

Solution to Problem Set 3: Laplace Transform

$$\begin{aligned}
1. \quad (a) \quad \mathcal{L}[\sin(\omega t + \theta)] &= \int_0^{\infty} e^{-st} \sin(\omega t + \theta) dt = \int_0^{\infty} e^{-st} d\left[\frac{-\cos(\omega t + \theta)}{\omega}\right] dt \\
&= \left[\frac{-e^{-st} \cos(\omega t + \theta)}{\omega}\right] \Big|_0^{\infty} - \int_0^{\infty} \frac{-\cos(\omega t + \theta)}{\omega} d(e^{-st}) \\
&= \frac{\cos \theta}{\omega} - \frac{s}{\omega^2} \int_0^{\infty} e^{-st} \cos(\omega t + \theta) dt = \frac{\cos \theta}{\omega} - \frac{s}{\omega^2} \int_0^{\infty} e^{-st} d[\sin(\omega t + \theta)] \\
&= \frac{\cos \theta}{\omega} - \frac{s}{\omega^2} \left(\left[e^{-st} \sin(\omega t + \theta) \right] \Big|_0^{\infty} - \int_0^{\infty} \sin(\omega t + \theta) d(e^{-st}) \right) = \frac{\cos \theta}{\omega} + \frac{s \sin \theta}{\omega^2} - \frac{s^2}{\omega^2} \int_0^{\infty} e^{-st} \sin(\omega t + \theta) dt \\
&\Rightarrow \left(1 + \frac{s^2}{\omega^2}\right) \int_0^{\infty} e^{-st} \sin(\omega t + \theta) dt = \frac{\cos \theta}{\omega} + \frac{s \sin \theta}{\omega^2} \\
&\Rightarrow \int_0^{\infty} e^{-st} \sin(\omega t + \theta) dt = \frac{\frac{\cos \theta}{\omega} + \frac{s \sin \theta}{\omega^2}}{1 + \frac{s^2}{\omega^2}} = \frac{\frac{\omega \cos \theta + s \sin \theta}{\omega^2}}{\frac{\omega^2 + s^2}{\omega^2}} = \frac{\omega \cos \theta + s \sin \theta}{\omega^2 + s^2} \\
\text{So } \mathcal{L}[\sin(\omega t + \theta)] &= \frac{\omega \cos \theta + s \sin \theta}{\omega^2 + s^2}.
\end{aligned}$$

$$(b) \quad \mathcal{L}[\cosh^2 3t] = \int_0^{\infty} e^{-st} \left(\frac{e^{3t} + e^{-3t}}{2} \right)^2 dt = \frac{1}{4} \int_0^{\infty} e^{-st} (e^{6t} + 2 + e^{-6t}) dt = \frac{1}{4} \left(\frac{1}{s-6} + \frac{2}{s} + \frac{1}{s+6} \right)$$

$$\begin{aligned}
(c) \quad f(t) &= \begin{cases} \frac{k}{c}t & 0 \leq t \leq c \\ 0 & \text{otherwise} \end{cases} \\
\mathcal{L}[f(t)] &= \int_0^{\infty} e^{-st} \frac{k}{c} t dt = \frac{k}{c} \left(\frac{te^{-st}}{-s} \Big|_0^c - \int_0^c \frac{e^{-st}}{-s} dt \right) = \frac{k}{c} \left[\frac{ce^{-sc}}{-s} + \frac{1}{s} \left(\frac{e^{-st}}{-s} \Big|_0^c \right) \right] \\
&= \frac{k}{c} \left[\frac{ce^{-sc}}{-s} - \frac{1}{s^2} (e^{-sc} - 1) \right] = \frac{k}{s^2 c} [1 - (1 + sc)e^{-sc}]
\end{aligned}$$

$$\begin{aligned}
(d) \quad f(t) &= \begin{cases} k & a \leq t \leq b \\ 0 & \text{otherwise} \end{cases} \\
\mathcal{L}[f(t)] &= \int_a^b e^{-st} k dt = k \frac{e^{-st}}{-s} \Big|_a^b = \frac{k}{s} (e^{-sa} - e^{-sb})
\end{aligned}$$

$$\begin{aligned}
(e) \quad \sin(\omega t + \theta) &= \sin \omega t \cos \theta + \cos \omega t \sin \theta \\
\Rightarrow \mathcal{L}[\sin(\omega t + \theta)] &= \mathcal{L}[\sin \omega t \cos \theta + \cos \omega t \sin \theta] = \cos \theta \mathcal{L}[\sin \omega t] + \sin \theta \mathcal{L}[\cos \omega t] \\
&= \frac{\omega \cos \theta + s \sin \theta}{s^2 + \omega^2}
\end{aligned}$$

