

EE3121

Differential Equations for Engineers

Lecture 2: First-Order Linear Ordinary Differential Equations



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Linear Equations

- Linear equations

- may be time-invariant or time-varying;
- may also be homogeneous or nonhomogeneous.

- A **nonhomogeneous** first-order linear ordinary differential equation (ODE) has the form:

$$\frac{dy}{dt} + p(t) y = g(t).$$

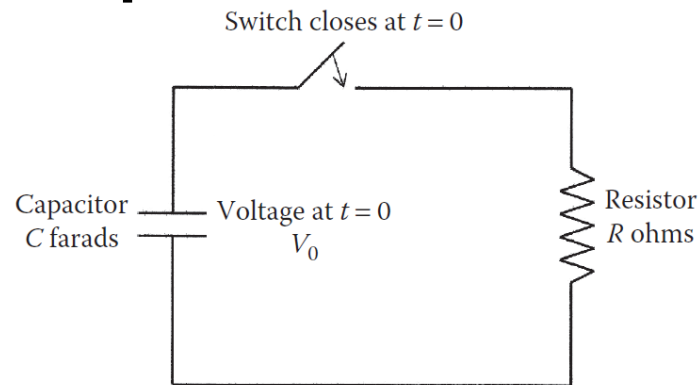
where $g(t)$ is nonzero.

- A **homogeneous** equation has $g(t) = 0$.

- The homogeneous equation has an interesting solution only if its initial condition is nonzero.

2.1 First-Order Linear Homogeneous Ordinary Differential Equations

■ Example: RC circuit



- The values R and C are known constants.

- By *Kirchhoff's voltage law*, the sum of the voltage drops around a closed circuit is zero. Then the charge on the capacitor is $q = CV$.
- By *Ohm's law*, the voltage drop over the resistor is IR .
- By *Kirchhoff's current law*, the current through the resistor must equal the rate of change of the charge on the capacitor. That is,

$$I = dq/dt = C dV/dt.$$

2.1.1 Example: RC circuit

- The sum of the voltage drops around the circuit after it is closed at $t = 0$ can be expressed as:

$$\begin{aligned} IR + V &= 0, \\ RC \frac{dV}{dt} + V &= 0. \\ \Rightarrow \frac{dV}{dt} + \left(\frac{1}{RC} \right) V &= 0, \quad V(0) = V_0. \end{aligned} \tag{1}$$

- A first-order ODE posed as an initial value problem must have one stated initial condition to yield a unique solution. In this case, $V(0) = V_0$.
- We solve Equation (1) by **separating variables**:
 - Moving all terms involving V to the left-hand side and all terms involving t to the other gives

$$\frac{dV}{V} = -\frac{dt}{RC}.$$

- Then performing the definite integrals from $V(0)$ to $V(t)$ and 0 to t leads to

$$\int_{V(0)}^{V(t)} \frac{dV}{V} = - \int_0^t \frac{dt}{RC}, \quad \Rightarrow \quad \ln \left(\frac{V(t)}{V(0)} \right) = -\frac{t}{RC}.$$

- Finally, taking the exponential of both sides, we obtain

$$\exp \left(\ln \left(\frac{V(t)}{V(0)} \right) \right) = \exp \left(-\frac{t}{RC} \right), \quad \Rightarrow \quad \frac{V(t)}{V(0)} = e^{-\frac{t}{RC}},$$

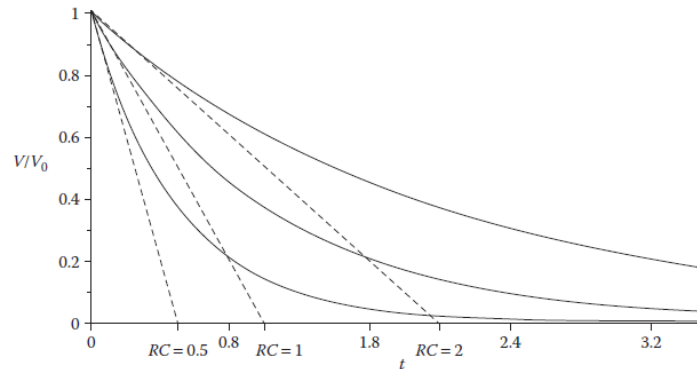
$$\Rightarrow V(t) = V_0 e^{-\frac{t}{RC}}, \quad V_0 = V(0). \quad (2)$$

Equation (2) is the solution to Equation (1).

