

EE6617: Tutorial 4

In what follows we assume signal model

$$\begin{aligned} x[n] &= A + w[n], \quad w[0] \sim \mathcal{N}(0, \sigma^2), \\ w[n] &= \rho w[n-1] + \sqrt{1 - \rho^2} z[n], \quad z[n] \sim \mathcal{N}(0, \sigma^2), \quad n = 0, \dots, N-1. \end{aligned}$$

The $0 \leq \rho \leq 1$ is the temporal correlation coefficient of noise, and $w[0], z[1], \dots, z[N-1]$ are i.i.d. and $w[-1] = 0$.

4.1 Suppose we want to estimate a DC signal A from $x[n]$, $n = 0, \dots, N-1$.

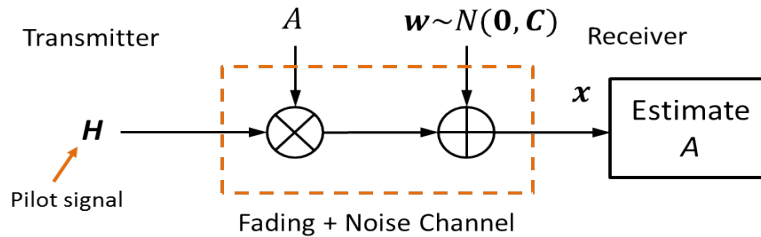
- (a) Find $E[w[n]]$ and $\text{var}(w[n])$. What is the distribution of $w[n]$?
- (b) Find $E[w[n]w[m]]$ and $E[\mathbf{w}\mathbf{w}^T]$ where $\mathbf{w} = [w[0] \ w[1] \ \dots \ w[N-1]]^T$. What is the distribution of $\mathbf{w}$?
- (c) One can model the observation in a linear form

$$\mathbf{x} = \mathbf{1}A + \mathbf{w} = \mathbf{H}\theta + \mathbf{w}$$

where $\mathbf{H} = \mathbf{1} \in \mathbb{R}^{N \times 1}$, $\theta = A \in \mathbb{R}$. Given the linear model, design MVU estimator $\hat{A}$. What is $\text{var}(\hat{A})$?

- (d) Set $N = 2$ and calculate $\text{var}(\hat{A})$ for $\rho = 0$ and $\rho = 1$. Is it more easier to estimate $\hat{A}$ for $\rho = 0$ than for $\rho = 1$? Provide your interpretation about the relation between ρ and the estimation performance.
- (e) For $N = 2$, compute eigenvalue decomposition of $\mathbf{C}^{-1}$ and express it in the form of $\mathbf{C}^{-1} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$.

4.2 Suppose the following communication system.



We call $\mathbf{H} \in \mathbb{R}^{N \times 1}$ the pilot signal, which is known at both the transmitter and receiver, and it is transmitted to estimate the scalar channel fading gain A at the receiver. The problem to estimate the channel A from the training channel output is called **channel estimation problem**.

We assume $N = 2$. In the previous problems, $\mathbf{H} = \mathbf{1} \in \mathbb{R}^{2 \times 1}$. Here we want to design $\mathbf{H}$ to minimize the variance of the scalar channel estimator $\hat{A}$,

$$\mathbf{H} = \underset{\mathbf{H}}{\text{argmin}} \text{var}(\hat{A}).$$

This problem is the **pilot signal optimization problem**. By applying Rayleigh quotient theorem, design the optimal pilot signal $\mathbf{H}$ that minimizes $\text{var}(\hat{A})$.

You will use the EVD results from **4.1.(e)**.

Can you explain why the designed $\mathbf{H}$ minimizes the variance by relating it to the noise covariance (or noise direction)?

Sol. 4.1 (a) $E[w[0]] = 0$, $E[w[1]] = E[\rho w[0] + \sqrt{1 - \rho^2} z[1]] = 0$, and clearly $E[w[n]] = E[\rho w[n-1] + \sqrt{1 - \rho^2} z[n]] = 0$.

$\text{var}(w[0]) = \sigma^2$, $\text{var}(w[1]) = \text{var}(\rho w[0] + \sqrt{1 - \rho^2} z[1]) = \rho^2 \sigma^2 + (1 - \rho^2) \sigma^2 = \sigma^2$, and clearly $\text{var}(w[n]) = \text{var}(\rho w[n-1] + \sqrt{1 - \rho^2} z[n]) = \rho^2 \sigma^2 + (1 - \rho^2) \sigma^2 = \sigma^2$.

Thus, the distribution of $w[n]$ is $w[n] \sim \mathcal{N}(0, \sigma^2)$.

