

EE 3121 - HW 1-3 Solution

Q1. (a) The characteristic equation for the given DE is

$$f(m) = m^2 + 3m + 2 = (m+2)(m+1) = 0, \text{ so the general solution is}$$

$$y = c_1 e^{-2x} + c_2 e^{-x}.$$

By applying the initial condition, we get

$$y(0) = 1 = c_1 + c_2, \quad y'(0) = 6 = -2c_1 - c_2 \rightarrow c_1 = -7, \quad c_2 = 8.$$

Hence, $y(x) = -7e^{-2x} + 8e^{-x}$ and to find the highest point, we need to find x^* that achieves $y'(x^*) = 0$.

$$y'(x) = 14e^{-2x} - 8e^{-x} = 0 \rightarrow 14e^{-2x} = 8e^{-x}, \quad x = \ln\left(\frac{14}{8}\right)$$

Therefore, the highest point on $y(x) = -7e^{-2x} + 8e^{-x}$ is

$$y\left(\ln\left(\frac{7}{4}\right)\right) = 8e^{-\ln(\frac{7}{4})} - 7e^{-2\ln(\frac{7}{4})} = 8 \cdot \frac{4}{7} - 7\left(\frac{4}{7}\right)^2 = \frac{32}{7} - \frac{16}{7} = \frac{16}{7}$$

$$\text{Given } x = x^* = \ln\left(\frac{14}{8}\right) = \ln\left(\frac{7}{4}\right)$$

∴ The highest point is $(x, y) = \left(\ln\left(\frac{7}{4}\right), \frac{16}{7}\right)$

$$\text{Q1. (b)} \quad 3y^{(3)} + 2y^{(2)} = 0 \rightarrow f(m) = 3m^3 + 2m^2 = m^2(3m+2) = 0$$

$$m = 0, 0, -\frac{2}{3}$$

Then, the general solution is

$$y(x) = C_1 + C_2 x + C_3 e^{-\frac{2}{3}x}$$

By initial conditions, we get

$$y(0) = -1 = C_1 + C_3, \quad y'(0) = 0 = C_2 - \frac{2}{3}C_3, \quad y''(0) = 1 = \frac{4}{9}C_3$$

$$\Rightarrow C_1 = -\frac{13}{4}, \quad C_2 = \frac{3}{2}, \quad C_3 = \frac{9}{4}$$

Therefore, the solution is

$$y(x) = -\frac{13}{4} + \frac{3}{2}x + \frac{9}{4}e^{-\frac{2}{3}x} = \frac{1}{4}(-13 + 6x + 9e^{-\frac{2}{3}x})$$

Q2. a) Since the general solution of the given DE is

$$y(x) = C_1 e^{-2x} + C_2 e^{-x},$$

we can find (C_1, C_2) for y_1 and y_2 , separately.

$$\text{For } y_1, \quad y_1(0) = 3 = C_1 + C_2, \quad y_1'(0) = 1 = -2C_1 - C_2,$$

$$\text{So, } C_1 = -4, \quad C_2 = 7 \text{ for } y_1(x)$$

$$\Rightarrow y_1(x) = -4e^{-2x} + 7e^{-x}$$

$$\text{For } y_2, \quad y_2(0) = 0 = C_1 + C_2, \quad y_2'(0) = 1 = -2C_1 - C_2$$

$$\text{So, } C_1 = -1, \quad C_2 = 1 \text{ for } y_2(x)$$

$$\Rightarrow y_2(x) = -e^{-2x} + e^{-x}$$

Q2. b) $y_1(x) = y_2(x)$

$$\Rightarrow -4e^{-2x} + 7e^{-x} = -e^{-2x} + e^{-x} \Leftrightarrow 2e^{-x} = e^{-2x}, e^x = \frac{1}{2}$$

$$\Rightarrow x = \ln\left(\frac{1}{2}\right) = -\ln 2 \text{ and } y_1\left(\ln\left(\frac{1}{2}\right)\right) = -4e^{2\ln 2} + 7e^{\ln 2} = -2$$

Hence, the intersection point is

$$(x, y) = (-\ln 2, -2)$$

Q3. a) $y'' - 2y' + 2y = 0 \rightarrow$ The characteristic equation $f(m) = m^2 - 2m + 2$

$$= (m - \alpha)(m - \bar{\alpha}) = 0$$

$$\text{where } \alpha = 1 + j, \bar{\alpha} = 1 - j.$$

So, the general solution is

$$y(x) = e^x (C_1 \cos x + C_2 \sin x) \text{ and}$$

by using the initial conditions,

$$y(0) = 0 = C_1$$

$$\begin{aligned} y'(0) = 5 &= e^x (C_1 \cos x + C_2 \sin x) + e^x (-C_1 \sin x + C_2 \cos x) \Big|_{x=0} \\ &= C_1 + C_2 \rightarrow C_1 = 0, C_2 = 5 \end{aligned}$$

$$\text{Hence, } y(x) = 5e^x \sin(x)$$

Q3. b) $9y'' - 12y' + 4y = 0$, whose characteristic equation is

$$f(m) = 9m^2 - 12m + 4$$

$$= (3m-2)^2 = 0 \rightarrow m = \frac{2}{3}, \frac{2}{3}$$

Hence, the general solution is

$$y(x) = C_1 e^{\frac{2}{3}x} + C_2 x e^{\frac{2}{3}x}$$

Q4. a) $y'' + 4y' + 3y = 2\sin(2t) \rightarrow$ The characteristic equation is given by
 $f(m) = (m+3)(m+1) = 0$

