

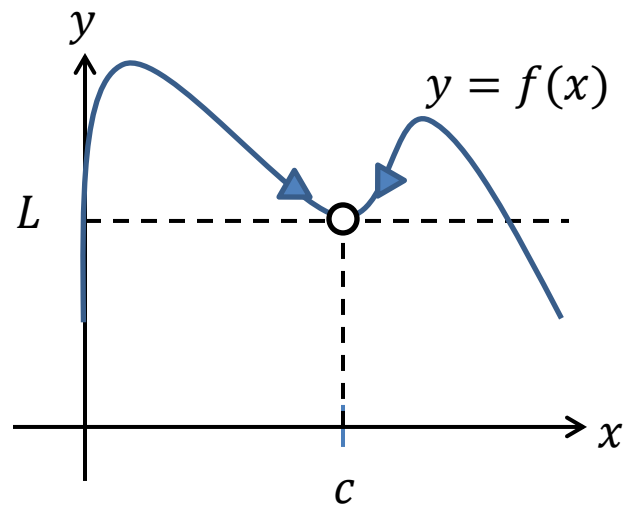
Review:

Definition of Limit

Intuitive definition of limit (informal one)

We say $f(x)$ has the limit L when x tends to c if the value of $f(x)$ gets close to L (as close as we like) if x is sufficiently close to c . Mathematically, we write

$$\lim_{x \rightarrow c} f(x) = L.$$

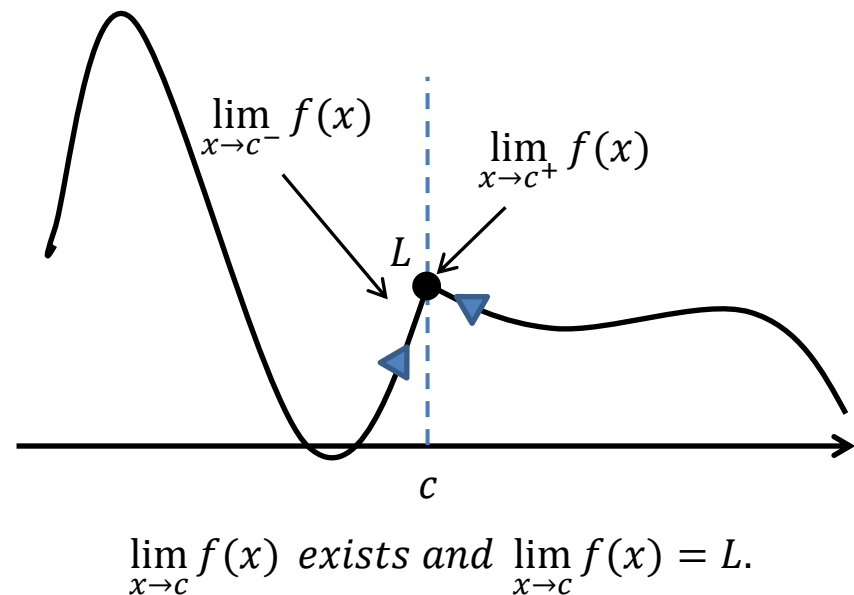
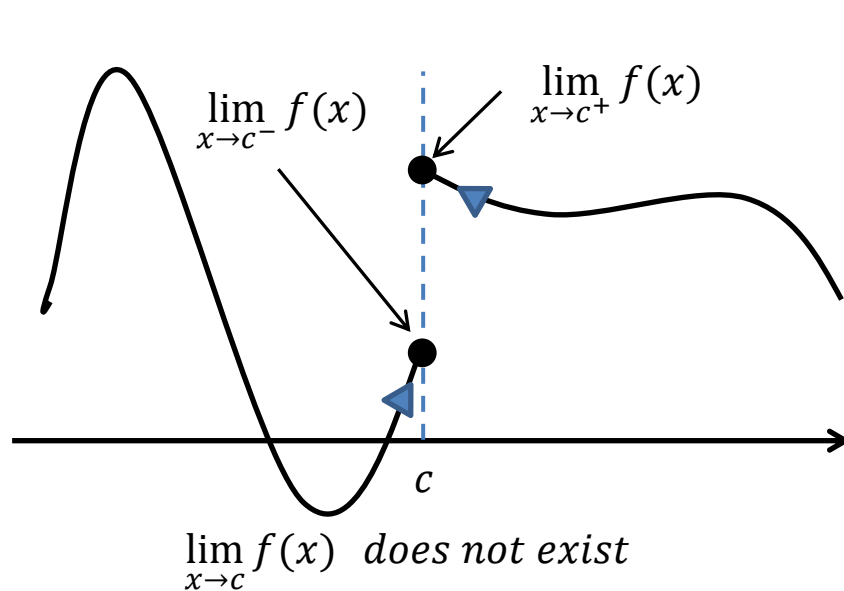