2. (a) $\mathcal{L}[t^6 - 3t^4 + t^2] = \frac{6!}{s^7} - 3\frac{4!}{s^5} + \frac{2!}{s^3}$
- (b) $\mathcal{L}[\sin^2 t] = \mathcal{L}\left[\frac{1}{2} - \frac{\cos 2t}{2}\right] = \frac{1}{2} \mathcal{L}[1] - \frac{1}{2} \mathcal{L}[\cos 2t] = \frac{1}{2s} - \frac{s}{2(s^2 + 4)}$
- (c) $\sin t \cos t = \frac{\sin 2t}{2} \Rightarrow \mathcal{L}[\sin t \cos t] = \frac{1}{2} \mathcal{L}[\sin 2t] = \frac{2}{2(s^2 + 4)}$
- (d) $\mathcal{L}[e^t \sin^2 t] = \frac{1}{2} \mathcal{L}[e^t] - \frac{1}{2} \mathcal{L}[e^t \cos 2t] = \frac{1}{2(s-1)} - \frac{1}{2} \left[\frac{s-1}{(s-1)^2 + 4} \right]$
- (e) $\mathcal{L}[e^{-\alpha t} (A \cos \omega t + B \sin \omega t)] = \frac{A(s+\alpha)}{(s+\alpha)^2 + \omega^2} + \frac{B\omega}{(s+\alpha)^2 + \omega^2}$ (First Shift Theorem)
3. (a) $\frac{s-4}{s^2-4} = \frac{3}{2} \frac{1}{s+2} - \frac{1}{2} \frac{1}{s-2} \Rightarrow \mathcal{L}^{-1}\left[\frac{s-4}{s^2-4}\right] = \frac{3}{2} \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2} \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = \frac{3}{2} e^{-2t} - \frac{1}{2} e^{2t}$
- (b) $\frac{1}{(s-a)(s-b)} = \frac{1}{a-b} \left(\frac{1}{s-a} - \frac{1}{s-b} \right)$
 $\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{(s-a)(s-b)}\right] = \frac{1}{a-b} \mathcal{L}^{-1}\left[\frac{1}{s-a} - \frac{1}{s-b}\right] = \frac{1}{a-b} (e^{at} - e^{bt})$
4. (a) Let $f(t) = t \cos \omega t$.
Then $f'(t) = \cos \omega t - t\omega \sin \omega t$, $f''(t) = -\omega \sin \omega t - \omega^2 t \cos \omega t = -\omega \sin \omega t - \omega^2 f(t)$
 $\mathcal{L}[f''(t)] = -\frac{2\omega^2}{s^2 + \omega^2} - \omega^2 \mathcal{L}[f(t)]$
In addition, $\mathcal{L}[f''] = s^2 \mathcal{L}[f] - sf'(0) - f'(0) = s^2 \mathcal{L}[f] - 1$.
Thus, $-\frac{2\omega^2}{s^2 + \omega^2} - \omega^2 \mathcal{L}[f(t)] = s^2 \mathcal{L}[f] - 1 \Rightarrow \mathcal{L}[f] = \mathcal{L}[t \cos \omega t] = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
- (b) Take Laplace transform
 $y'' - 3y' + 2y = 4t \Rightarrow s^2 Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{4}{s^2}$
 $\Rightarrow (s^2 - 3s + 2)Y(s) = \frac{4}{s^2} + 4s - 9$
 $\Rightarrow Y(s) = \frac{4}{s^2(s-2)(s-1)} + \frac{4s-9}{(s-2)(s-1)} = \frac{2}{s^2} + \frac{3}{s} + \frac{1}{s-2} - \frac{4}{s-1} - \frac{1}{s-2} + \frac{5}{s-1}$
 $\Rightarrow Y(s) = \frac{2}{s^2} + \frac{3}{s} + \frac{1}{s-1}$
So $y(t) = 2t + 3 + e^t$.
5. (a) Observe that $\mathcal{L}[te^{\alpha t}] = \frac{1}{(s-\alpha)^2}$.
 $\mathcal{L}^{-1}\left[\frac{2}{(s-2)^2}\right] = 2 \mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right] = 2te^{2t}$

