

EE 3109 Applied Electromagnetics
2022/23 Semester B
Week 1: Tutorial Solution

1. If $\varphi(x, y, z) = 5xz + x^3y^2$, find $\nabla\varphi$ (or $\text{grad } \varphi$) at the point $(-1, 2, 2)$.

Solution

The gradient of a scalar field φ is a vector that represents both the magnitude and direction of the maximum space rate of increase of φ . Mathematically:

$$\nabla\varphi = \frac{\partial\varphi}{\partial x} \hat{a}_x + \frac{\partial\varphi}{\partial y} \hat{a}_y + \frac{\partial\varphi}{\partial z} \hat{a}_z$$

Given $\varphi(x, y, z) = 5xz + x^3y^2$,

$$\begin{aligned}\frac{\partial\varphi}{\partial x} &= \frac{\partial}{\partial x}(5xz + x^3y^2) = 5z + 3x^2y^2 \\ \frac{\partial\varphi}{\partial y} &= \frac{\partial}{\partial y}(5xz + x^3y^2) = 2x^3y \\ \frac{\partial\varphi}{\partial z} &= \frac{\partial}{\partial z}(5xz + x^3y^2) = 5x\end{aligned}$$

Thus, $\nabla\varphi = (5z + 3x^2y^2)a_x + (2x^3y)a_y + (5x)a_z$

Substitute $x = -1$, $y = 2$ and $z = 2$ to get $\nabla\varphi = 22a_x - 4a_y - 5a_z$

2. If $\mathbf{A} = y^2z\hat{i} + xy^3z^2\hat{j} - x^3yz^3\hat{k}$, find $\nabla \cdot \mathbf{A}$ (or $\text{div } \mathbf{A}$) at the point $(1, -2, 2)$.

Solution

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ \frac{\partial A_x}{\partial x} &= \frac{\partial}{\partial x}(y^2z) = 0; \quad \frac{\partial A_y}{\partial y} = \frac{\partial}{\partial y}(xy^3z^2) = 3xy^2z^2; \quad \frac{\partial A_z}{\partial z} = \frac{\partial}{\partial z}(-x^3yz^3) = -3x^3yz^2\end{aligned}$$

Thus;

$$\nabla \cdot \mathbf{A} = 3xy^2z^2 - 3x^3yz^2$$

Substitute $x = 1$, $y = -2$ and $z = 2$:

$$\nabla \cdot \mathbf{A} = 72$$

3. If $\mathbf{A} = x^2y\hat{i} + 3xy^3z^2\hat{j} - 5xz^2\hat{k}$, find $\nabla \times \mathbf{A}$ (or curl $\mathbf{A}$) at the point (2,1,-1).

Solution

$$\nabla \times \bar{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \bar{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{a}_x - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \hat{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{a}_z$$

where $A_x = x^2y$, $A_y = 3xy^3z^2$ and $A_z = -5xz^2$

$$\nabla \times \bar{A} = \left(\frac{\partial(-5xz^2)}{\partial y} - \frac{\partial(3xy^3z^2)}{\partial z} \right) \hat{a}_x - \left(\frac{\partial(-5xz^2)}{\partial x} - \frac{\partial(x^2y)}{\partial z} \right) \hat{a}_y + \left(\frac{\partial(3xy^3z^2)}{\partial x} - \frac{\partial(x^2y)}{\partial y} \right) \hat{a}_z$$

$$\nabla \times \bar{A} = (0 - 6xy^3z) \hat{a}_x - (-5z^2 - 0) \hat{a}_y + (3y^3z^2 - x^2) \hat{a}_z$$

$$\nabla \times \bar{A} = (-6xy^3z) \hat{a}_x + (5z^2) \hat{a}_y + (3y^3z^2 - x^2) \hat{a}_z$$

Substitute $x = 2$, $y = 1$ and $z = -1$

$$\nabla \times \bar{A} = 12\hat{a}_x + 5\hat{a}_y - \hat{a}_z$$

4. Find the time domain expression of

(a) $I = 0.6\angle 67^\circ$ [A]; (b) $V = 26j\angle - (3\pi/4)$ [mV]; (c) $H = 43 \exp(-j(37z + 90^\circ))$ [A/m]

Solution

(a) $I = 0.6\angle 67^\circ$ [A]

A given a phasor quantity $I = I_m \angle \theta$ has a time domain expression $I = I_m \cos(\omega t + \theta)$

Where $I_m = 0.6$ and $\theta = 67^\circ$

Thus;

$$I = 0.6 \cos(\omega t + 67^\circ) \text{ [A]}$$

(b) $V = 26j\angle - (3\pi/4)$ [mV]