- Clearly, the Equation (1) is a time-invariant ordinary differential equation (ODE).

2.1.1 Example: RC circuit

- This following figure graphs Equation (2).



- Note that the time constant of the circuit, RC , can be determined graphically as the intercept on the time axis of the voltage's initial slope.
- It follows from Equation (2) that the solution to any **first-order linear time-invariant (LTI) ODE** of the form

$$\frac{dy}{dt} + \lambda y = 0, \quad y_0 = y(0),$$

is

$$y(t) = y_0 e^{-\lambda t}.$$

2.1.2 Time-Varying Equations, Justification of Separation of Variables Procedure

- We solve time-varying first-order homogeneous ODEs in the same way; that is, via **separation of variables**.
- Consider the general form:

$$\frac{dy}{dt} + p(t) y = 0, \quad y(t_0) = y_0. \quad (3)$$

- We separate variables

$$\frac{dy}{y} = -p(t) dt,$$

- and perform definite integrals on the left from $y(t_0) = y_0$ and on the right from t_0 to t :

$$\int_{y_0}^{y(t)} \frac{dy}{y} = - \int_{t_0}^t p(\tau) d\tau, \quad \Rightarrow \quad \ln \left(\frac{y(t)}{y_0} \right) = - \int_{t_0}^t p(\tau) d\tau.$$

- Then we take the exponential of both sides:

$$\exp \left(\ln \left(\frac{y(t)}{y_0} \right) \right) = \exp \left(- \int_{t_0}^t p(\tau) d\tau \right), \quad \Rightarrow \quad \frac{y(t)}{y_0} = \exp \left(- \int_{t_0}^t p(\tau) d\tau \right),$$

$$y(t) = y_0 \exp \left(- \int_{t_0}^t p(\tau) d\tau \right). \quad (4)$$

Equation (4) is the solution to Equation (3).

- Success depends entirely on being able to perform the integration $\int_{t_0}^t p(\tau) d\tau$.

2.1.2 Time-Varying Equations, Justification of Separation of Variables Procedure

- We now justify the shortcut separation of variables procedure by solving Equation (3) with more attention to mathematical rigor.
- Dividing Equation (3) by y gives,

$$\frac{1}{y} \frac{dy}{dt} + p(t) = 0. \quad (5)$$

- Integrating Equation (5) over time from t_0 to t leads to:

$$\begin{aligned} \int_{t_0}^t \left(\frac{1}{y} \frac{dy}{d\tau} + p(\tau) \right) d\tau &= \int_{t_0}^t (0) d\tau = 0, \\ \Rightarrow \int_{t_0}^t \left(\frac{1}{y} \frac{dy}{d\tau} \right) d\tau &= - \int_{t_0}^t (p(\tau)) d\tau. \end{aligned} \quad (6)$$

■ Accordingly,

$$\int_{t_0}^t \left(\frac{1}{y} \frac{dy}{d\tau} \right) d\tau = \int_{t_0}^t \left(\frac{d}{d\tau} (\ln y(\tau)) \right) d\tau = \ln(y(t)) - \ln(y(t_0)) = \ln \left(\frac{y(t)}{y_0} \right). \quad (7)$$

■ From Equations (6) and (7), taking the exponential of both sides, we arrive at

$$y(t) = y_0 \exp \left(- \int_{t_0}^t p(\tau) d\tau \right). \quad (8)$$

■ Equation (8) matches Equation(4), so the separation of variables procedure gives the correct answer.

