

Problem Set 6:

Partial Differential Equations

1. Solve the boundary value problems

$$(a) \begin{cases} \frac{\partial^2 u}{\partial x \partial y} = 4xy + e^x \\ \frac{\partial u}{\partial y}(0, y) = y, u(x, 0) = 2 \end{cases}$$

$$(b) \begin{cases} \frac{\partial^2 u}{\partial x \partial y} = 3 \frac{\partial u}{\partial y} + 2y \\ \frac{\partial u}{\partial y}(0, y) = y^2 - 2y, u(x, 0) = x + 3e^{-x} \end{cases}$$

2. By making the change of variables $r = x + y$, $s = -x + y$, find the general solution of

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = w.$$

3. Find a solution for the partial differential equation $3 \frac{\partial u}{\partial x} - 2 \frac{\partial u}{\partial y} - 5 \frac{\partial u}{\partial z} = 0$ by the method of separation of variables.

4. Use the method of separation of variables to solve the partial differential equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c} \frac{\partial \psi}{\partial t}, \quad 0 < x < \pi, t > 0, c \text{ is a non-zero positive constant, with the following sets of conditions:}$$

$$(a) \quad \psi(0, t) = \psi(\pi, t) = 0, t > 0 \text{ and } \psi(x, t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

$$(b) \quad \frac{\partial \psi}{\partial x}(0, t) = \frac{\partial \psi}{\partial x}(\pi, t) = 0, t > 0 \text{ and } \psi(x, t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

5. A metal bar for length π is placed on the x -axis. Assume that the bar is thermally insulated and there is no external heat source. Then the temperature distribution function $u(x, t)$ satisfies the

$$\text{diffusion equation } \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, t > 0 \text{ with boundary conditions}$$

$$u(0, t) = u(\pi, t) = 0, t \geq 0.$$

- (a) Classify the type of the above second order linear partial differential equation.
 (b) Solve the above boundary value problem by the method of separation of variables and find the general solution by the principle of superposition.
 (c) If the initial condition is $u(x, 0) = f(x)$, determine the solution in terms of $f(x)$.
 (d) Suppose $f(x) = 3\sin x - 2\sin 2x + 10\sin 5x$, $0 < x < \pi$. Express the solution explicitly.

6. A thin rectangular plate is bounded by the lines $x = 0, x = 3, y = 0$ and $y = 2$. The temperature is zero along the edges $x = 0, x = 3, y = 0$ and has the constant value V_0 along the edge $y = 2$. In

the steady state the temperature V at the point (x, y) of the plate satisfies the equation

$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$. Assuming that V is periodic in x , use the method of separation of variables to find the solution of this problem.

7. Consider the boundary-value problem
$$\begin{cases} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \\ \frac{\partial v}{\partial x} \Big|_{x=0} = 0, \quad \frac{\partial v}{\partial x} \Big|_{x=l} = 0 \end{cases} \dots\dots(I).$$

Let $V(x, t) = X(x)T(t)$ be a nontrivial solution to
$$\begin{cases} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \\ \frac{\partial v}{\partial x} \Big|_{x=0} = 0, \quad \frac{\partial v}{\partial x} \Big|_{x=l} = 0 \end{cases} \dots\dots(I)$$

(a) Show that $X(x)$ is a solution to the following boundary-value problem

$$\begin{cases} \frac{d^2 X}{dx^2} + \lambda X = 0 \\ \frac{dX}{dx} \Big|_{x=0} = 0, \quad \frac{dX}{dx} \Big|_{x=l} = 0 \end{cases} \dots\dots(II)$$

(b) Show that if $X(x)$ is a nontrivial solution to the boundary-value problem (II), then

$$\lambda = \left(\frac{n\pi}{l} \right)^2, \quad n = 0, 1, 2, \dots, \text{ and } X(x) \text{ can be } \cos \frac{n\pi x}{L}, \quad n = 0, 1, 2, \dots.$$

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