \\ &\xrightarrow{R_1 + R_3 \rightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 6 & 0 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{array} \right) \end{aligned}$$

$$\begin{aligned}
&\xrightarrow{R_2 - R_1 \rightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{array} \right) \\
&\xrightarrow{R_3 - 2R_2 \rightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array} \right) \\
&\xrightarrow{R_2 - 2R_3 \rightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array} \right) \\
&\xrightarrow{R_1 - 4R_3 \rightarrow R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -11 & 8 & -4 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array} \right) \\
&\xrightarrow{R_1 - 2R_2 \rightarrow R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & -7 & 5 & -2 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array} \right)
\end{aligned}$$

This is now in the form $(I|B)$, so we deduce that A is invertible and that

$$A^{-1} = \begin{pmatrix} 3 & -2 & 0 \\ -7 & 5 & -2 \\ 3 & -2 & 1 \end{pmatrix}$$

Question 4: Suppose that the numbers x_1 and x_2 satisfy the equations

$$x_1 - 2x_2 = 3$$

Find x_1 and x_2 using Cramer's rule. $3x_1 + 5x_2 = 20$

We have the following system of linear equations

$$\begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 20 \end{pmatrix}$$

By Cramer's rule, we have

$$x_1 = \frac{\begin{vmatrix} 3 & -2 \\ 20 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix}} = \frac{(3)(5) - (-2)(20)}{(1)(5) - (-2)(3)} = \frac{55}{11} = 5$$

$$x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 3 & 20 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix}} = \frac{(1)(20) - (3)(3)}{11} = \frac{11}{11} = 1$$

Question 5: An economy has four industries: coal, iron, electricity, and railways. To produce \$1 of coal requires \$0.10 worth of iron, \$0.20 of electricity and \$0.50 rail costs for transportation. To produce \$1 of iron requires \$0.60 worth of coal for fuel, \$0.01 of electricity and \$0.20 for transport. To produce \$1 of electricity requires \$0.50 worth of coal for fuel, \$0.02 of electricity for the auxiliary equipment, and \$0.01 for transport. To provide \$1 worth of transport, the railway requires \$0.50 coal for fuel, \$0.20 iron, and \$0.25 electricity. Each week, the external demand for coal is \$8000, the external demand for iron is \$1000, the external demand for electricity is \$4000, and the external demand for the railway is \$6000. Write down the equation which determines the production schedule for these industries.

Let x_1, x_2, x_3 and x_4 be the amount of coal, iron, electricity, and railways measured in dollars. We then have

$$x_1 = 8000 + 0.6x_2 + 0.5x_3 + 0.5x_4$$

$$x_2 = 1000 + 0.1x_1 + 0.2x_4$$

$$x_3 = 4000 + 0.2x_1 + 0.01x_2 + 0.02x_3 + 0.25x_4$$

$$x_4 = 6000 + 0.5x_1 + 0.2x_2 + 0.01x_3$$

In matrix form, the system is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8000 \\ 1000 \\ 4000 \\ 6000 \end{pmatrix} + \begin{pmatrix} 0 & 0.6 & 0.5 & 0.5 \\ 0.1 & 0 & 0 & 0.2 \\ 0.2 & 0.01 & 0.02 & 0.25 \\ 0.5 & 0.2 & 0.01 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \mathbf{d} + A\mathbf{x}$$

and so we need to solve the equation $(I - A)\mathbf{x} = \mathbf{d}$, that is

$$\begin{pmatrix} 1 & -0.6 & -0.5 & -0.5 \\ -0.1 & 1 & 0 & -0.2 \\ -0.2 & -0.01 & 0.98 & -0.25 \\ -0.5 & -0.2 & -0.01 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8000 \\ 1000 \\ 4000 \\ 6000 \end{pmatrix}$$