$$(b) \quad \frac{3}{2s^2 - 8s + 10} = \frac{3}{2[(s-2)^2 + 1]} = \frac{3}{2} \left[\frac{1}{(s-2)^2 + 1} \right]$$

$$\mathcal{L}^{-1} \left[\frac{3}{2s^2 - 8s + 10} \right] = \frac{3}{2} \mathcal{L}^{-1} \left[\frac{1}{(s-2)^2 + 1} \right] = \frac{3e^{2t} \sin t}{2}$$

$$(c) \quad \frac{s+6}{s^2 + 4s + 12} = \frac{(s+2)+4}{(s+2)^2 + 8} = \frac{s+2}{(s+2)^2 + 8} + \frac{4}{(s+2)^2 + 8}$$

$$\mathcal{L}^{-1} \left[\frac{s+6}{s^2 + 4s + 12} \right] = \mathcal{L}^{-1} \left[\frac{s+2}{(s+2)^2 + 8} \right] + \mathcal{L}^{-1} \left[\frac{4}{(s+2)^2 + 8} \right] = e^{-2t} \cos \sqrt{8}t + e^{-2t} \frac{4}{\sqrt{8}} \sin \sqrt{8}t$$

$$(d) \quad \frac{1+e^{-2s}}{s^2} = \frac{1}{s^2} + \frac{e^{-2s}}{s^2} \Rightarrow \mathcal{L}^{-1} \left[\frac{1+e^{-2s}}{s^2} \right] = \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] + \mathcal{L}^{-1} \left[\frac{e^{-2s}}{s^2} \right].$$

$$\text{Observe that } \mathcal{L}[t] = \frac{1}{s^2} \text{ so } \mathcal{L}^{-1} \left(\frac{1}{s^2} \right) = t.$$

By Second Shift Theorem, $g(t) = \mathcal{L}^{-1} [e^{-as} F(s)]$, where $F(s) = \mathcal{L}[f]$ and $g(t) = f(t-a)u_a(t)$.

$$\text{It follows that } \mathcal{L}^{-1} \left[\frac{e^{-2s}}{s^2} \right] = (t-2)u_2(t) = \begin{cases} 0 & 0 \leq t < 2 \\ t-2 & t \geq 2 \end{cases}.$$

$$\begin{aligned} \mathcal{L}^{-1} \left[\frac{1+e^{-2s}}{s^2} \right] &= \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] + \mathcal{L}^{-1} \left[\frac{e^{-2s}}{s^2} \right] = t + (t-2)u_2(t) = t + \begin{cases} 0 & 0 \leq t < 2 \\ t-2 & t \geq 2 \end{cases} \\ &= \begin{cases} t & 0 \leq t < 2 \\ 2t-2 & t \geq 2 \end{cases} = \begin{cases} t & 0 \leq t < 2 \\ 2(t-1) & t \geq 2 \end{cases} \end{aligned}$$

6. (a)

$$\begin{aligned} \sin t * \cos t &= \int_0^t \sin \tau \cos(t-\tau) d\tau = \int_0^t \sin \tau (\cos t \cos \tau + \sin t \sin \tau) d\tau \\ &= \int_0^t \cos t \sin \tau \cos \tau d\tau + \int_0^t \sin t \sin \tau \sin \tau d\tau = \cos t \int_0^t \sin \tau d(\sin \tau) + \sin t \int_0^t \frac{1 - \cos 2\tau}{2} d\tau \\ &= \cos t \frac{\sin^2 \tau}{2} \Big|_0^t + \sin t \left(\int_0^t \frac{1}{2} d\tau - \int_0^t \frac{\cos 2\tau}{2} d\tau \right) = \frac{\cos t \sin^2 t}{2} + \frac{t \sin t}{2} - \frac{\sin t \sin 2\tau}{4} \Big|_0^t \\ &= \frac{\cos t \sin^2 t}{2} + \frac{t \sin t}{2} - \frac{\sin t \sin 2t}{4} = \frac{\cos t \sin^2 t}{2} + \frac{t \sin t}{2} - \frac{\cos t \sin^2 t}{2} = \frac{t \sin t}{2} \end{aligned}$$