Relationship between one-sided limit and existence of limits

Theorem (Existence of limits)

The limits $\lim_{x \rightarrow c} f(x)$ exists if and only if ($\Leftrightarrow$) both left-hand limits and right hand limits exists and

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x).$$



| End (review)

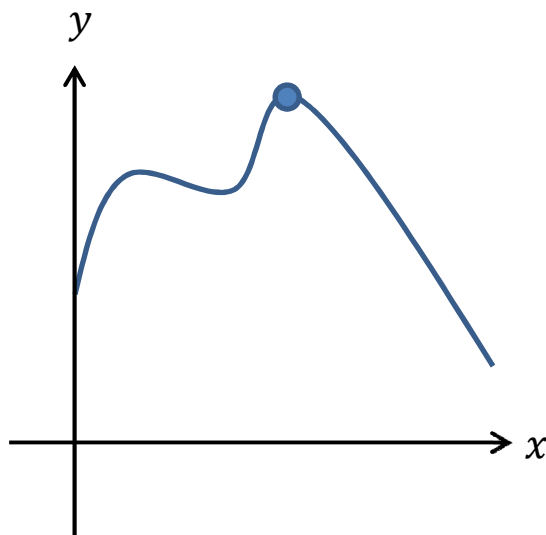
MA1200 Calculus and Basic Linear Algebra I

Lecture Note 7

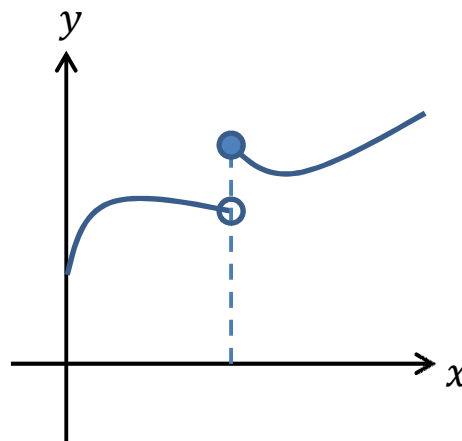
Continuity of functions

What is continuous function?

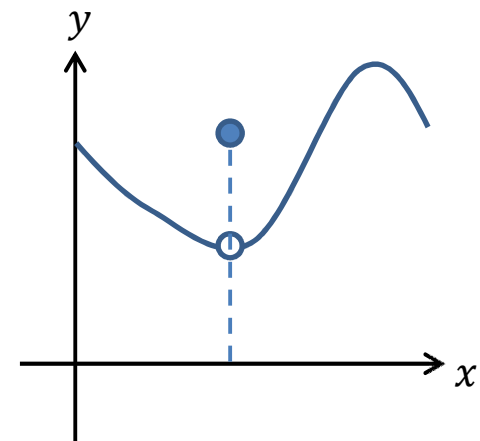
Roughly speaking, a continuous function $f(x)$ is a function which the graph $y = f(x)$ is continuous (no jumps, no breaks).



(continuous function)



(not continuous function)



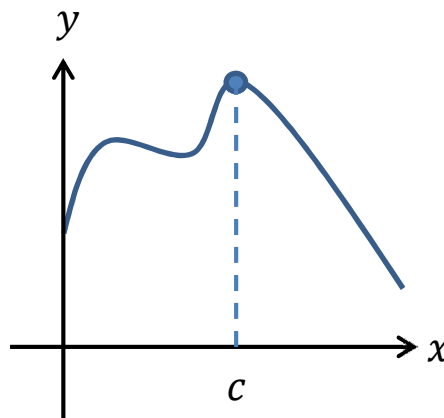
Mathematical definition of continuity

Definition (Continuity of function $f(x)$)