$\rightarrow$ The complementary solution is

$$y_c = C_1 e^{-3t} + C_2 e^{-t},$$

where the particular solution is

$$y_p = \alpha_1 \sin(2t) + \alpha_2 \cos(2t).$$

By substituting y_p back to the given DE, we get

$$(8\alpha_1 - \alpha_2) \cos(2t) - (\alpha_1 + 8\alpha_2) \sin(2t) = 2\sin(2t)$$

$$\rightarrow \alpha_1 = -\frac{2}{65}, \alpha_2 = -\frac{16}{65}.$$

Hence, the general solution $y(t)$ is given by

$$y(t) = y_c + y_p = C_1 e^{-3t} + C_2 e^{-t} - \frac{2}{65} \left[\sin(2t) + 8 \cos(2t) \right]$$

Q 4. b). $y'' + 7y' + 10y = 6e^{-2t} \rightarrow f(m) = (m+5)(m+2) = 0$

$\rightarrow$ The complementary solution is $y_c = C_1 e^{-5t} + C_2 e^{-2t}$

$\rightarrow$ Then, the particular solution is $y_p = \alpha_1 t e^{-2t}$

By substituting y_p back to the given DE,

$$\Rightarrow \{-4\alpha_1(1-t) + 7\alpha_1(1-2t) + 10\alpha_1 t\} e^{-2t} = 6e^{-2t}$$

$$\Rightarrow 3\alpha_1 = 6, \text{ so } \alpha_1 = 2$$

Hence, the general solution $y(t)$ is given by

$$y(t) = y_c + y_p = C_1 e^{-5t} + C_2 e^{-2t} + 2t e^{-2t}$$

Q 4. c) $y'' - 10y' + 25y = 17 \cdot t e^{5t} \rightarrow f(m) = (m-5)^2 = 0$

$\rightarrow$ The complementary solution is

$$y_c = C_1 e^{5t} + C_2 t \cdot e^{5t} \quad \text{and}$$

We will use Variation of Parameter method to

find the particular solution. ($P(t) = -10$, $Q(t) = 25$, $R(t) = 17 \cdot t e^{5t}$)

$$\begin{aligned} W &= y_1 y_2' - y_2 y_1' = e^{5t} (1+5t) e^{5t} - 5t e^{5t} \cdot e^{5t} \\ &= e^{10t} \end{aligned}$$

Q4. c) (Continue)

$$u(t) = - \int \frac{t \cdot e^{5t} \cdot 17 \cdot t e^{5t}}{e^{10t}} dt = -\frac{17}{3} t^3$$

$$v(t) = \int \frac{e^{5t} \cdot 17 \cdot t e^{5t}}{e^{10t}} dt = \frac{17}{2} t^2$$

→ Hence, the general solution $y(t)$ is given by

$$\begin{aligned} y &= y_c + y_p = C_1 e^{5t} + C_2 t e^{5t} - \frac{17}{3} t^3 e^{5t} - \frac{17}{2} t^3 e^{5t} \\ &= C_1 e^{5t} + C_2 t e^{5t} - \frac{85}{6} t^3 e^{5t} \end{aligned}$$

Q4. d) $t^2 y'' - t y' + y = 0$, $y(1) = 3$, $y'(1) = -2$

Assume $t = \exp(\ell)$, i.e., $\frac{dt}{d\ell} = e^\ell = t$

$$y' = \frac{dy}{dt} = \frac{d\ell}{dt} \cdot \frac{dy}{d\ell} = \frac{1}{t} \cdot \frac{dy}{d\ell} \rightarrow t y' = \frac{dy}{d\ell}$$

$$\begin{aligned} y'' &= \frac{d}{dt} \left(\frac{1}{t} \cdot \frac{dy}{d\ell} \right) = -\frac{1}{t^2} \cdot \frac{dy}{d\ell} + \frac{1}{t} \cdot \frac{d}{dt} \cdot \frac{dy}{d\ell} = -\frac{1}{t^2} \cdot \frac{dy}{d\ell} + \frac{1}{t^2} \cdot \frac{d^2 y}{d\ell^2} \\ &\rightarrow t^2 y'' = -\frac{dy}{d\ell} + \frac{d^2 y}{d\ell^2} \end{aligned}$$

Then, the given DE can be changed as follows

$$\frac{d^2 y}{d\ell^2} - 2 \frac{dy}{d\ell} + y = 0 \rightarrow y(\ell) = C_1 e^\ell + C_2 \ell \cdot e^\ell$$

Since $t = e^\ell$, $\ell = \ln t$, the general solution is given in terms of t as

Q4. d) (Continue)

$$y(t) = C_1 t + C_2 t \cdot \ln t.$$

Due to the initial conditions, we have

$$y(1) = C_1 = 3, \quad y'(1) = C_1 + C_2 \ln t + C_2 \Big|_{t=1} \\ = C_1 + C_2 = -2$$