$$(b) \quad \frac{s^2}{(s^2 + 4)^2} = \frac{s}{s^2 + 4} \times \frac{s}{s^2 + 4}$$

$$\mathcal{L}^{-1} \left[\frac{s^2}{(s^2 + 4)^2} \right] = \mathcal{L}^{-1} \left[\frac{s}{s^2 + 2^2} \right] * \mathcal{L}^{-1} \left[\frac{s}{s^2 + 4} \right] = \cos 2t * \cos 2t.$$

$$\begin{aligned}
\cos 2t * \cos 2t &= \int_0^t \cos 2\tau \cos 2(t-\tau) d\tau = \int_0^t \cos 2\tau (\cos 2t \cos 2\tau + \sin 2t \sin 2\tau) d\tau \\
&= \int_0^t \cos 2t \cos^2 2\tau d\tau + \int_0^t \sin 2t \sin 2\tau \cos 2\tau d\tau \\
&= \cos 2t \int_0^t \frac{1+\cos 4\tau}{2} d\tau + \sin 2t \int_0^t \frac{\sin 4\tau}{2} d\tau = \cos 2t \left(\frac{\tau}{2} + \frac{\sin 4\tau}{8} \right) \Big|_0^t + \sin 2t \left(-\frac{\cos 4\tau}{8} \right) \Big|_0^t \\
&= \cos 2t \left(\frac{t}{2} + \frac{\sin 4t}{8} \right) + \sin 2t \left(\frac{1-\cos 4t}{8} \right) = \frac{t \cos 2t}{2} + \frac{\sin 4t \cos 2t}{8} + \frac{\sin 2t}{8} - \frac{\sin 2t \cos 4t}{8} \\
&= \frac{t \cos 2t}{2} + \frac{2 \sin 2t \cos^2 2t}{8} + \frac{\sin 2t}{8} - \frac{\sin 2t (2 \cos^2 2t - 1)}{8} = \frac{t \cos 2t}{2} + \frac{\sin 2t}{8} + \frac{\sin 2t}{8} = \frac{t \cos 2t}{2} + \frac{\sin 2t}{4}
\end{aligned}$$

7. Take Laplace transform

$$y'' - 2y' + y = e^x \Rightarrow s^2 Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + Y(s) = \frac{1}{s-1}$$

$$\Rightarrow (s^2 - 2s + 1)Y(s) = \frac{1}{s-1} - 2s + 1 = \frac{1}{s-1} - 2(s-1) - 1$$

$$\Rightarrow Y(s) = \frac{1}{(s^2 - 2s + 1)(s-1)} - \frac{2(s-1)}{(s^2 - 2s + 1)} - \frac{1}{(s^2 - 2s + 1)} = \frac{1}{(s-1)^3} - \frac{2(s-1)}{(s-1)^2} - \frac{1}{(s-1)^2}$$

$$= \frac{1}{2} \frac{2!}{(s-1)^3} - \frac{2}{s-1} - \frac{1}{(s-1)^2}$$

$$\begin{aligned}
\text{So } y(x) &= \mathcal{L}^{-1}[Y(s)] = \frac{1}{2} \mathcal{L}^{-1} \left[\frac{2!}{(s-1)^3} \right] - 2 \mathcal{L}^{-1} \left[\frac{1}{s-1} \right] - \mathcal{L}^{-1} \left[\frac{1}{(s-1)^2} \right] \\
&= \frac{e^x x^2}{2} - 2e^x - e^x x
\end{aligned}$$

8.