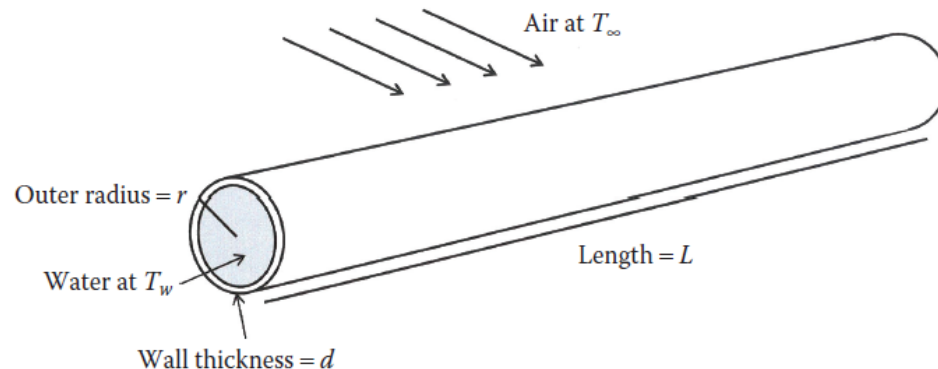
2.2 First-Order Linear Nonhomogeneous Ordinary Differential Equations

- A *nonhomogeneous* first-order linear ODE has the form

$$\frac{dy}{dt} + p(t)y = g(t). \quad (9)$$

where $g(t)$ is nonzero.

- **Example: Uninsulated water pipe in air draft**



- The pipe is full of water, and the water does not run for days.

Note: Over this time the pipe experiences diurnal variation in the ambient air temperature (warmer in the day, cooler at night).

Example: Uninsulated water pipe in air draft

- We model the air temperature variation as sinusoidal

$$T_{\infty} = \bar{T} + T_0 \sin wt, \quad (10)$$

where, T_{∞} is the ambient air temperature, $\bar{T}$ is the daily mean temperature, T_0 is the maximum variation from the mean.

- We assume that ventilation causes a slight draft of air over the pipe and that the pipe's wall thickness d is much less than its outer radius r .
- Under these circumstances, by Isaac Newton's law of cooling, the rate of heat going into the water from the airstream is

$$q = hA (T_{\infty} - T_W), \quad (11)$$

where, $A = 2\pi rL$ is the exposed surface area of the pipe, T_W is the temperature of the water, h is a constant known as the *convection coefficient*.

Example: Uninsulated water pipe in air draft

- Also, assuming that the water does not freeze, the thermal energy stored in the water is denoted by

$$E = mcT_W, \quad (12)$$

where, m is the mass, c is the specific heat of water.

- Now let

$$T_W = \bar{T} + T. \quad (13)$$

- Then, using Equations (10) and (13), Equation (11) becomes

$$q = hA (T_0 \sin wt - T). \quad (14)$$

- Applying the key physical principle—the rate of change of the thermal energy in the water equals the rate of heat exchanged between airstream and water gives:

$$\frac{dE}{dt} = q. \quad (15)$$

Example: Uninsulated water pipe in air draft

- From Equations (12) through (15),

$$mc \frac{dT}{dt} = hA (T_0 \sin wt - T). \quad (16)$$

- Rearranging Equation (16),

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin wt, \quad (17)$$

where

$$\lambda = \frac{hA}{mc}. \quad (18)$$

2.2.1 Integrating Factor

- How do we solve Equation (17)??

- Let us be more inclusive and ask how we solve a linear nonhomogeneous first-order ODE in its general form:

$$\frac{dy}{dt} + p(t) y = g(t), \quad y(t_0) = y_0. \quad (19)$$

- For reasons that will be clear momentarily, we seek an **integrating factor** $\mu(t)$ that, when multiplied into the equation turns the left-hand side into a derivative.