→ So, $C_1 = 3$, $C_2 = -5$ and the solution is

$$y(t) = 3t - 5t \cdot \ln(t)$$

Q5. a) $y'' - 2x \cdot y' + 2y = 0$

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$, where $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Then, the given DE can be expressed as follows

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

↓

for this term, denote $n-2=l$, then

$$\sum_{l=0}^{\infty} (l+2)(l+1) a_{l+2} x^l - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n$$

$$= (2a_2 + 2a_0) + \sum_{l=1}^{\infty} \left[(l+2)(l+1) a_{l+2} - 2(l-1) a_l \right] x^l = 0$$

$$\rightarrow a_2 = -a_0, \quad a_{n+2} = \frac{2(n-1) a_n}{(n+2)(n+1)}$$

→ Hence, $y(x) = \sum_{n=0}^{\infty} a_n x^n$ where $a_2 = -a_0$, $a_{n+2} = \frac{2(n-1) a_n}{(n+2)(n+1)}$

Q5. a) (continue)

or we can also express as

$$y(x) = \sum_{n=0,2,4,\dots} a_n x^n + \sum_{n=1,3,5,\dots} a_n x^n = a_0 y_1(x) + a_1 y_2(x),$$

$$\text{where } y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{a_{2k}}{a_0} x^{2k},$$

$$y_2(x) = 1 + \sum_{k=1}^{\infty} \frac{a_{2k+1}}{a_1} x^{2k+1}$$

Q5. b) $(1-x^2) y'' - 2x y' + 20y = 0$

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$, where $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Then, the given DE is

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 20 a_n x^n = 0$$

Use $n-2=l$ ($n=l+2$)

$$\Rightarrow \sum_{l=0}^{\infty} (l+2)(l+1) a_{l+2} x^l - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 20 a_n x^n = 0$$

$$\Rightarrow (2a_2 + 20a_0) + (6a_3 - 2a_1 + 20a_1)x +$$

$$\sum_{l=2}^{\infty} \left[(l+2)(l+1)a_{l+2} - l(l-1)a_l - 2la_l + 20a_l \right] x^l = 0$$

Q5. b) (Continue)

Then, the solution is $y(x) = \sum_{n=0}^{\infty} a_n x^n$, where

$$a_2 = -10a_0, \quad a_3 = -3a_1, \quad a_{n+2} = \frac{(n+5)(n-4)}{(n+2)(n+1)} a_n.$$

Q6. a) $y'' + 2y' + 2y = 2e^{2x} + e^{-3x}$

For complementary solution, $m^2 + 2m + 2 = (m+1)^2 + 1 = 0$
 $m = -1 \pm j$

$$\text{So, } y_c(x) = e^{-x} (C_1 \cos x + C_2 \sin x)$$

For particular solution,
$$\begin{cases} y_p = \alpha_1 e^{2x} + \alpha_2 e^{-3x} \\ y_p' = 2\alpha_1 e^{2x} - 3\alpha_2 e^{-3x} \\ y_p'' = 4\alpha_1 e^{2x} + 9\alpha_2 e^{-3x} \end{cases}$$

$$\Rightarrow y_p'' + 2y_p' + 2y_p = 2e^{2x} + e^{-3x}$$

$$\Leftrightarrow \underbrace{(4\alpha_1 + 4\alpha_1 + 2\alpha_1)}_{10\alpha_1 = 2} e^{2x} + \underbrace{(9\alpha_2 - 6\alpha_2 + 2\alpha_2)}_{5\alpha_2 = 1} e^{-3x} = 2e^{2x} + e^{-3x}$$

$\Leftrightarrow \alpha_1 = \alpha_2 = \frac{1}{5}$

$$\text{Hence, } y(x) = y_c + y_p = e^{-x} \{ C_1 \cos x + C_2 \sin x \} + \frac{1}{5} (e^{2x} + e^{-3x})$$

Q6. b) $y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}$

For complementary solution, $m^2 + 4m + 4 = (m+2)^2 = 0 \rightarrow m = -2$

$$\text{So, } y_1 = e^{-2x}, \quad y_2 = x e^{-2x}$$

Q6. b) (Continue)

For particular solution, we use variation of parameters

$$W = \begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix} = e^{-4x}$$

$$\Rightarrow u = - \int \frac{x e^{-2x} \cdot \frac{e^{-2x}}{x^2}}{e^{-4x}} dx = -\ln x, \quad v = \int \frac{e^{-2x} \cdot \frac{e^{-2x}}{x^2}}{e^{-4x}} dx = -\frac{1}{x}$$

$$\text{Hence, } y(x) = y_c + y_p = (C_1 + C_2 x - \ln x) e^{-2x}$$

Q7. $\frac{dN_1(t)}{dt} = \frac{2}{24} N_2(t) - \frac{8}{24} N_1(t)$

$$\frac{dN_2(t)}{dt} = \frac{8}{24} N_1(t) - \frac{2}{24} N_2(t)$$

$$\frac{d}{dt} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{1}{3} & \frac{1}{12} \\ \frac{1}{3} & -\frac{1}{12} \end{pmatrix}}_{=A} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}, \quad \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} N_1(0) \\ N_2(0) \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda \left(\lambda + \frac{5}{12} \right) = 0. \rightarrow \lambda_1 = 0, \quad \lambda_2 = -\frac{5}{12}.$$

$$\text{For } \lambda_1 = 0 \rightarrow y_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \text{For } \lambda_2 = -\frac{5}{12}, \quad y_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{Then, } V = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{5}{12} \end{pmatrix}, \quad V^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$$

and the general solutions are given by

$$\begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} = e^{At} \begin{pmatrix} N_1(0) \\ N_2(0) \end{pmatrix} = V \cdot e^{Dt} \cdot V^{-1} \begin{pmatrix} N_1(0) \\ N_2(0) \end{pmatrix}$$