$$\begin{cases} s^2 X - 2sX + 3sY + 2Y = \frac{4}{s} \\ 2sY - sX + 3Y = 0 \end{cases} \Rightarrow X = \frac{2sY + 3Y}{s}$$

$$\Rightarrow s^2 \left(\frac{2sY + 3Y}{s} \right) - 2s \left(\frac{2sY + 3Y}{s} \right) + 3sY + 2Y = \frac{4}{s}$$

$$\Rightarrow 2s^2 Y + 3sY - 4sY - 6Y + 3sY + 2Y = \frac{4}{s}$$

$$\Rightarrow Y = \frac{4}{s(2s^2 + 2s - 4)} = \frac{2}{s(s+2)(s-1)}$$

Then

$$\begin{cases} X = \frac{2sY + 3Y}{s} \\ Y = \frac{2}{s(s+2)(s-1)} \end{cases} \Rightarrow X = \frac{2s \left[\frac{2}{s(s+2)(s-1)} \right] + 3 \left[\frac{2}{s(s+2)(s-1)} \right]}{s} = \frac{4s+6}{s^2(s+2)(s-1)}$$

$$X = \frac{4s+6}{s^2(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s-1}$$

$$\Rightarrow 4s+6 = As(s+2)(s-1) + B(s+2)(s-1) + Cs^2(s-1) + Ds^2(s+2)$$

$$\Rightarrow \begin{cases} 6 = -2B \\ 10 = 3D \\ -2 = -12C \\ 14 = 8A + 4B + 4C + 16D \end{cases} \Rightarrow \begin{cases} B = -3 \\ D = \frac{10}{3} \\ C = \frac{1}{6} \\ A = -\frac{7}{2} \end{cases} \Rightarrow X(s) = -\frac{7}{2} \frac{1}{s} - 3 \frac{1}{s^2} + \frac{1}{6} \frac{1}{s+2} + \frac{10}{3} \frac{1}{s-1}$$

$$Y = \frac{2}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

$$\Rightarrow 2 = A(s+2)(s-1) + Bs(s-1) + Cs(s+2) \Rightarrow \begin{cases} 2 = -2A \\ 2 = 3C \\ 2 = 6B \end{cases} \Rightarrow \begin{cases} A = -1 \\ C = \frac{2}{3} \\ B = \frac{1}{3} \end{cases}$$

$$\Rightarrow Y(s) = -\frac{1}{s} + \frac{1}{3} \frac{1}{s+2} + \frac{2}{3} \frac{1}{s-1}$$

And

$$\begin{cases} X(s) = -\frac{7}{2} \frac{1}{s} - 3 \frac{1}{s^2} + \frac{1}{6} \frac{1}{s+2} + \frac{10}{3} \frac{1}{s-1} \\ Y(s) = -\frac{1}{s} + \frac{1}{3} \frac{1}{s+2} + \frac{2}{3} \frac{1}{s-1} \end{cases} \Rightarrow \begin{cases} x(t) = -\frac{7}{2} - 3t + \frac{1}{6} e^{-2t} + \frac{10}{3} e^t \\ y(t) = -1 + \frac{1}{3} e^{-2t} + \frac{2}{3} e^t \end{cases}$$

9. (a) (i), (ii)

$$\text{Let } t = x+1, \text{ then } \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} \underset{\substack{t=x+1 \\ \Rightarrow \frac{dt}{dx}=1}}{=} \frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{dy}{dt} \right) = \frac{d}{dt} \left(\frac{dy}{dt} \right) \frac{dt}{dx} \underset{\substack{t=x+1 \\ \Rightarrow \frac{dt}{dx}=1}}{=} \frac{d^2y}{dt^2}$$

$$\text{and } x = -1 \Rightarrow t = 0.$$

$$\begin{cases} \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = \alpha(x) \\ y(-1) = y'(-1) = 0 \end{cases} \xrightarrow{t=x+1} \begin{cases} \frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \beta(t) \\ y(0) = y'(0) = 0 \end{cases},$$

$$\text{where } \beta(t) = \alpha(t-1) = \begin{cases} 1 & \text{for } 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(b) \begin{cases} \frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \beta(t) \\ y(0) = y'(0) = 0 \end{cases} \Rightarrow s^2 Y(s) + sY(s) - 2Y(s) = B(s) \Rightarrow (s^2 + s - 2)Y(s) = B(s)$$