- We now seek a function $\mu(t)$ such that

$$\mu(t) \left(\frac{dy}{dt} + p(t) y \right) = \frac{d}{dt} (\mu(t) y). \quad (20)$$

2.2.1 Integrating Factor

- Multiplying through on the left-hand side of Equation (20),

$$\mu(t) \left(\frac{dy}{dt} + p(t)y \right) = \mu(t) \frac{dy}{dt} + \mu(t)p(t)y. \quad (21)$$

- By the product rule of differentiation, then the right-hand side of Equation (20) becomes

$$\frac{d}{dt} (\mu(t)y) = \mu(t) \frac{dy}{dt} + \frac{d\mu}{dt} y. \quad (22)$$

- We want to find a function μ such that **the right-hand sides of Equations (21) and (22) are equal**. The first term on the right-hand side is common to both equations.
- Therefore, the integrating factor we seek must satisfy

$$\mu(t)p(t)y = \frac{d\mu}{dt}y,$$

or

$$\frac{d\mu}{dt} = p(t) \mu(t). \quad (23)$$

- We can apply the results of Section 2.1 (Equations (3) and (4)) to Equation (23) and write

$$\mu(t) = \mu(t_0) \exp \left(\int_{t_0}^t p(\tau) d\tau \right). \quad (24)$$

- We do not know what value to assign to $\mu(t_0)$, but it turns out not to matter for an integrating factor (as we will see, the value cancels out), so we set

$$\mu(t_0) = 1. \quad (25)$$

- Furthermore, it suffices, and is most convenient, to use the indefinite integral form for the integrating factor:

$$\mu(t) = \exp \left(\int p(\tau) d\tau \right). \quad (26)$$

2.2.1 Integrating Factor

- From Equations (19) and (20) we arrive at

$$\frac{d}{dt} (\mu(t) y(t)) = \mu(t) g(t). \quad (27)$$

- The value of the integrating factor is now clear: with $\mu(t)$ given by Equation (26) we can solve Equation (27) with an integration.
- For the waterpipe temperature example, using Equations (17), (19), and (26),

$$\mu(t) = e^{\lambda t}, \quad (28)$$

and we have

$$\frac{d}{dt} (e^{\lambda t} T) = e^{\lambda t} \lambda T_0 \sin wt. \quad (29)$$

2.2.1 Integrating Factor

- Integrating both sides,

$$\int_0^t \frac{d}{d\tau} (e^{\lambda\tau} T) d\tau = \int_0^t e^{\lambda\tau} \lambda T_0 \sin(w\tau) d\tau. \quad (30)$$

- From the fundamental theorem of calculus and assuming zero initial condition, the left-hand side becomes

$$\int_0^t \frac{d}{d\tau} (e^{\lambda\tau} T) d\tau = e^{\lambda t} T(t). \quad (31)$$

- From Equations (30) and (31),

$$T(t) = e^{-\lambda t} \int_0^t e^{\lambda\tau} \lambda T_0 \sin(w\tau) d\tau. \quad (32)$$

2.2.1 Integrating Factor

- Performing the integral in Equation (32) is a little involved and it is convenient to postpone dealing with it explicitly until **Lecture 7**.
- We offer it to the student as challenge problem CP2.2 and state without proof here that

$$T(t) = e^{-\lambda t} \lambda T_0 \int_0^t e^{\lambda \tau} \sin(w\tau) d\tau = \left(\frac{\lambda}{\lambda^2 + w^2} \right) T_0 (\lambda \sin wt - w \cos wt + we^{-\lambda t}). \quad (33)$$