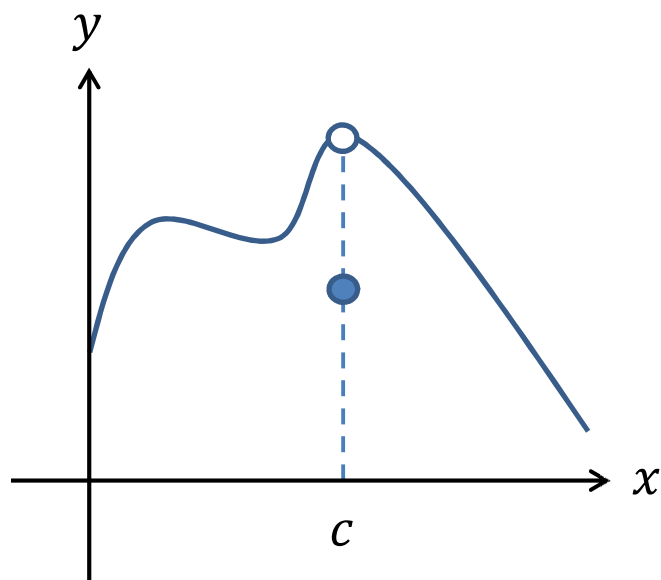
We say a function is continuous at $x = c$ if both $\lim_{x \rightarrow c} f(x)$ and $f(c)$ exist and

$$\lim_{x \rightarrow c} f(x) = f(c).$$

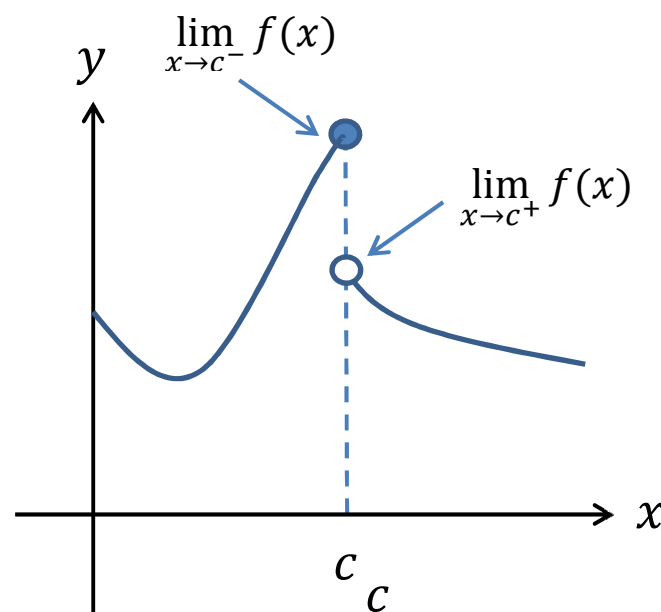
Furthermore, we say a function is continuous on its domain if it is continuous at every point of its domain.



If the condition " $\lim_{x \rightarrow c} f(x) = f(c)$ " does not satisfy, we say the function is not continuous at $x = c$. For example



$$\lim_{x \rightarrow c} f(x) \neq f(c)$$



$$\lim_{x \rightarrow c} f(x) \text{ does not exist}$$

$$\left(\lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x) \right)$$

Notes on continuity

- Most of the elementary functions such as $y = x^3$, $y = e^x$, $y = \cos x$, $y = \sqrt{x}$, $y = |x|$ are all continuous on its domain.
- To check the continuity of a function at $x = c$, we may follow the following procedure:

Step 1: Compute $f(c)$

Step 2: Compute $\lim_{x \rightarrow c} f(x)$

(Note: If necessary, one needs to consider the left-hand limit and right-hand limit when computing $\lim_{x \rightarrow c} f(x)$)

Step 3: Compare the limits with $f(c)$.

Example 1

Consider the function

$$f(x) = \begin{cases} \frac{\sin 3x}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}.$$

Is $f(x)$ continuous at $x = 0$?

☺Solution:

Step 1: First, note that $f(0) = 2$ by definition.

$$\text{Step 2: } \lim_{x \rightarrow 0} f(x) \stackrel{x \neq 0}{\cong} \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \lim_{x \rightarrow 0} 3 \left(\frac{\sin 3x}{3x} \right) \stackrel{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1}{\cong} 3 \times 1 = 3.$$

Step 3: $\lim_{x \rightarrow 0} f(x) = 3 \neq 2 = f(0)$.

