

Complex Networks:

Lecture 3: Introduction to Graph Theory

EE 6605

Instructor: G Ron Chen



What is a Graph?

- A **graph** is a diagrammatical representation of some physical structure

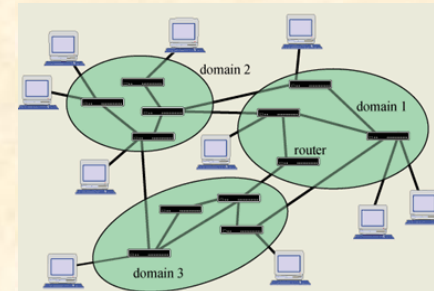
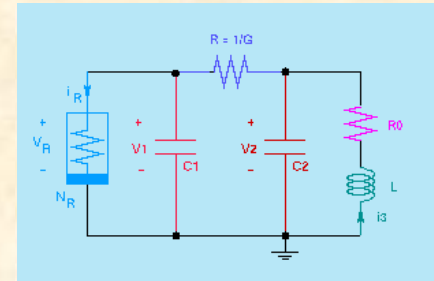
such as:

a circuit

a computer network

a human relation network

... and so on.



Example:

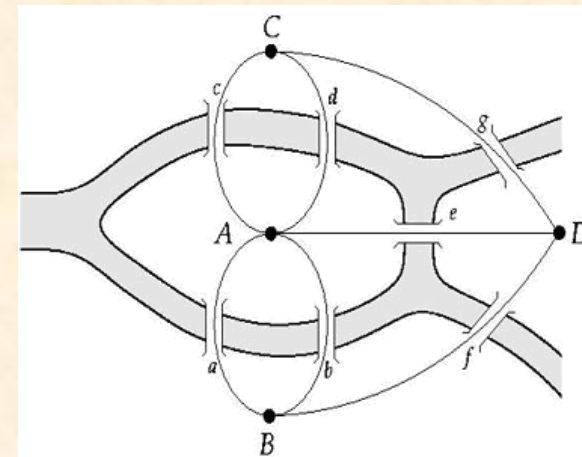
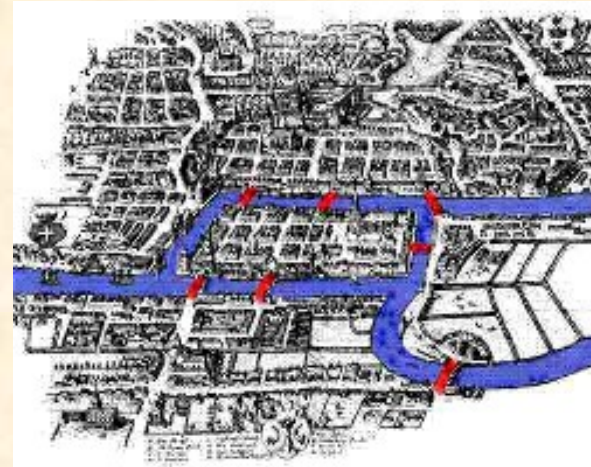
The seven-bridge problem of Königsburg

Euler (1707-1783)

studied and solved this famous seven-bridge problem of Königsburg

-- no solution!

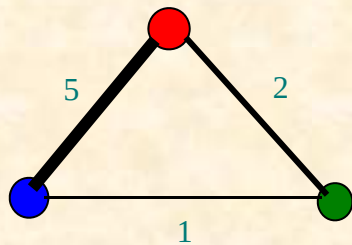
Euler was the pioneer of
Graph Theory



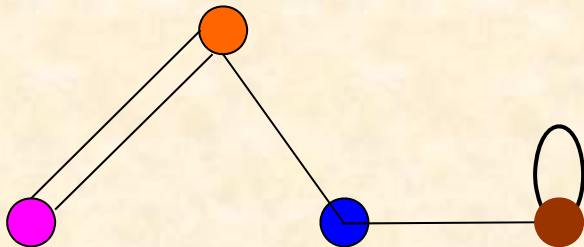
Notation

- Let G be a non-empty graph with at least one **node** (or, **vertex**).
- In a non-isolated case, G has at least one **edge** (or, **link**); thus, it has at least two nodes.
- Let $N(G)$ and $M(E)$ denote the set of its nodes and the set of its edges, respectively.
- In general, $N(G)$ and $M(E)$ are finite sets.
- Such a non-empty pair $(N(G), M(E))$ is referred to as a **simple graph** G