Q7. (Continue)

$$\begin{aligned}
 \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} &= \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{5}{2}t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} N_1(0) \\ N_2(0) \end{pmatrix} \\
 &= \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4e^{-\frac{5}{2}t} & -e^{-\frac{5}{2}t} \end{pmatrix} \begin{pmatrix} N_1(0) \\ N_2(0) \end{pmatrix} \\
 &= \frac{1}{5} \begin{bmatrix} 1 + 4e^{-\frac{5}{2}t} & 1 - e^{-\frac{5}{2}t} \\ 4(1 - e^{-\frac{5}{2}t}) & 4 + e^{-\frac{5}{2}t} \end{bmatrix} \begin{bmatrix} N_1(0) \\ N_2(0) \end{bmatrix}
 \end{aligned}$$

Hence

$$\begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} = \frac{1}{5} \begin{bmatrix} (N_1(0) + N_2(0)) + (4N_1(0) - N_2(0))e^{-\frac{5}{2}t} \\ 4(N_1(0) + N_2(0)) + (N_2(0) - 4N_1(0))e^{-\frac{5}{2}t} \end{bmatrix}$$

Q8. a) $\ddot{y} - 3\dot{y} + 6y = \sin(t)$

Let's denote $\bar{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, then

$$\frac{d}{dt} \bar{Y} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 3 \end{bmatrix} \bar{Y} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$$

$$\rightarrow \frac{d\bar{Y}}{dt} = A\bar{Y} + \bar{f}, \text{ where } A = \begin{bmatrix} 0 & 1 \\ -6 & 3 \end{bmatrix}, \bar{f} = \begin{bmatrix} 0 \\ \sin t \end{bmatrix}$$

Q 8. b) $\ddot{y} + 4y = t^2 - e^{3t}$

Let's denote $\bar{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, then

$$\frac{d\bar{Y}}{dt} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \bar{Y} + \begin{bmatrix} 0 \\ t^2 - e^{3t} \end{bmatrix}$$

$$\rightarrow \frac{d\bar{Y}}{dt} = A\bar{Y} + \bar{f}, \text{ where } A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}, \bar{f} = \begin{bmatrix} 0 \\ t^2 - e^{3t} \end{bmatrix}$$

Q 8. c) $\ddot{\ddot{y}} - 8\ddot{y} + 5\dot{y} + 2y = 7t$

Let's denote $\bar{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}$, then

$$\frac{d\bar{Y}}{dt} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \ddot{\ddot{y}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & 8 \end{bmatrix} \bar{Y} + \begin{bmatrix} 0 \\ 0 \\ 7t \end{bmatrix}$$

$$\rightarrow \frac{d\bar{Y}}{dt} = A\bar{Y} + \bar{f}, \text{ where } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & 8 \end{bmatrix}, \bar{f} = \begin{bmatrix} 0 \\ 0 \\ 7t \end{bmatrix}$$

Q9. $A = \begin{bmatrix} 6 & -3 \\ 1 & 2 \end{bmatrix}$, then $\det(A - \lambda I)$

$$= (6-\lambda)(2-\lambda) + 3$$

$$= \lambda^2 - 8\lambda + 15 = (\lambda-3)(\lambda-5) = 0$$

For $\lambda_1 = 5$, $\rightarrow (A - \lambda_1 I) \mathbf{v}_1 = 0$

$$\Rightarrow \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \rightarrow v_{11} = 3v_{12}, \text{ so}$$

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} v_{11}$$

For $\lambda_2 = 3$, $\rightarrow (A - \lambda_2 I) \mathbf{v}_2 = 0$

$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \rightarrow v_{21} = v_{22}, \text{ so}$$

$$\mathbf{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_{21}$$

let's assume $v_{11} = v_{21} = 1$ for simplicity.