(b) First we assume $n \geq m$, then

$$\begin{aligned} E[w[n]w[m]] &= E\left[\left(\rho w[n-1] + \sqrt{1 - \rho^2} z[n]\right) w[m]\right] \\ &= E[\rho w[n-1]w[m]] \\ &= E\left[\rho\left(\rho w[n-2] + \sqrt{1 - \rho^2} z[n-1]\right) w[m]\right] \\ &= E[\rho^2 w[n-2]w[m]] \\ &= \rho^{n-m} \sigma^2. \end{aligned}$$

When $m \geq n$, it is easy to see $E[w[n]w[m]] = \rho^{m-n} \sigma^2$. Thus, we have

$$E[w[n]w[m]] = \rho^{|m-n|} \sigma^2$$

and results in $\mathbf{C} = E[\mathbf{w}\mathbf{w}^T]$, $[\mathbf{C}]_{i,j} = \rho^{|i-j|} \sigma^2$. Thus, the distribution of $\mathbf{w}$ is $\mathcal{N}(\mathbf{0}, \mathbf{C})$.

(c) The MVU estimator is

$$\begin{aligned} \hat{A} &= (\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{C}^{-1} \mathbf{x} = \frac{\mathbf{1}^T \mathbf{C}^{-1} \mathbf{x}}{\mathbf{1}^T \mathbf{C}^{-1} \mathbf{1}} \\ \text{var}(\hat{A}) &= (\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H})^{-1} = \frac{1}{\mathbf{1}^T \mathbf{C}^{-1} \mathbf{1}}. \end{aligned}$$

(d) For $N = 2$, $C = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$. The inverse of $\mathbf{C}$ is $C^{-1} = \frac{1}{\sigma^2(1-\rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$. The variance of $\hat{A}$ is now calculated as

$$\text{var}(\hat{A}) = \left(\frac{2(1-\rho)}{\sigma^2(1-\rho^2)} \right)^{-1} = \frac{\sigma^2(1+\rho)}{2}$$

$\text{var}(\hat{A})_{\rho=0} = \frac{\sigma^2}{2}$ and $\text{var}(\hat{A})_{\rho=1} = \sigma^2$. Since $\text{var}(\hat{A})_{\rho=0} \leq \text{var}(\hat{A})_{\rho=1}$, estimating A when $\rho = 0$ is easier than $\rho = 1$. This is because as ρ increases, the level of noise correlation increases such that $C \stackrel{\rho \rightarrow 1}{\approx} \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \ 1] = \sigma^2 \mathbf{1}\mathbf{1}^T$ and the signal direction $\mathbf{1}A$ matched to the direction of the dominant eigenvector of C .

(e) From $C^{-1} = \frac{1}{\sigma^2(1-\rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$, it is enough to find the eigenvalues and eigenvectors for $\Phi = \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$. First equate $\Phi \mathbf{v} = \lambda \mathbf{v}$ and this is equivalent to $(\Phi - \lambda \mathbf{I}) \mathbf{v} = \mathbf{0}$

$\Rightarrow \det(\Phi - \lambda \mathbf{I}) = 0$, i.e., the matrix $\Phi - \lambda \mathbf{I}$ should be singular. The characteristic polynomial is

$$\begin{aligned}\det(\Phi - \lambda \mathbf{I}) &= \det \left(\begin{bmatrix} 1-\lambda & -\rho \\ -\rho & 1-\lambda \end{bmatrix} \right) = (1-\lambda)^2 - \rho^2 = \lambda^2 - 2\lambda + 1 - \rho^2 \\ &= (\lambda - (1+\rho))(\lambda - (1-\rho)) = 0\end{aligned}$$

Then, the roots of the above characteristic polynomial are the eigenvalues of Φ , i.e., $\lambda_1 = 1 + \rho$, $\lambda_2 = 1 - \rho$.

For $\lambda_1 = 1 + \rho$, the eigenvector $\mathbf{v}_1$ is found from $(\Phi - \lambda_1 \mathbf{I}) \mathbf{v}_1 = \mathbf{0}$, i.e.,

$$(\Phi - (1 + \rho)\mathbf{I}) \mathbf{v}_1 = \begin{bmatrix} -\rho & -\rho \\ -\rho & -\rho \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Similarly, for $\lambda_2 = 1 - \rho$,

$$(\Phi - (1 - \rho)\mathbf{I}) \mathbf{v}_2 = \begin{bmatrix} \rho & -\rho \\ -\rho & \rho \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus,

$$\begin{aligned}\mathbf{C}^{-1} &= \frac{1}{\sigma^2(1-\rho^2)} \Phi \\ &= \frac{1}{2\sigma^2(1-\rho^2)} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sigma^2(1-\rho)} & 0 \\ 0 & \frac{1}{2\sigma^2(1+\rho)} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad (1)\end{aligned}$$

Sol. 4.2 Notice that the variance of $\hat{A}$ is $\text{var}(\hat{A}) = (\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H})^{-1}$. For $0 \leq \rho \leq 1$, we have from (1) that the eigenvalues of $\mathbf{C}^{-1}$ is ordered such that $\frac{1}{2\sigma^2(1+\rho)} \leq \frac{1}{2\sigma^2(1-\rho)}$. Thus, based on the Rayleigh quotient theorem, the $(\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H})^{-1}$ is minimized when $\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H}$ is maximized and this is achieved by choosing $\mathbf{H} = [1 \ -1]^T$. The minimum variance is $\text{var}(\hat{A}) = \frac{\sigma^2(1-\rho)}{2}$.