↑
 $\theta = 3x$

Therefore, we conclude that $f(x)$ is not continuous at $x = 0$.

Example 2

Consider the function

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 3x^2 & \text{if } x \geq 1 \end{cases}$$

Determine whether the function is continuous at $x = 1$.

☺Solution:

Step 1: First, note that $f(1) = 3(1)^2 = 3$.

Step 2: Note that

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x + 1) = 3, \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 3x^2 = 3$$

So the limits $\lim_{x \rightarrow 1} f(x)$ exists and $\lim_{x \rightarrow 1} f(x) = 3$.

Step 3: $\lim_{x \rightarrow 1} f(x) = 3 = f(1)$.

Therefore, we conclude that $f(x)$ is continuous at $x = 1$.

Example 3

Consider the function

$$f(x) = \begin{cases} \frac{x^2 - 3x - 10}{x - 5} & \text{if } x \neq 5 \\ a & \text{if } x = 5 \end{cases}$$

where a is real number.

- (a) If $a = 4$, is $f(x)$ continuous at $x = 5$?
- (b) What is the value of a so that $f(x)$ is continuous at $x = 5$?

☺Solution:

(a) Step 1: By definition, we get $f(5) = a = 4$.

Step 2: For $x \neq 5$, we have $f(x) = \frac{x^2 - 3x - 10}{x - 5}$.

$$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5} = \lim_{x \rightarrow 5} \frac{(x - 5)(x + 2)}{x - 5} = \lim_{x \rightarrow 5} (x + 2) = 7.$$

Step 3: $\lim_{x \rightarrow 5} f(x) = 7 \neq 4 = f(5)$.

Hence, the function is not continuous at $x = 5$ in this case.

(b) If $f(x)$ is continuous at $x = 5$, then we must have

$$\lim_{x \rightarrow 5} f(x) = f(5).$$

Using the result in (a), we obtain

$$\underbrace{a}_{f(5)} = \underbrace{7}_{\lim_{x \rightarrow 5} f(x)}.$$

Some properties of continuous functions

Theorem 1 (Basic algebraic operation of continuous functions)

If $f(x)$ and $g(x)$ be continuous at $x = c$, then the function

$$kf(x), f(x) + g(x), f(x) - g(x), f(x)g(x), \frac{f(x)}{g(x)} \text{ (if } g(c) \neq 0), |f(x)|$$

are all continuous at $x = c$. (k is constant)

Theorem 2 (Composition of continuous function)

If $f(x)$ is continuous at c and $g(x)$ is continuous at $f(c)$, then the composition $(g \circ f)(x)$ is also continuous at $x = c$

$$\lim_{x \rightarrow c} (g \circ f)(x) = \lim_{x \rightarrow c} g(f(x)) = g\left(\lim_{x \rightarrow c} f(x)\right) = g(f(c)).$$

(*Note: Theorem 2 is quite useful in computing limits)

Example 4

Let $f(x) = \cos x$ and $g(x) = e^x$ are continuous function over the real number. Using Theorem 1 and 2, we can conclude that the following functions

$$kf(x) = k \cos x, \quad f(x) \pm g(x) = \cos x \pm e^x$$

$$f(x)g(x) = e^x \cos x, \quad \frac{f(x)}{g(x)} = \frac{\cos x}{e^x} \quad (e^x \neq 0)$$

$$|f(x)| = |\cos x|, \quad (g \circ f)(x) = g(f(x)) = g(\cos x) = e^{\cos x}$$

are all continuous over real number.

Theorem 3

If $f(x)$ is a continuous function and $g(x)$ is a function (may not be continuous), then we have

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right)$$

provided that the limits $\lim_{x \rightarrow c} g(x)$ exists.