Then, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 1 \\ \frac{1}{3} & 1 \end{bmatrix}$ and

the solution of the given DEs is

$$\bar{\mathbf{y}} = e^{At} \mathbf{y}_{(0)}, \text{ where}$$

Q 9. (Continue)

$$\begin{aligned}
 \bar{Y} &= e^{At} Y_{(0)} = V \cdot e^{Dt} \cdot V^{-1} Y_{(0)} \\
 &= \frac{3}{2} \begin{bmatrix} 1 & 1 \\ \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\
 &= \frac{3}{2} \begin{bmatrix} 1 & 1 \\ \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -2 \\ \frac{8}{3} \end{bmatrix} \\
 &= \frac{3}{2} \begin{bmatrix} 1 & 1 \\ \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} -2e^{5t} \\ \frac{8}{3}e^{3t} \end{bmatrix} = \begin{bmatrix} -3e^{5t} + 4e^{3t} \\ -e^{5t} + 4e^{3t} \end{bmatrix}
 \end{aligned}$$

Q 10. Denote $\bar{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $\frac{d\bar{X}}{dt} = A\bar{X}$, where $A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$

$$\det(A - \lambda I) = (\lambda - 3)(\lambda - 2) - 12 = \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1)$$

For $\lambda_1 = 6$, $(A - \lambda_1 I) \mathcal{V}_1 = 0$

$$\Rightarrow \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix} = 0 \rightarrow \mathcal{V}_1 = \begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} \mathcal{V}_1$$

Q10. (Continue)

For $\lambda_2 = -1$, $(A - \lambda_2 I) \mathbf{v}_2 = 0$

$$\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \rightarrow \mathbf{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} v_{21}$$

let's assume $v_{11} = v_{21} = 1$ for simplicity.

Then, the solution of $\frac{d\bar{\mathbf{x}}}{dt} = A\bar{\mathbf{x}}$ is

$$\bar{\mathbf{x}} = e^{At} \bar{\mathbf{x}}_{(0)} = V \cdot e^{Dt} \cdot V^{-1} \begin{bmatrix} x_{(0)} \\ y_{(0)} \end{bmatrix} \text{ where}$$

$$\begin{aligned} \rightarrow e^{At} &= -\frac{4}{7} \begin{bmatrix} 1 & 1 \\ \frac{3}{4} & -1 \end{bmatrix} \begin{bmatrix} e^{6t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -\frac{3}{4} & 1 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 4e^{6t} + 3e^{-t} & 4(e^{6t} - e^{-t}) \\ 3(e^{6t} - e^{-t}) & 3e^{6t} + 4e^{-t} \end{bmatrix} \end{aligned}$$

(Alternative solution for Q10)

Either forms are ok.

$$\bar{\mathbf{x}} = c_1 e^{6t} \mathbf{v}_1 + c_2 e^{-t} \mathbf{v}_2 \Rightarrow \bar{\mathbf{x}} = c_1 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

where c_1 and c_2 are constant

$$Q11. a) \quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \rightarrow \frac{d}{dt} \bar{x} = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{\text{denote } A} \bar{x}$$

$$\det(A - \lambda I) = (\lambda - 1)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

$$\text{For } \lambda_1 = 3, \quad \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \rightarrow v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} v_{11}$$

$$\text{For } \lambda_2 = -1, \quad \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \rightarrow v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} v_{21}$$

Then, the general solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= c_1 e^{3t} \cdot v_1 + c_2 e^{-t} \cdot v_2 \\ &= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} \quad \text{--- form (1)} \end{aligned}$$

or you may use Matrix Exponential as follows

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{At} \begin{bmatrix} x_{1(0)} \\ x_{2(0)} \end{bmatrix} = V \cdot e^{Dt} \cdot V^{-1} \begin{bmatrix} x_{1(0)} \\ x_{2(0)} \end{bmatrix} \quad \text{--- form (2)}$$

(If the problem asks you to use Matrix Exponential, you need to use form(2). Otherwise, form(1) is ok)

$$\text{In this problem, } e^{At} = \frac{1}{4} \begin{bmatrix} 2(e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ 4(e^{3t} - e^{-t}) & 2(e^{3t} + e^{-t}) \end{bmatrix}$$

Q 11. b) Denote $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $\frac{d\bar{X}}{dt} = A\bar{X}$,

$$A = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \rightarrow \det(A - \lambda I) = (\lambda + \frac{1}{2})^2 + 1 = (\lambda - \alpha)(\lambda - \bar{\alpha})$$

$$\text{where } \alpha = -\frac{1}{2} + j, \bar{\alpha} = -\frac{1}{2} - j$$

$$\text{For } \lambda_1 = \alpha, (A - \lambda_1 I) \mathcal{V}_1 = \begin{bmatrix} -j & 1 \\ -1 & -j \end{bmatrix} \begin{bmatrix} \mathcal{V}_{11} \\ \mathcal{V}_{12} \end{bmatrix} = 0 \rightarrow \mathcal{V}_1 = \begin{bmatrix} 1 \\ j \end{bmatrix} \mathcal{V}_{11}$$

$$\text{For } \lambda_2 = \bar{\alpha}, (A - \lambda_2 I) \mathcal{V}_2 = \begin{bmatrix} j & 1 \\ -1 & j \end{bmatrix} \begin{bmatrix} \mathcal{V}_{21} \\ \mathcal{V}_{22} \end{bmatrix} = 0 \rightarrow \mathcal{V}_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix} \mathcal{V}_{21}$$

Let's assume $\mathcal{V}_{11} = \mathcal{V}_{21} = 1$ for simplicity. Then, the general solution is given by