Examples of General (non-simple) Graphs



weighted graphs
sometimes considered

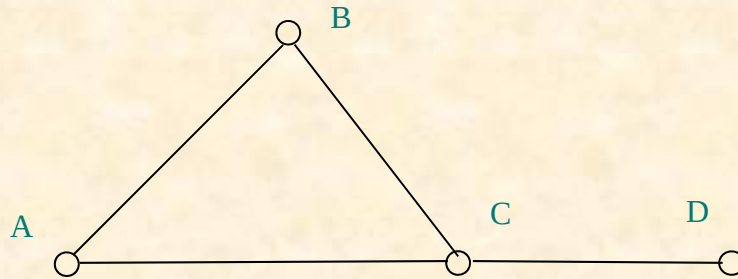


graph with self-loop
or multiple edges
are not allowed

Convention:  not allowed

When a node is removed, all its edges will also be removed

Examples of Simple Graph



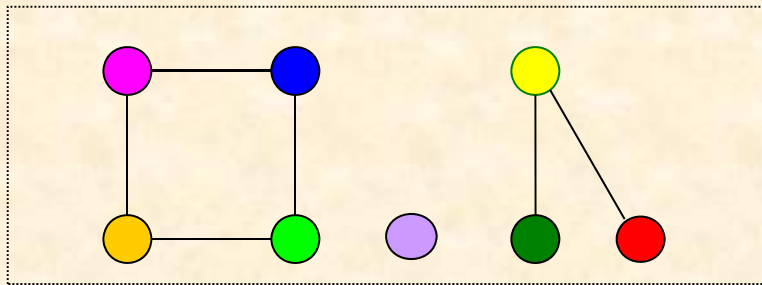
$$N(G) = \{A, B, C, D\}$$

$$M(E) = \{AB, AC, BC, CD\}$$

Subgraph: ABC AB D etc. (needed not be connected)

Loop (Circuit): ABC

Component: A self-connected subgraph, but un-connected with other parts of the same graph



← A simple graph with 3 components

- **Theorem:** *If a simple graph G with N nodes has K components, then the number of edges, M , of G satisfies*

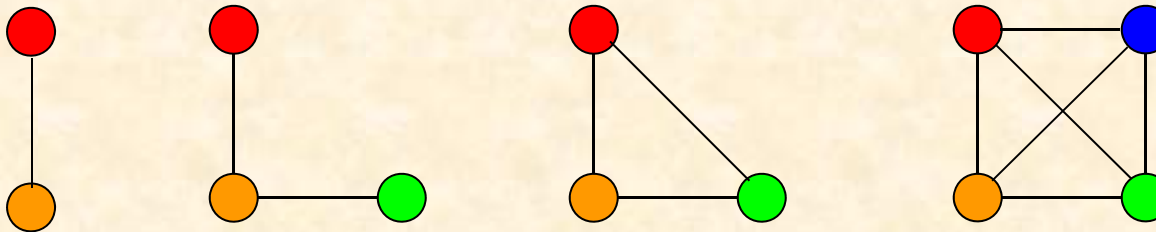
$$N - K \leq M \leq \frac{1}{2}(N - K)(N - K + 1)$$

In particular, for a connected graph, it reduces to

$$N - 1 \leq M \leq \frac{1}{2}N(N - 1)$$

- **Proof.** The general case is proved in the lecture notes, while the case of $K = 1$ is obvious: A connected graph with N nodes has at least $N - 1$ edges and at most $\frac{1}{2}N(N - 1)$ edges.

Examples:



Corollary: *If a simple graph of N nodes satisfies*

$M > \frac{1}{2}(N-1)(N-2)$ then it must be connected.

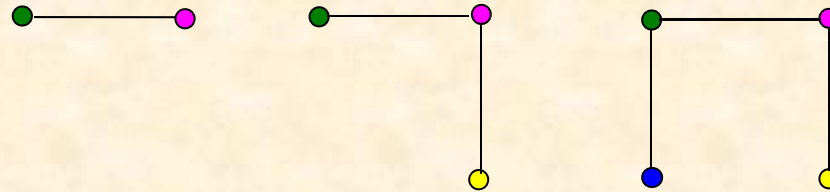
Proof. If not connected, then $K \geq 2$ in $N-K \leq M \leq \frac{1}{2}(N-K)(N-K+1)$

In case of $K = 2$: $N-2 \leq M \leq \frac{1}{2}(N-1)(N-2)$

But, now it is assumed $M > \frac{1}{2}(N-1)(N-2)$

This is a contradiction.