Example 5

Compute $\lim_{x \rightarrow \pi} \sin(x + \cos x)$ and $\lim_{x \rightarrow 1} \cos \frac{\sqrt{x+3}-2}{x-1}$

☺Solution:

1st limit

Note that $f(x) = \sin x$ is continuous, then

$$\lim_{x \rightarrow \pi} \sin(x + \cos x) = \sin\left(\lim_{x \rightarrow \pi} (x + \cos x)\right) = \sin(\pi - 1) \approx 0.8415.$$

2nd limit

Note that $g(x) = \cos x$ is continuous, then

$$\lim_{x \rightarrow 1} \cos \frac{\sqrt{x+3}-2}{x-1} = \cos \left(\lim_{x \rightarrow 1} \frac{\sqrt{x+3}-2}{x-1} \right) \dots (*)$$

Note that

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x+3}-2}{x-1} &= \lim_{x \rightarrow 1} \frac{\sqrt{x+3}-2}{x-1} \left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} \right) \stackrel{(a-b)(a+b)=a^2-b^2}{=} \lim_{x \rightarrow 1} \frac{\overbrace{x+3-2^2}^{=x-1}}{(x-1)(\sqrt{x+3}+2)} \\ &= \lim_{x \rightarrow 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{1+3}+2} = \frac{1}{4}. \end{aligned}$$

From (*), we conclude that

$$\lim_{x \rightarrow 1} \cos \frac{\sqrt{x+3}-2}{x-1} = \cos \left(\frac{1}{4} \right) = 0.9689.$$

Example 6

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function (may or may not continuous) such that

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^3} = \frac{\pi}{2}. \text{ Compute the limits}$$

$$(a) \lim_{x \rightarrow 0} f(x) \quad \text{and} \quad (b) \lim_{x \rightarrow 0} e^{\cos\left(\frac{f(x)}{x^2}\right)}$$

☺Solution:

$$(a) \text{ Note that } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{f(x)}{x^3} x^3 = \frac{\pi}{2} \times 0 = 0.$$

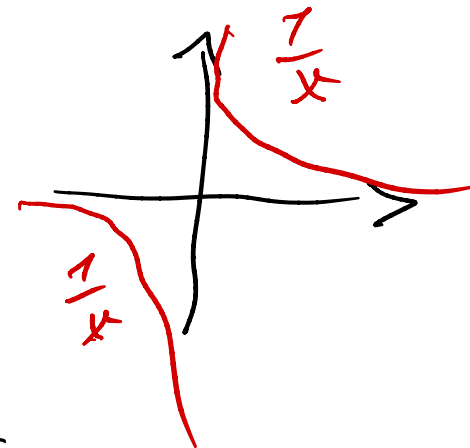
(b) Note that the function $e^{\cos x}$ is continuous (see Example 4) and

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = \lim_{x \rightarrow 0} \frac{f(x)}{x^3} x = \frac{\pi}{2} \times 0 = 0.$$

So using Theorem 3, we have

$$\lim_{x \rightarrow 0} e^{\cos\left(\frac{f(x)}{x^2}\right)} = e^{\cos\left(\lim_{x \rightarrow 0} \frac{f(x)}{x^2}\right)} = e^{\cos 0} = e^1 = e.$$

Example 7



- (a) Find the limits $\lim_{x \rightarrow 0^+} \frac{1}{x}$ and $\lim_{x \rightarrow 0^-} \frac{1}{x}$.
- (b) Hence, determine if the limits $\lim_{x \rightarrow 0} \frac{1+2^{1/x}}{3+2^{1/x}}$ exists.

☺Solution:

- (a) Using the graph of $y = \frac{1}{x}$, one can see that $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$ and

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

- (b) We consider the left-hand limits and right-hand limits.

- Left-hand limits

When $x \rightarrow 0^-$, then $\frac{1}{x} \rightarrow -\infty$. So we have

$$\lim_{x \rightarrow 0^-} \frac{1 + 2^{1/x}}{3 + 2^{1/x}} = \frac{1 + 2^{\lim_{x \rightarrow 0^-} \frac{1}{x}}}{3 + 2^{\lim_{x \rightarrow 0^-} \frac{1}{x}}} = \frac{1 + 2^{-\infty}}{3 + 2^{-\infty}} \stackrel{||}{=} \frac{1 + 2^{-\infty} = \frac{1}{2^{\infty}} \rightarrow 0}{3 + 0} = \frac{1}{3}.$$

$$2^{-x} = \left(\frac{1}{2}\right)^x \\ \Rightarrow \lim_{x \rightarrow \infty} 2^{-x} = 0$$

- Right-hand limits

When $x \rightarrow 0^+$, then $\frac{1}{x} \rightarrow +\infty$, $2^{\frac{1}{x}} \rightarrow 2^{+\infty} = +\infty$ and hence $\frac{1}{2^{\frac{1}{x}}} = 0$.

