
EE3121

Differential Equations for Engineers

Tutorial 2: First-Order Linear Ordinary Differential Equations



Instructor: Prof. Jie Chen

DEPARTMENT OF ELECTRICAL ENGINEERING

Summary

- Linear equations

- may be time-invariant or time-varying;
- may also be homogeneous or nonhomogeneous.

- A **nonhomogeneous** first-order linear ordinary differential equation (ODE) has the form:

$$\frac{dy}{dt} + p(t) y = g(t).$$

where $g(t)$ is nonzero.

- A **homogeneous** equation has $g(t) = 0$.

- The homogeneous equation has an interesting solution only if its initial condition is nonzero.

Summary

- We solve the equation

$$\frac{dy}{dt} + p(t) y = 0, \quad y(t_0) = y_0.$$

with a useful, easy-to-remember procedure that we needed to justify because it takes some license with mathematical rigor.

- We **separate variables**

$$\frac{dy}{y} = -p(t) dt,$$

- Performing definite integrals on the left from $y(t_0)$ to $y(t)$ and on the right from t_0 to t gives:

$$\int_{y_0}^{y(t)} \frac{dy}{y} = - \int_{t_0}^t p(\tau) d\tau,$$

$$\Rightarrow \ln \left(\frac{y(t)}{y_0} \right) = - \int_{t_0}^t p(\tau) d\tau.$$

■ It leads to a solution that can be written as

$$y(t) = y_0 \exp \left(- \int_{t_0}^t p(\tau) d\tau \right).$$

Summary

- To solve

$$\frac{dy}{dt} + p(t)y = g(t), \quad y(t_0) = y_0.$$

- We look for an integrating factor $\mu(t)$ such that

$$\mu(t) \left(\frac{dy}{dt} + p(t)y \right) = \frac{d}{dt}(\mu y).$$

- We find that

$$\mu(t) = \exp \left(\int^t p(\tau) d\tau \right),$$

in its indefinite integral form suffices.

- We then integrate

$$\frac{d}{dt}(\mu y) = \mu g,$$

to a solution that can be written as

$$y(t) = y_0 \exp \left(- \int_{t_0}^t p(\tau) d\tau \right) + \int_{t_0}^t \exp \left(- \int_{\tau}^t p(\tau) d\tau \right) g(\tau) d\tau.$$

- The term $\exp \left(- \int_{\tau}^t p(\tau) d\tau \right) = K(t, \tau)$ is called the *kernel* of the ODE.
- For an LTI ODE the kernel has the form $K(t - \tau)$.

Summary

- In an engineering-oriented viewpoint, one considers whatever is being modeled by the ODE as a **system** defined by its
 - inputs,
 - outputs,
 - the rules (the ODE) by which the inputs determine the outputs.
- The output of a system can be parsed into a
 - forced response and natural response, or alternatively,
 - steady-state response and transient response.

Summary

- When a system's steady-state response to a unit-amplitude sine-wave input is

$$y_{SS}(s) = A \sin(wt) - B \cos(wt).$$

- It can be written as

$$y_{SS}(s) = G \sin(wt - \theta),$$

where $G = \sqrt{A^2 + B^2}$, and $\theta = \arctan(B, A)$ are the system's *gain* and *phase lag*.

- The **gain** of a first-order time-invariant system with sine-wave input in the form

$$\frac{dy}{dt} + \lambda y = C \sin wt,$$

decreases monotonically to zero as w goes to infinity,

- Its **phase lag** will *increase* monotonically from zero to $\pi/2$.

Summary

- A linear system is said to be **stable** if its response to bounded initial conditions is bounded.
- An LTI system is said to be **damped** if its response to bounded initial conditions goes to zero as time goes to infinity.
- A first-order time-invariant system described by

$$\frac{dy}{dt} + \lambda y = C \sin wt,$$

is stable and damped if and only if $\lambda > 0$.

Summary: Solved Problems

■ SP2.1 Solve

$$\frac{dy}{dt} + \left(\frac{4t}{1+t^2} \right) y = 0, \quad y(0) = 3.$$

■ *Solution:*

- Since the equation is homogeneous we separate variables:

$$\frac{dy}{y} = \frac{-4t dt}{1+t^2}.$$

- Integrating both sides and using the dummy variable τ under the integral on the right-hand side to avoid confusion with the integration end point t :

$$\int_{y(0)}^{y(t)} \frac{dy}{y} = - \int_0^t \frac{4\tau}{1+\tau^2} d\tau,$$

$$\begin{aligned}\Rightarrow \ln y(t) - \ln y(0) &= -2 \ln(1 + t^2) + 2 \ln(1), \\ \Rightarrow \ln \left(\frac{y(t)}{3} \right) &= -\ln(1 + t^2)^2 = \ln \frac{1}{(1 + t^2)^2},\end{aligned}$$

■ Taking the exponent of both sides:

$$\exp \left(\ln \left(\frac{y(t)}{3} \right) \right) = \exp \left(\ln \frac{1}{(1 + t^2)^2} \right),$$

$$\Rightarrow \frac{y(t)}{3} = \frac{1}{(1 + t^2)^2},$$

$$\Rightarrow y(t) = \frac{3}{(1 + t^2)^2}.$$

Summary: Solved Problems

■ SP2.2 Solve

$$\frac{dy}{dt} + (2 - \tan t) y = 0, \quad y(0) = 4.$$

■ *Solution:*

- Since the equation is homogeneous we separate variables:

$$\frac{dy}{y} = -(2 - \tan t) dt.$$

- Integrating both sides gives rise to:

$$\int_{y(0)}^{y(t)} \frac{dy}{y} = - \int_0^t (2 - \tan \tau) d\tau,$$

$$\Rightarrow \ln \left(\frac{y(t)}{y(0)} \right) = -2t - \ln \left(\frac{\cos t}{1} \right) = -2t - \ln(\cos t) = -2t + \ln \left(\frac{1}{\cos t} \right),$$