- Solved Problems SP2.3 and SP2.4 at the Tutorial are additional examples of how to solve ODEs with an integrating factor.
- If we carry through the integration of

$$\frac{d}{dt} (\mu(t) y(t)) = \mu(t) g(t),$$

in the general case we find

$$\begin{aligned}\mu(t) y(t) - \mu(t_0) y(t_0) &= \int_{t_0}^t \mu(\tau) g(\tau) d\tau, \\ \Rightarrow y(t) &= \frac{\mu(t_0)}{\mu(t)} y(t_0) + \int_{t_0}^t \left(\frac{\mu(\tau)}{\mu(t)} \right) g(\tau) d\tau.\end{aligned}$$

■ Recalling Equation (24), we have the general solution

$$y(t) = y_0 \exp \left(- \int_{t_0}^t p(\tau) d\tau \right) + \int_{t_0}^t \exp \left(- \int_{\tau}^t p(\tau) d\tau \right) g(\tau) d\tau.$$

■ Note that the first term is just the homogeneous solution, Equation (8).

■ It is important for engineers to find ways to check their own work.

■ We can always substitute a supposed solution into the ODE to verify that the ODE is satisfied, including the initial conditions.

2.2.2 Kernel and Conclusion

- Before leaving this topic, we have several important observations.

- First, note that Equation (32) can be written as

$$T(t) = \lambda T_0 \int_0^t e^{-\lambda(t-\tau)} \sin(w\tau) d\tau. \quad (34)$$

- This is of the form

$$y(t) = \int_0^t K(t-\tau) g(\tau) d\tau. \quad (35)$$

where, $y(t)$ is the solution of the ODE, $g(\tau)$ is the input, $K(t-\tau)$ is called the **kernel** for the ODE.

- In this example, $g(\tau) = \lambda T_0$ (recall Equation (17)) and

$$K(t-\tau) = e^{-\lambda(t-\tau)}.$$

2.2.2 Kernel and Conclusion

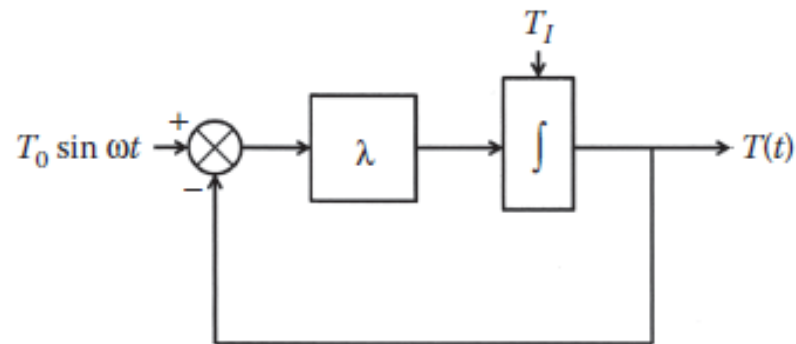
- Equation (35) is a particularly important equation.
 - It applies to much more than our waterpipe temperature example.
 - The solution to **any** LTI nonhomogeneous ODE with zero initial conditions, regardless of its order, can be written in the form of Equation (35).
- The form is called the *convolution* of $K(t)$ and $g(t)$.
 - For LTI nonhomogeneous ODEs, the **convolution** and **kernel** determine the degree to which the present value of the solution depends on values of past inputs.
 - The value of the solution at any time is a weighted sum of values of the input at times past.
 - In the waterpipe temperature example (examining Equation (34)), the weight given to an input decreases exponentially with the distance in time of that input from the current time.

2.2.3 System Viewpoint

- For engineers, whatever is being modeled by the ODE is the **system**.
- A system is defined by its inputs, outputs, and the rules by which the inputs determine the outputs.
- In the waterpipe temperature example, the system has
 - one input ($T_0 \sin wt$, the temperature variation of the ambient air about the daily mean),
 - one output ($T(t)$, the temperature variation of the water about the daily mean),
 - input–output rule (Equation (17)).

2.2.3 System Viewpoint

- Systems governed by linear ODEs lend themselves to block diagrams as a means of depicting clearly the inputs, outputs, and interaction rules.



- Waterpipe temperature example.