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= c_1 e^{(-\frac{1}{2}+j)t} \mathcal{V}_1 + c_2 e^{(-\frac{1}{2}-j)t} \mathcal{V}_2 \\ &= e^{-\frac{1}{2}t} \left[c_1 e^{jt} \begin{pmatrix} 1 \\ j \end{pmatrix} + c_2 e^{-jt} \begin{pmatrix} 1 \\ -j \end{pmatrix} \right] \quad (\text{use } e^{jt} = \cos t + j \sin t) \\ &= e^{-\frac{1}{2}t} \left[A \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + B \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \right] \quad \text{where } A = c_1 + c_2 \\ &\hspace{15em} B = j(c_1 - c_2) \end{aligned}$$

Q 11. c) Denote $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $\frac{d\bar{X}}{dt} = A\bar{X}$, $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

$$\begin{aligned} \det(A - \lambda I) &= (1 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2 = 0 \text{ which is a repeated eigenvalue} \end{aligned}$$

Q11. c) (Continue)

$$\text{For } \lambda_1 = 2, (A - \lambda_1 I) \mathbf{v}_1 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_{11}$$

Let's assume $v_{11} = 1$ for simplicity. Then, the first solution of $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ is $c_1 e^{2t} \mathbf{v}_1 = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

which means e^{2t} is one solution of $x_1(t)$, that has a repeated root. So, the general solution of $x_1(t)$ can be expressed as $x_1(t) = c_1 e^{2t} + c_2 t e^{2t}$. Since

$$\begin{aligned} x_2(t) &= x_1(t) - \frac{dx_1}{dt} = c_1 e^{2t} + c_2 t e^{2t} \\ &\quad - 2c_1 e^{2t} - c_2 (1+t) e^{2t} \\ &= -c_1 e^{2t} - c_2 (1+t) e^{2t}, \end{aligned}$$

the general solution of $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is given by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} t \\ 1+t \end{pmatrix} e^{2t}$$

Q12. a) By substituting $y = \frac{1}{2}(\dot{x} - x)$ into $\frac{dy}{dt} = 2x - 2y$,

Q 12. a) (Continue) we get

$$\frac{1}{2} \frac{d}{dt} (\dot{x} - x) = \frac{1}{2} (\ddot{x} - \dot{x}) = 2x - (\dot{x} - x) = 3x - \dot{x}$$

$$\ddot{x} - \dot{x} = 6x - 2\dot{x}$$

$$\Rightarrow \ddot{x} + \dot{x} - 6x = 0, \text{ whose characteristic equation is}$$

$$f(m) = (m+3)(m-2) = 0$$

So, $x(t) = C_1 e^{-3t} + C_2 e^{2t}$ and

$$y(t) = \frac{1}{2} (\dot{x} - x) = \frac{1}{2} (-3C_1 e^{-3t} + 2C_2 e^{2t} - C_1 e^{-3t} - C_2 e^{2t})$$

$$= -2C_1 e^{-3t} + \frac{C_2}{2} e^{2t}. \text{ By the initial conditions,}$$

$$x(0) = C_1 + C_2 = 1, \quad y(0) = -2C_1 + \frac{C_2}{2} = 2 \rightarrow C_1 = -\frac{3}{5}, \quad C_2 = \frac{8}{5}.$$

Therefore, the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -\frac{3}{5} e^{-3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{4}{5} e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Q 12. b) By substituting $x_2 = \frac{1}{4} (\dot{x}_1 + 5x_1)$ into $\frac{d^2 x_2}{dt^2} = 4x_1 - 5x_2$,

$$\frac{1}{4} \frac{d^2}{dt^2} (\dot{x}_1 + 5x_1) = 4x_1 - \frac{5}{4} (\dot{x}_1 + 5x_1)$$

$$\rightarrow x_1^{(4)} + 5x_1^{(2)} = 16x_1 - 5\dot{x}_1 - 25x_1 = -5\dot{x}_1 - 9x_1$$

$$\rightarrow x_1^{(4)} + 10x_1^{(2)} + 9x_1 = 0 \rightarrow \text{whose characteristic equation is } f(m) = (m^2 + 9)(m^2 + 1) = 0$$

Q 12, b) (Continue). The roots are

$$m_1 = 3j, m_2 = -3j, m_3 = j, m_4 = -j.$$

Hence $x_1(t) = C_1 \cos(3t) + C_2 \sin(3t) + C_3 \cos t + C_4 \sin t$ and

$$x_2(t) = \frac{1}{4} (\ddot{x}_1 + 5x_1)$$

$$= \frac{1}{4} \begin{bmatrix} -9C_1 \cos(3t) - 9C_2 \sin(3t) - C_3 \cos t - C_4 \sin t \\ + 5C_1 \cos(3t) + 5C_2 \sin(3t) + 5C_3 \cos t + 5C_4 \sin t \end{bmatrix}$$

$$= -C_1 \cos(3t) - C_2 \sin(3t) + C_3 \cos(t) + C_4 \sin(t).$$

Q 13. a) Denote $\mathcal{L}(x(t)) = X(s)$, then

$$\mathcal{L}(x'(t)) = sX(s) - 2, \mathcal{L}(x''(t)) = s^2 X(s) - 2s + 1.$$

By substituting these expressions into the DE,

$$s^2 X(s) - 2s + 1 - sX(s) + 2 - 6X(s)$$

$$= (s^2 - s - 6) X(s) - 2s + 3 = 0$$

$$\Rightarrow X(s) = \frac{2s - 3}{(s-3)(s+2)} = \frac{\alpha_1}{s-3} + \frac{\alpha_2}{s+2}, \text{ where } \alpha_1 = \frac{3}{5}, \alpha_2 = \frac{7}{5}.$$

$$\Rightarrow \frac{3}{5} \frac{1}{s-3} + \frac{7}{5} \frac{1}{s+2}$$

$$\text{Hence, } x(t) = \mathcal{L}^{-1}(X(s)) = \frac{3}{5} e^{3t} + \frac{7}{5} e^{-2t}$$