So we have

$$\lim_{x \rightarrow 0^+} \frac{1 + \overbrace{2^{1/x}}^{\rightarrow \infty}}{\underbrace{3 + 2^{1/x}}_{\rightarrow \infty}} = \lim_{x \rightarrow 0^+} \frac{1}{\frac{2^{1/x}}{2^{1/x}} + 1} = \frac{0 + 1}{0 + 1} = 1.$$

Since $\lim_{x \rightarrow 0^-} \frac{1 + 2^{1/x}}{3 + 2^{1/x}} \neq \lim_{x \rightarrow 0^+} \frac{1 + 2^{1/x}}{3 + 2^{1/x}}$, so we conclude that the limits

$$\lim_{x \rightarrow 0} \frac{1 + 2^{1/x}}{3 + 2^{1/x}}$$

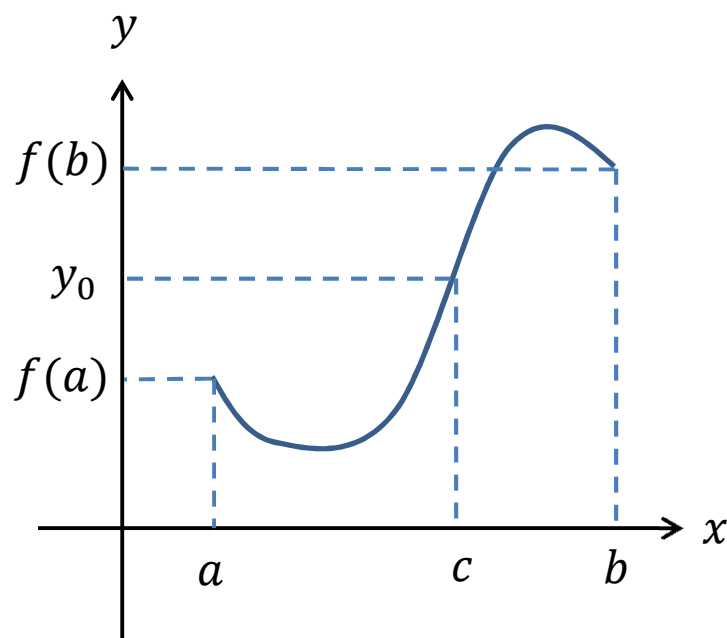
does not exist.

Important property of continuous function

Theorem 4 (Intermediate Value Theorem)

If $f(x)$ is a continuous function on an interval $[a, b]$ and y_0 is a real number between $f(a)$ and $f(b)$, then there is a number c ($a \leq c \leq b$) such that

$$f(c) = y_0.$$



The intermediate value theorem is useful in checking whether a given equation has solution. It provides a way to find the solution of the equation.

Example 8

Consider the equation $x^5 + 2x - 1 = 0$, show that there is a solution between 0 and 1.

Handwritten notes: $= f(x)$ (above the equation), y_0 (with an arrow pointing to the 0), and $[a,b] \rightarrow [0,1]$ (on the left).

☺Solution:

One may rephrase the statement as

“There is a number $0 \leq z \leq 1$ such that $z^5 + 2z - 1 = 0$.”

We let $f(x) = x^5 + 2x - 1$ and $f(x)$ is continuous. By simple calculation, we get $f(0) = -1 < 0$ and $f(1) = 2 > 0$.

By intermediate value theorem, there is z ($0 \leq z \leq 1$) such that

$$f(z) = z^5 + 2z - 1 = 0.$$

Application of intermediate value theorem: Method of Bisection

- It is a root-finding technique by using intermediate value theorem repeatedly.
- In Example 8, we have shown that the solution lies between 0 and 1. The bisection method aims to obtain the solution by narrowing this range.

Step 1:

We pick the mid-point between 0 and 1. That is, $x = 0.5$. We compute the value of $f(0.5)$.

$$f(x) = x^5 + 2x - 1$$

Since $f(0.5) = 0.03125 > 0$, then the solution lies between 0 and 0.5.