$$\Rightarrow Y(s) = \frac{B(s)}{s^2 + s - 2} = \frac{B(s)}{(s+2)(s-1)}$$

$$(c) \quad \frac{1}{(s+2)(s-1)} = \frac{a}{s-1} + \frac{b}{s+2} \Rightarrow 1 = a(s+2) + b(s-1) \Rightarrow \begin{cases} a = \frac{1}{3} \\ b = -\frac{1}{3} \end{cases}$$

$$\text{So } Y(s) = \frac{1}{s-1} B(s) - \frac{1}{s+2} B(s).$$

(d)

$$Y(s) = \frac{1}{s-1} B(s) - \frac{1}{s+2} B(s) = \mathcal{L}\left[\frac{e^t}{3}\right] B(s) - \mathcal{L}\left[\frac{e^{-2t}}{3}\right] B(s) = \mathcal{L}\left[\frac{e^t}{3} - \frac{e^{-2t}}{3}\right] B(s)$$

$$\Rightarrow y(t) = \left(\frac{e^t}{3} - \frac{e^{-2t}}{3}\right) * \beta(t)$$

$$\text{And } f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau, \quad t \geq 0$$

$$\left(\frac{e^t}{3} - \frac{e^{-2t}}{3}\right) * \beta(t) = \begin{cases} \int_0^t \left(\frac{e^{t-\tau}}{3} - \frac{e^{-2(t-\tau)}}{3}\right) \beta(\tau) d\tau & \text{for } 0 < t < 1 \\ \int_0^1 \left(\frac{e^{t-\tau}}{3} - \frac{e^{-2(t-\tau)}}{3}\right) \beta(\tau) d\tau & \text{for } t \geq 1 \end{cases}$$

$$\begin{aligned} \int_0^t \left(\frac{e^{t-\tau}}{3} - \frac{e^{-2(t-\tau)}}{3}\right) \beta(\tau) d\tau &= \int_0^t \left(\frac{e^{t-\tau}}{3} - \frac{e^{-2(t-\tau)}}{3}\right) d\tau = \int_0^t \frac{e^{t-\tau}}{3} d\tau - \int_0^t \frac{e^{-2(t-\tau)}}{3} d\tau \\ &= -\int_0^t \frac{e^{t-\tau}}{3} d(t-\tau) + \int_0^t \frac{e^{-2(t-\tau)}}{3} d(t-\tau) = -\frac{e^{t-\tau}}{3} \Big|_0^t + \frac{e^{-2(t-\tau)}}{6} \Big|_0^t = -\frac{1}{3} + \frac{e^t}{3} - \frac{1}{6} + \frac{e^{-2t}}{6} = \frac{e^t}{3} + \frac{e^{-2t}}{6} - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \int_0^1 \left(\frac{e^{t-\tau}}{3} - \frac{e^{-2(t-\tau)}}{3}\right) \beta(\tau) d\tau &= \int_0^1 \left(\frac{e^{t-\tau}}{3} - \frac{e^{-2(t-\tau)}}{3}\right) d\tau = \int_0^1 \frac{e^{t-\tau}}{3} d\tau - \int_0^1 \frac{e^{-2(t-\tau)}}{3} d\tau \\ &= -\int_0^1 \frac{e^{t-\tau}}{3} d(t-\tau) + \int_0^1 \frac{e^{-2(t-\tau)}}{3} d(t-\tau) = -\frac{e^{t-\tau}}{3} \Big|_0^1 + \frac{e^{-2(t-\tau)}}{6} \Big|_0^1 = -\frac{e^{t-1}}{3} + \frac{e^t}{3} - \frac{e^{-2(t-1)}}{6} + \frac{e^{-2t}}{6} \end{aligned}$$