Q13. b) Denote $\mathcal{L}(x(t)) = X(s)$, then $\mathcal{L}(x''(t)) = s^2 X(s)$

By applying LT to the DE, we get

$$(s^2 + 4)X(s) = \frac{3}{s^2 + 9} \rightarrow X(s) = \frac{\alpha_1}{(s^2 + 4)} + \frac{\alpha_2}{(s^2 + 9)}, \text{ where}$$

$$\alpha_1 = \frac{3}{5}, \alpha_2 = -\frac{3}{5}$$

Since $\mathcal{L}^{-1}\left(\frac{1}{s^2 + \omega^2}\right) = \frac{1}{\omega} \sin(\omega t)$, the solution is

$$x(t) = \mathcal{L}^{-1}(X(s)) = \frac{3}{10} \sin(2t) - \frac{1}{5} \sin(3t).$$

Q14. a) Let's denote $\mathcal{L}(y(t)) = Y(s)$, $\mathcal{L}(z(t)) = Z(s)$, then

$\mathcal{L}(y') = sY(s) - 8$, $\mathcal{L}(z') = sZ(s) - 3$. The given DEs can be expressed as follows

$$\begin{aligned} (s-2)Y(s) + 3Z(s) &= 8 \\ 2Y(s) + (s-1)Z(s) &= 3 \end{aligned} \Rightarrow \begin{bmatrix} s-2 & 3 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\text{Then, } Y(s) = \frac{1}{(s-4)(s+1)} (8s-17), \quad Z(s) = \frac{1}{(s-4)(s+1)} (3s-22)$$

$$\text{where } Y(s) = \frac{3}{s-4} + \frac{5}{s+1}, \quad Z(s) = \frac{-2}{s-4} + \frac{5}{s+1}.$$

Hence, the solutions are

$$y(t) = \mathcal{L}^{-1}(Y(s)) = 3e^{4t} + 5e^{-t}$$

$$z(t) = \mathcal{L}^{-1}(Z(s)) = -2e^{4t} + 5e^{-t}$$

Q14. b) Denote $Y(s) = \mathcal{L}(y(t))$, then

$$\mathcal{L}(y') = sY(s) - 2, \quad \mathcal{L}(y'') = s^2 Y(s) - 2s + 2$$

Applying LT to the given DE leads to

$$(s^2 Y(s) - 2s + 2) + 2(sY(s) - 2) + Y(s) = \frac{1}{s}$$

$$\Rightarrow (s^2 + 2s + 1)Y(s) = 2s + 2 + \frac{1}{s}$$

$$\Rightarrow Y(s) = \frac{2(s+1)}{(s+1)^2} + \frac{1}{s(s+1)^2} = \frac{2}{s+1} + \left(\frac{-1}{(s+1)^2} + \frac{-1}{(s+1)} + \frac{1}{s} \right)$$

$$= -\frac{1}{(s+1)^2} + \frac{1}{s+1} + \frac{1}{s}$$

As $\mathcal{L}^{-1}\left(\frac{1}{(s-a)^2}\right) = te^{at}$, $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$, the solution is given by

$$y(t) = \mathcal{L}^{-1}(Y(s)) = 1 + e^{-t} - te^{-t} = 1 + (1-t)e^{-t}$$

Q14. c) Denote $Y(s) = \mathcal{L}(y(t))$, where $\mathcal{L}(y'') = s^2 Y(s)$

Applying LT to the given DE leads to

$$(s^2 + 1)Y(s) = \frac{3}{s^2 + 9}$$

Q 14. c) (Continue)

$$Y(s) = \frac{3}{(s^2+1)(s^2+9)} = \frac{\frac{8}{3}}{s^2+1} - \frac{\frac{8}{3}}{s^2+9}$$

Since $\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{1}{\omega} \sin(\omega t)$,

the solution is given by

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{8}{3} \left[\sin(t) - \frac{1}{3} \sin(3t) \right]$$

Q 15. a) Denote $Y(s) = \mathcal{L}(y(t))$, where $\mathcal{L}(\dot{y}) = sY$, $\mathcal{L}(\ddot{y}) = s^2Y$

Then, the given DE can be expressed after LT as

$$(s^2 + 7s + 12)Y(s) = \frac{3}{s+2}$$

$$\rightarrow Y(s) = \frac{3}{(s+4)(s+3)(s+2)} = \frac{\frac{3}{2}}{s+4} + \frac{-3}{s+3} + \frac{\frac{3}{2}}{s+2}$$

Hence, the solution is

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{3}{2} (e^{-4t} + e^{-2t} - 2e^{-3t})$$