- Had there been an initial variation T_I in the water temperature from the daily mean, the output of the system would have been

$$T(t) = \left(\frac{\lambda}{\lambda^2 + w^2} \right) T_0 (\lambda \sin(wt) - w \cos(wt) + we^{-\lambda t}) + T_I e^{-\lambda t}. \quad (36)$$

2.2.3 System Viewpoint

- Equation (36) can be parsed in two ways, each of them important, depending on the engineer's interest at the time.
- The first way parses the solution into *forced response* and *natural response*.
 - The forced response arises as a result of the input and the natural response results from initial conditions.
 - In Equation (36), the forced response is

$$\left(\frac{\lambda}{\lambda^2 + w^2} \right) T_0 (\lambda \sin (wt) - w \cos (wt)) + \left(\frac{\lambda w}{\lambda^2 + w^2} \right) T_0 e^{-\lambda t},$$

and the natural response is $T_I e^{-\lambda t}$.

2.2.3 System Viewpoint

- A second, equally valid, way parses the solution into a *steady-state response* and *transient response*.
- The transient response dies away in time and the steady-state response remains.
- In Equation (36), the steady-state response is

$$\left(\frac{\lambda}{\lambda^2 + w^2} \right) T_0 (\lambda \sin (wt) - w \cos (wt)) , \quad (37)$$

and the transient response is

$$T_I e^{-\lambda t} + \left(\frac{\lambda w}{\lambda^2 + w^2} \right) T_0 e^{-\lambda t}.$$

2.2.3 System Viewpoint

- For engineers, Equation (37) is an inadequate expression for the steady-state response. For example, an engineer may wonder if
 - the amplitude of the output is greater or less than the amplitude of the input,
 - and how this ratio depends on w , the frequency of the input.
- To improve on Equation (37) we can define

$$\begin{aligned}\cos \theta &= \frac{\lambda}{\sqrt{\lambda^2 + w^2}}, \\ \sin \theta &= \frac{w}{\sqrt{\lambda^2 + w^2}}.\end{aligned}\tag{38}$$

- Substituting Equations (38) into Equation (37) results in the expression

$$T_{SS}(t) = \frac{\lambda}{\sqrt{\lambda^2 + w^2}} T_0 \sin (wt - \theta),\tag{39}$$

for the steady-state response of the waterpipe temperature.

2.2.3 System Viewpoint

- Equations (39) can be written as

$$T_{SS}(t) = GT_0 \sin(wt - \theta), \quad (40)$$

where

$$G = \frac{\lambda}{\sqrt{\lambda^2 + w^2}}, \quad (41)$$

$$\theta = \arctan\left(\frac{w}{\lambda}\right). \quad (42)$$

- The coefficient G is called the *gain* of the system and θ is called the *phase lag*.
- These are the defining elements in the *frequency response* of the system.

2.2.3 System Viewpoint

■ Several observations:

- From Equations (40) and (41) engineers can immediately see that the gain is always less than one and hence the steady-state amplitude of the output is always less than amplitude of the input.
- Equation (41) tells us that the gain diminishes monotonically from one to zero as w , the frequency of the input, goes from zero to infinity.
- Moreover, it follows from Equation (42) that the phase lag increases monotonically from zero to $\pi/2$.

2.3 System Stability

- A **linear system** is said to be **stable** if its response to bounded initial conditions is bounded.
- An **linear time-invariant (LTI) system** is said to be **damped** if its response to bounded initial conditions goes to zero as time goes to infinity.
- A first-order time-invariant system described by the differential equation

$$\frac{dy}{dt} + \lambda y = 0$$

is **stable** and **damped** if and only if $\lambda > 0$.

- The definition of stability just given suffices for our purposes for linear systems, but not for nonlinear ones.
 - We will refine our definition of stability in Lecture 4.
 - For other systems the term used is *asymptotically stable*, as will be discussed in Lecture 4.

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