$$\begin{aligned} \text{Hence, } \left(\frac{e^t}{3} - \frac{e^{-2t}}{3}\right) * \beta(t) &= \begin{cases} \int_0^t (t-\tau) \left(\frac{e^\tau}{3} - \frac{e^{-2\tau}}{3}\right) d\tau & \text{for } 0 < t < 1 \\ \int_0^1 (t-\tau) \left(\frac{e^\tau}{3} - \frac{e^{-2\tau}}{3}\right) d\tau & \text{for } t \geq 1 \end{cases} \\ &= \dots = \begin{cases} \frac{e^t}{3} + \frac{e^{-2t}}{6} - \frac{1}{2} & \text{for } 0 < t < 1 \\ -\frac{e^{t-1}}{3} + \frac{e^t}{3} - \frac{e^{-2(t-1)}}{6} + \frac{e^{-2t}}{6} & \text{for } t \geq 1 \end{cases} \end{aligned}$$

$$\text{So } y(t) = \begin{cases} \frac{e^t}{3} + \frac{e^{-2t}}{6} - \frac{1}{2} & \text{for } 0 < t < 1 \\ -\frac{e^{t-1}}{3} + \frac{e^t}{3} - \frac{e^{-2(t-1)}}{6} + \frac{e^{-2t}}{6} & \text{for } t \geq 1 \end{cases}.$$

(e) It follows that

$$y(x) = \begin{cases} \frac{e^{x+1}}{3} + \frac{e^{-2(x+1)}}{6} - \frac{1}{2} & \text{for } -1 < x < 0 \\ -\frac{e^x}{3} + \frac{e^{x+1}}{3} - \frac{e^{-2x}}{6} + \frac{e^{-2(x+1)}}{6} & \text{for } x \geq 0 \end{cases}$$

$$10. \quad f(t) = at + \int_0^t \sin(t-\tau) f(\tau) d\tau = at + f(t) * \sin t.$$

It follows that

$$\mathcal{L}[f(t)] = \mathcal{L}[at + f(t) * \sin t] = \mathcal{L}[at] + \mathcal{L}[f(t) * \sin t] = \mathcal{L}[at] + \mathcal{L}[f(t)] \mathcal{L}[\sin t]$$

$$\mathcal{L}[f(t)] = \frac{a}{s^2} + \frac{1}{s^2+1} \mathcal{L}[f(t)] \Rightarrow \mathcal{L}[f(t)] \left(1 - \frac{1}{s^2+1}\right) = \frac{a}{s^2}$$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{\frac{a}{s^2}}{1 - \frac{1}{s^2+1}} = \frac{\frac{a}{s^2}}{\frac{s^2}{s^2+1}} = \frac{a(s^2+1)}{s^4} = \frac{a}{s^2} + \frac{a}{s^4}$$

$$\text{Therefore, } f(t) = a \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \frac{a}{6} \mathcal{L}^{-1}\left[\frac{6}{s^4}\right] = at + \frac{at^3}{6}.$$

11.

$$y' - 4y + 4 \int_0^t y dt = \frac{1}{3} t^3 \Rightarrow \mathcal{L}[y'] - 4 \mathcal{L}[y] + 4 \mathcal{L}\left[\int_0^t y dt\right] = \mathcal{L}\left[\frac{1}{3} t^3\right]$$

$$\Rightarrow sY(s) - y(0) - 4Y(s) + \frac{4Y(s)}{s} = \frac{2}{s^4}$$

$$\Rightarrow_{y(0)=0} Y(s) = \frac{2}{s^3(s-2)^2}$$

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{2}{s^3(s-2)^2}\right] = \frac{1}{8}t^2 + \frac{1}{4}t + \frac{3}{16} - \frac{3}{16}e^{2t} + \frac{1}{8}te^{2t}.$$

12.

$$\begin{aligned} \mathcal{L}[f(t)] &= \mathcal{L}\left[\int_0^t ue^{-2u} \cos 3u du\right] = \frac{1}{s} \mathcal{L}[te^{-2t} \cos 3t] = -\frac{1}{s} \frac{d}{ds} \mathcal{L}[e^{-2t} \cos 3t] \\ &= -\frac{1}{s} \frac{d}{ds} (\mathcal{L}[\cos 3t])|_{s+2} \underset{\mathcal{L}[\cos 3t] = \frac{s}{s^2+9}}{=} -\frac{1}{s} \frac{d}{ds} \left[\frac{s+2}{(s+2)^2+9} \right] = \frac{(s+2)^2-9}{s[(s+2)^2+9]^2} \end{aligned}$$

-End-