- Taking the exponent of both sides:

$$\exp \left(\ln \left(\frac{y(t)}{y(0)} \right) \right) = \exp \left(-2t + \ln \left(\frac{1}{\cos t} \right) \right),$$

$$\Rightarrow \frac{y(t)}{4} = \exp(-2t) \cdot \exp \left(\ln \left(\frac{1}{\cos t} \right) \right) = \frac{e^{-2t}}{\cos t},$$

$$\Rightarrow y(t) = 4 \frac{e^{-2t}}{\cos t}.$$

Summary: Solved Problems

■ SP2.3 Solve

$$\frac{dy}{dt} + \left(\frac{3}{t}\right) y = t^2, \quad y(1) = 2. \quad (1)$$

■ *Solution:*

- Since Equation (1) is nonhomogeneous, we search for an integrating factor $\mu(t)$, such that

$$\mu(t) \left(\frac{dy}{dt} + \left(\frac{3}{t}\right) y \right) = \frac{d}{dt} (\mu(t) y).$$

- We can determine $\mu(t)$ either by *working through the requirements on it*, as we did in Equations $\mu(t) \left(\frac{dy}{dt} + p(t) y \right) = \mu(t) \frac{dy}{dt} + \mu(t) p(t) y$ through $\frac{d\mu}{dt} = p(t) \mu(t)$ earlier,

Summary: Solved Problems

- Or by recognizing that $p(t) = 3/t$ in Equation (1) and applying the formula given in Equation $\mu(t) = \exp\left(\int^t p(\tau) d\tau\right)$, we are then led to:

$$\mu(t) = \exp\left(\int^t p(\tau) d\tau\right) = \exp\left(\int^t \frac{3d\tau}{\tau}\right) = \exp(3 \ln t) = \exp(\ln(t^3)) = t^3.$$

- Note that we are using the indefinite integral here.
- Multiplying both sides of Equation (1) by $\mu(t) = t^3$, we have

$$t^3 \left(\frac{dy}{dt} + \left(\frac{3}{t} \right) y \right) = t^3 \cdot t^2,$$

$$\Rightarrow \frac{d}{dt} (t^3 y) = t^{3+2} = t^5.$$

Summary: Solved Problems

■ Integrating both sides:

$$\begin{aligned}\int_1^t \frac{d}{d\tau} (\tau^3 y) d\tau &= \int_1^t \tau^5 d\tau, \\ \Rightarrow \tau^3 y \Big|_1^t &= \frac{\tau^6}{6} \Big|_1^t, \\ \Rightarrow t^3 y(t) - 1 \cdot y(1) &= \frac{t^6}{6} - \frac{1}{6}, \\ \Rightarrow t^3 y(t) - 2 &= \frac{t^6}{6} - \frac{1}{6}, \\ \Rightarrow y(t) &= \frac{t^6}{6} + \frac{11}{6t^3}.\end{aligned}$$

Summary: Solved Problems

■ SP2.4 Solve

$$\frac{dy}{dt} + e^{\lambda t} y = k e^{\lambda t}, \quad y(0) = y_0. \quad (2)$$

■ *Solution:*

- Since Equation (2) is nonhomogeneous, we search for an integrating factor $\mu(t)$, such that

$$\mu(t) \left(\frac{dy}{dt} + e^{\lambda t} y \right) = \frac{d}{dt} (\mu(t) y).$$

- Recognizing that $p(t) = e^{\lambda t}$ in Equation (2) and applying Equation $\mu(t) = \exp \left(\int^t p(\tau) d\tau \right)$:

$$\mu(t) = \exp \left(\int^t e^{\lambda \tau} d\tau \right) = \exp \left(\frac{e^{\lambda t}}{\lambda} \right).$$

- Multiplying both sides of Equation (2) by $\mu(t) = \exp(e^{\lambda t}/\lambda)$,

$$\exp\left(\frac{e^{\lambda t}}{\lambda}\right) \left(\frac{dy}{dt} + e^{\lambda t} y\right) = \exp\left(\frac{e^{\lambda t}}{\lambda}\right) (k e^{\lambda t}),$$

$$\Rightarrow \frac{d}{dt} \left(\exp\left(\frac{e^{\lambda t}}{\lambda}\right) y \right) = k e^{\lambda t} \exp\left(\frac{e^{\lambda t}}{\lambda}\right).$$

- Integrating both sides:

$$\int_0^t \frac{d}{d\tau} \left(y \exp\left(\frac{e^{\lambda \tau}}{\lambda}\right) \right) d\tau = \int_0^t k e^{\lambda \tau} \exp\left(\frac{e^{\lambda \tau}}{\lambda}\right) d\tau,$$

$$\Rightarrow y \exp\left(\frac{e^{\lambda \tau}}{\lambda}\right) \Big|_0^t = k \exp\left(\frac{e^{\lambda \tau}}{\lambda}\right) \Big|_0^t,$$

$$y(t) \exp\left(\frac{e^{\lambda t}}{\lambda}\right) - y(0) \exp\left(\frac{1}{\lambda}\right) = k \left(\exp\left(\frac{e^{\lambda t}}{\lambda}\right) - \exp\left(\frac{1}{\lambda}\right) \right),$$

$$\Rightarrow y(t) = k + (y_0 - k) \exp\left(- (e^{\lambda t} - 1) / \lambda\right).$$

End