Recall that $j = e^{j\pi/2}$

$$\text{Then } V = 26j\angle - (3\pi/4) = 26e^{j\pi/2}e^{-j3\pi/4} = 26e^{-j(3\pi/4-\pi/2)} \text{ [mV]}$$

$$V = 26 \cos(\omega t + \pi/2 - 3\pi/4) = 26 \cos(\omega t - \pi/4) \text{ [mV]}$$

(c) $H = 43 \exp(-j(37z + 90^\circ))$ [A/m]

$$H = 43e^{-j(37z+\frac{\pi}{2})} = 43 \cos(\omega t - 37z - \pi/2) \text{ [A/m]}$$

$$H = 43 \sin(\omega t - 37z) \text{ [A/m]}$$

5. Find the phasor expression of

- (a) $\mathbf{H}(\mathbf{r}, t) = 0.6 \cos(\omega t - x - 2y)$ [A/m]
- (b) $\mathbf{E}(\mathbf{r}, t) = 0.6 \sin(\omega t - 3x + 4y + 34^\circ)$ [E/m]
- (c) $V(z, t) = A \cos(\omega t - z) + A \sin(\omega t + z)$
- (d) $V(z, t) = A \cos(\omega t - z + 60^\circ) + A \sin(\omega t + 2z - 30^\circ)$

Solution

- (a) $\mathbf{H}(\mathbf{r}, t) = 0.6 \cos(\omega t - x - 2y)$ [A/m]

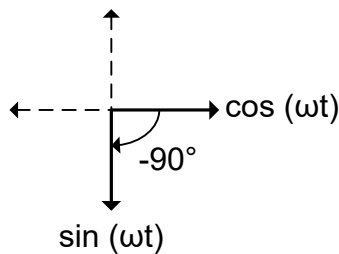
A time-domain expression of the form $H(t) = H_m \cos(\omega t + \theta)$ has a phasor form

$$E = E_m \angle \theta$$

By comparison; $H_m = 0.6$ and $\theta = -x - 2y$

$$\mathbf{H} = 0.6 e^{-jx} e^{-j2y}$$

- (b) $\mathbf{E}(\mathbf{r}, t) = 0.6 \sin(\omega t - 3x + 4y + 34^\circ)$



Thus $\sin \omega t = \cos(\omega t - 90^\circ)$

We can therefore rewrite the given equation as:

$$\mathbf{E} = 0.6 \cos(\omega t - 3x + 4y + 34^\circ - 90^\circ) = 0.6 \cos(\omega t - 3x + 4y - 56^\circ)$$

By comparison; $E_m = 0.6$ and $\theta = -3x + 4y - 56^\circ$

The corresponding phasor expression is therefore;

$$\mathbf{E} = 0.6 \angle (-3x + 4y - 56^\circ)$$

$$\mathbf{E} = 0.6 e^{-j56} e^{j(-3x+4y)}$$

$$\mathbf{E} = 0.6 e^{-j56} e^{-j3x} e^{j4y}$$

- (c) $V(z, t) = A \cos(\omega t - z) + A \sin(\omega t + z) = A[\cos(\omega t - z) + \cos(\omega t + z - \pi/2)]$
 $V = A(e^{-jz} + e^{jz} e^{-j\pi/2})$

- (d) $V(z, t) = A \cos(\omega t - z + 60^\circ) + A \sin(\omega t + 2z - 30^\circ)$
 $= A \cos(\omega t - z + 60^\circ) + A \cos(\omega t + 2z - 30^\circ - 90^\circ)$
 $= A \cos(\omega t - z + 60^\circ) + A \cos(\omega t + 2z - 120^\circ)$
 $= A e^{-jz} e^{j\pi/3} + A e^{j2z} e^{-j2\pi/3}$

NB:

- (i) $e^x e^y = e^{x+y}$ and $e^x / e^y = e^{x-y}$
- (ii) $j = e^{j\pi/2}$ and $-j = e^{-j\pi/2}$
- (iii) $\sqrt{j} = \pm e^{j\pi/4}$ and $\sqrt{-j} = \pm e^{-j\pi/4}$
- (iv) $e^{j\pi} = e^{-j\pi} = -1$

6. If the voltage across a 10 [nF] capacitor is $V(z, t) = A \cos(\omega t - \beta z)$, write the phasor for the current flowing into and out of the capacitor (remember, $I = C \frac{dV}{dt}$).