Q 15. b) Denote $Y(s) = \mathcal{L}(y(t))$, where $\mathcal{L}(y) = sY$, $\mathcal{L}(y') = s^2Y$,

and $\mathcal{L}(\delta(t-c)) = e^{-sc}$. By applying LT

to the given DE, we get

$$(s^2 + 8s + 15)Y(s) = -4e^{-7s}$$

$$\rightarrow Y(s) = -\frac{4e^{-7s}}{(s+5)(s+3)} = 2e^{-7s} \left[\frac{1}{s+5} - \frac{1}{s+3} \right]$$

$$\text{Since } \mathcal{L}^{-1}\left(\frac{e^{-sc}}{s-a}\right) = e^{a(t-c)} u(t-c),$$

the solution is derived as

$$y(t) = \mathcal{L}^{-1}(Y(s)) = 2 \left[e^{-5(t-7)} u(t-7) - e^{-3(t-7)} u(t-7) \right]$$

Q 15. c) $\mathcal{L}(y(t)) = Y(s)$, $\mathcal{L}(y') = sY(s) + 2$, $\mathcal{L}(y'') = s^2Y(s) + 2s - 6$,
 $\mathcal{L}(\delta(t-c)) = e^{-sc}$

Apply these results into the given DE to get

$$\rightarrow (s^2 + 8s + 20)Y(s) + 2s - 6 + 16 = 4e^{-3s} - 12e^{-5s}$$

$$\rightarrow (s-\alpha)(s-\bar{\alpha})Y(s) = -2(s+5) + 4e^{-3s} - 12e^{-5s},$$

$$\text{where } \alpha = -4 + 2j, \bar{\alpha} = -4 - 2j$$

Q15. c) (Continue)

$$Y(s) = \frac{-2(s+5)}{(s-\alpha)(s-\bar{\alpha})} + \frac{4e^{-3s}}{(s-\alpha)(s-\bar{\alpha})} - \frac{12e^{-5s}}{(s-\alpha)(s-\bar{\alpha})}$$

$$= \frac{1}{2} \left[\frac{-2+j}{s-\alpha} + \frac{-2-j}{s-\bar{\alpha}} \right] - j(e^{-3s} - 3e^{-5s}) \left[\frac{1}{s-\alpha} - \frac{1}{s-\bar{\alpha}} \right]$$

By using the following LT pair $\mathcal{L}^{-1}\left(\frac{e^{-sc}}{s-a}\right) = e^{a(t-c)} u(t-c)$,

we get the solution $y(t)$ as follows

$$\Rightarrow y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{1}{2} \left((-2+j)e^{\alpha t} + (-2-j)e^{\bar{\alpha} t} \right)$$

$$- j e^{\alpha(t-3)} u(t-3) + 3j e^{\alpha(t-5)} u(t-5)$$

$$+ j e^{\bar{\alpha}(t-3)} u(t-3) - 3j e^{\bar{\alpha}(t-5)} u(t-5) \quad \left. \vphantom{\frac{1}{2} \left((-2+j)e^{\alpha t} + (-2-j)e^{\bar{\alpha} t} \right)} \right\} \text{form (1)}$$

By using Euler theorem and the following relations,

$$e^{\alpha t} = e^{-4t} e^{2jt}, \quad e^{2jt} + e^{-2jt} = 2\cos(2t), \quad e^{2jt} - e^{-2jt} = 2j\sin(2t)$$

we can further simplify as

$$\Rightarrow y(t) = -e^{-4t} \left[2\cos(2t) + \sin(2t) \right] + 2e^{-4(t-3)} \sin(2(t-3)) u(t-3)$$

$$- 6e^{-4(t-5)} \sin(2(t-5)) u(t-5) \quad \left. \vphantom{-e^{-4t} \left[2\cos(2t) + \sin(2t) \right]} \right\} \text{form (2)}$$

Both form (1) and form (2) are ok.

Q 15. d) Denote $X(s) = \mathcal{L}(x(t))$, $Y(s) = \mathcal{L}(y(t))$,
 where $\mathcal{L}(x'(t)) = s X(s) - 1$, $\mathcal{L}(y') = s Y(s)$.

Then, by applying LT to the given DE,

$$\begin{cases} (s-1)X(s) - 2Y(s) = 1 \\ -2X(s) + (s+2)Y(s) = 0 \end{cases} \Rightarrow \begin{bmatrix} s-1 & -2 \\ -2 & (s+2) \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{where } \begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{W} \begin{bmatrix} s+2 & 2 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{W} \begin{bmatrix} s+2 \\ 2 \end{bmatrix},$$

$$W = (s+1)(s+2) - 4 = s^2 + s - 6 = (s+3)(s-2)$$

$$X(s) = \frac{s+2}{(s+3)(s-2)} = \frac{1}{5} \left(\frac{1}{s+3} + \frac{4}{s-2} \right)$$

$$Y(s) = \frac{2}{(s+3)(s-2)} = \frac{2}{5} \left(-\frac{1}{s+3} + \frac{1}{s-2} \right)$$

Therefore, the solutions are

$$x(t) = \mathcal{L}^{-1}(X(s)) = \frac{1}{5} (e^{-3t} + 4e^{2t})$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{2}{5} (e^{2t} - e^{-3t})$$