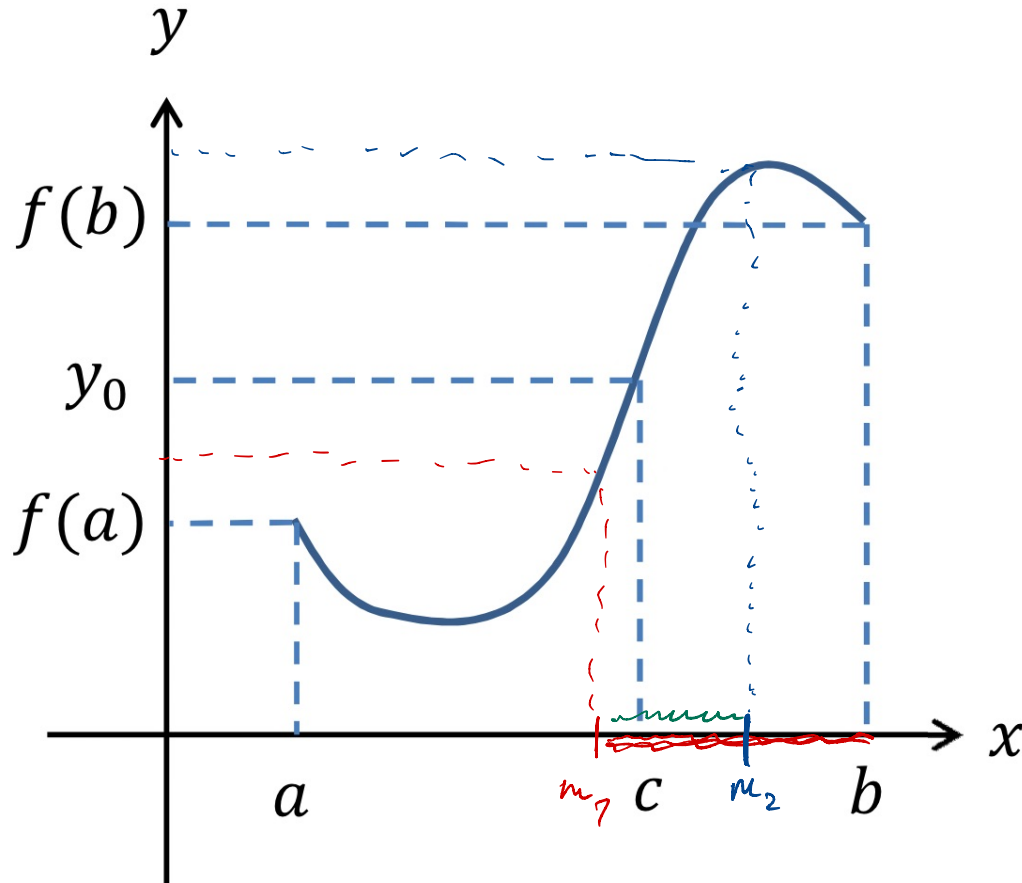
Step 2:

We pick the mid-point between 0 and 0.5. That is, $x = 0.25$. We compute the value of $f(0.25)$.

Since $f(0.25) = -0.4990 < 0$, then the solution lies between 0.25 and 0.5.

Method of Bisection:

Solve $f(x) = y_0$!



One can repeat this process and obtain the approximated solution of the equation:

$$f(x) = x^2 + 2x - 1$$

Midpoint x	$f(x)$	Updated range of z
0.5	0.03125	$0 \leq z \leq 0.5$
0.25	-0.4990	$0.25 \leq z \leq 0.5$
0.375	-0.24258	$0.375 \leq z \leq 0.5$
0.4375	-0.10897	$0.4375 \leq z \leq 0.5$
0.46875	-0.03987	$0.46875 \leq z \leq 0.5$
0.484375	-0.00459	$0.484375 \leq z \leq 0.5$
0.492188	0.01326	$0.484375 \leq z \leq 0.492188$
0.488282	0.004319	$0.484375 \leq z \leq 0.488282$
0.486329	-0.00014	$0.486329 \leq z \leq 0.488282$
0.487306	0.00209	$0.486329 \leq z \leq 0.487306$
0.486818	0.000977	$0.486329 \leq z \leq 0.486818$
0.486574	0.000421	$0.486329 \leq z \leq 0.486574$
0.486452	0.000142	$0.486329 \leq z \leq 0.486452$

The approximated solution is $x \approx 0.486$.