Solution

$$V(z, t) = A \cos(\omega t - \beta z)$$

In phasor notation;

$$V = Ae^{-j\beta z}$$

$$I = C \frac{dV}{dt} = C \frac{d}{dt} Ae^{-j\beta z}$$

where $\frac{d}{dt} X(t) \Rightarrow j\omega X$

$$I = j\omega CAe^{-j\beta z} = j\omega \times 10 \times 10^{-9} Ae^{-j\beta z} \quad [A]$$

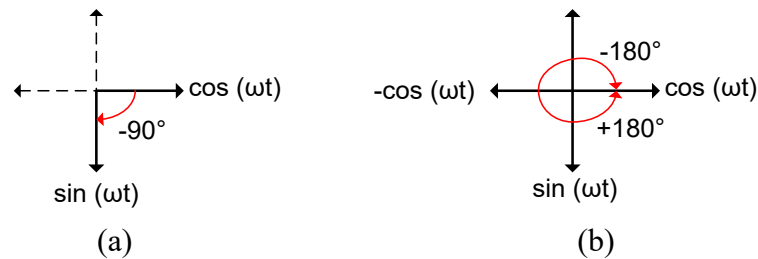
7. The electric field of an electromagnetic wave $\mathbf{E}(\mathbf{r}, t) = \hat{a}_x E_0 \cos \left[10^8 \pi \left(t - \frac{z}{c} \right) + \theta \right]$ is the sum of

$$\mathbf{E}_1(\mathbf{r}, t) = \hat{a}_x 0.03 \sin \left[10^8 \pi \left(t - \frac{z}{c} \right) \right] \text{ and}$$

$$\mathbf{E}_2(\mathbf{r}, t) = \hat{a}_x 0.04 \cos \left[10^8 \pi \left(t - \frac{z}{c} \right) - \frac{\pi}{3} \right]$$

(i.e. $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_1(\mathbf{r}, t) + \mathbf{E}_2(\mathbf{r}, t)$). Find E_0 and θ .

Solution



Recall the following:

From fig. (a)

$$\sin \omega t = \cos(\omega t - 90^\circ)$$

From fig. (b)

$$\cos \omega t = -\cos(\omega t - 180^\circ)$$

$$\cos \omega t = -\cos(\omega t + 180^\circ)$$

Also:

$$Ae^{j\theta} = A(\cos \theta + j \sin \theta)$$

$$\text{Re}[Ae^{j\theta}] = A \cos \theta$$

Thus,

$$\begin{aligned} \bar{E}_1(\mathbf{r}, t) &= \hat{a}_x 0.03 \sin \left[10^8 \pi \left(t - \frac{z}{c} \right) \right] \\ &= \hat{a}_x 0.03 \cos \left[10^8 \pi \left(t - \frac{z}{c} \right) - \frac{\pi}{2} \right] \\ &= \hat{a}_x \text{Re} \left[0.03 e^{-j\frac{\pi}{2}} e^{j10^8 \pi \left(t - \frac{z}{c} \right)} \right] \end{aligned}$$

Likewise:

$$\begin{aligned}\bar{E}_2(\mathbf{r}, t) &= \hat{a}_x 0.04 \cos \left[10^8 \pi \left(t - \frac{z}{c} \right) - \frac{\pi}{3} \right] \\ &= \hat{a}_x \operatorname{Re} \left[0.04 e^{-j\frac{\pi}{3}} e^{j10^8 \pi \left(t - \frac{z}{c} \right)} \right]\end{aligned}$$

In phasor form;

$$\begin{aligned}\bar{E} &= \hat{a}_x \left[E_0 e^{-j\theta} e^{-j\frac{10^8 \pi z}{c}} \right] \\ \bar{E}_1 &= \hat{a}_x \left[0.03 e^{-j\frac{\pi}{2}} e^{-j\frac{10^8 \pi z}{c}} \right] \\ \bar{E}_2 &= \hat{a}_x \left[0.04 e^{-j\frac{\pi}{3}} e^{-j\frac{10^8 \pi z}{c}} \right]\end{aligned}$$

Hence,

$$\begin{aligned}\bar{E} &= \bar{E}_1 + \bar{E}_2 \\ \hat{a}_x \left[E_0 e^{j\theta} e^{-j\frac{10^8 \pi z}{c}} \right] &= \hat{a}_x \left[0.03 e^{-j\frac{\pi}{2}} e^{-j\frac{10^8 \pi z}{c}} \right] + \hat{a}_x \left[0.04 e^{-j\frac{\pi}{3}} e^{-j\frac{10^8 \pi z}{c}} \right] \\ \hat{a}_x [E_0 e^{j\theta}] e^{-j\frac{10^8 \pi z}{c}} &= \hat{a}_x \left[0.03 e^{-j\frac{\pi}{2}} + 0.04 e^{-j\frac{\pi}{3}} \right] e^{-j\frac{10^8 \pi z}{c}} \\ \hat{a}_x [E_0 e^{j\theta}] e^{-j\frac{10^8 \pi z}{c}} &= \hat{a}_x (0.068 e^{-j1.27}) e^{-j\frac{10^8 \pi z}{c}}\end{aligned}$$

Comparing both sides of the equation yields;

$$E_0 = 0.068 \quad \text{and} \quad \theta = -1.27 \text{ rad}$$