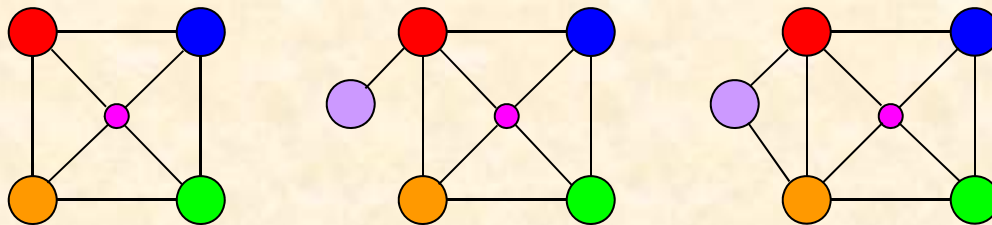
Examples:



Some Basic Results

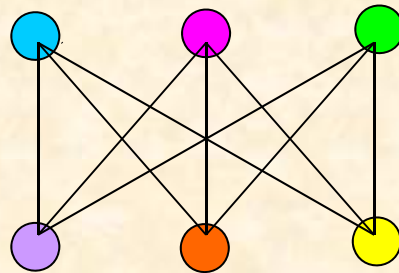
- ❖ **Theorem** (Handshaking Lemma) *The total node degree of a graph is always an even number.*
- ❖ **Proof.** Since every edge joins two nodes, so the total node degree is twice of the number of edges.
- ❖ **Corollary:** *In any graph, the number of nodes of odd degrees must be even.*

Examples:

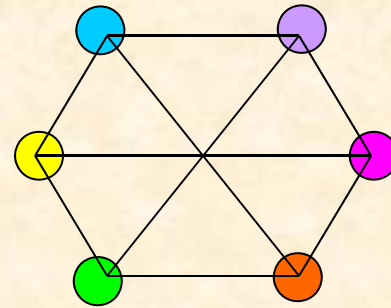


Isomorphism

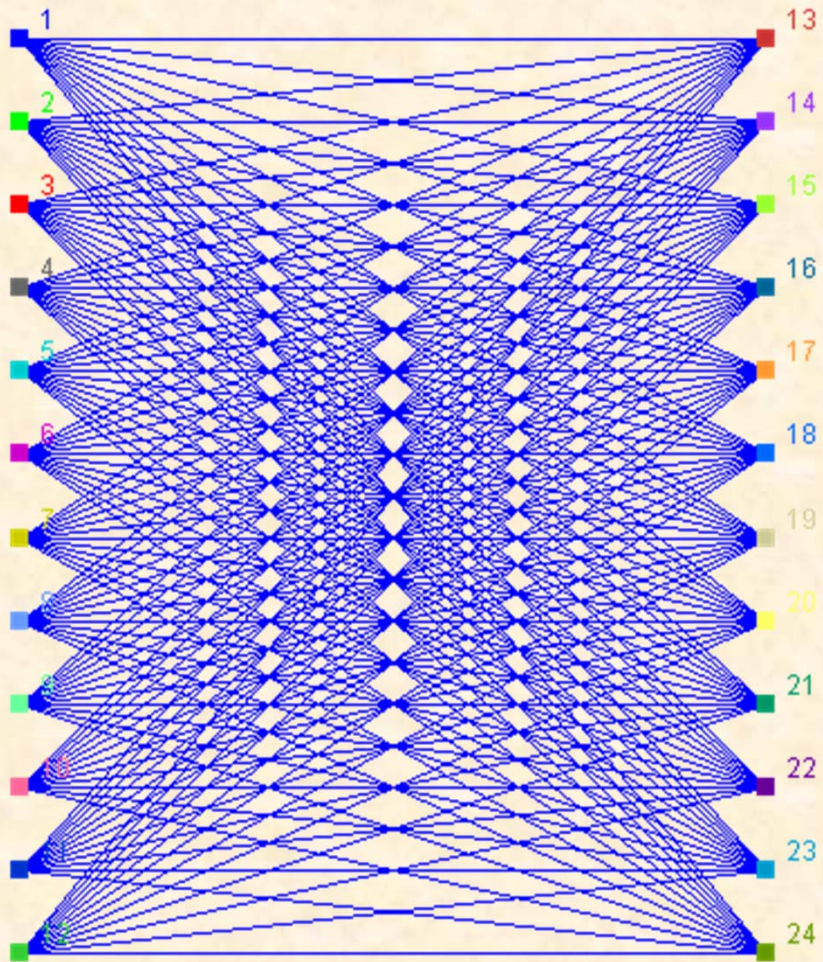
- Two graphs G_1 and G_2 are said to be **isomorphic**, if there is a **one-one** correspondence between the **nodes** of G_1 and those of G_2 , with the **property** that the number of **edges** joining any two nodes of G_1 is equal to the number of **edges** joining the two corresponding nodes of G_2 .
- Example:**



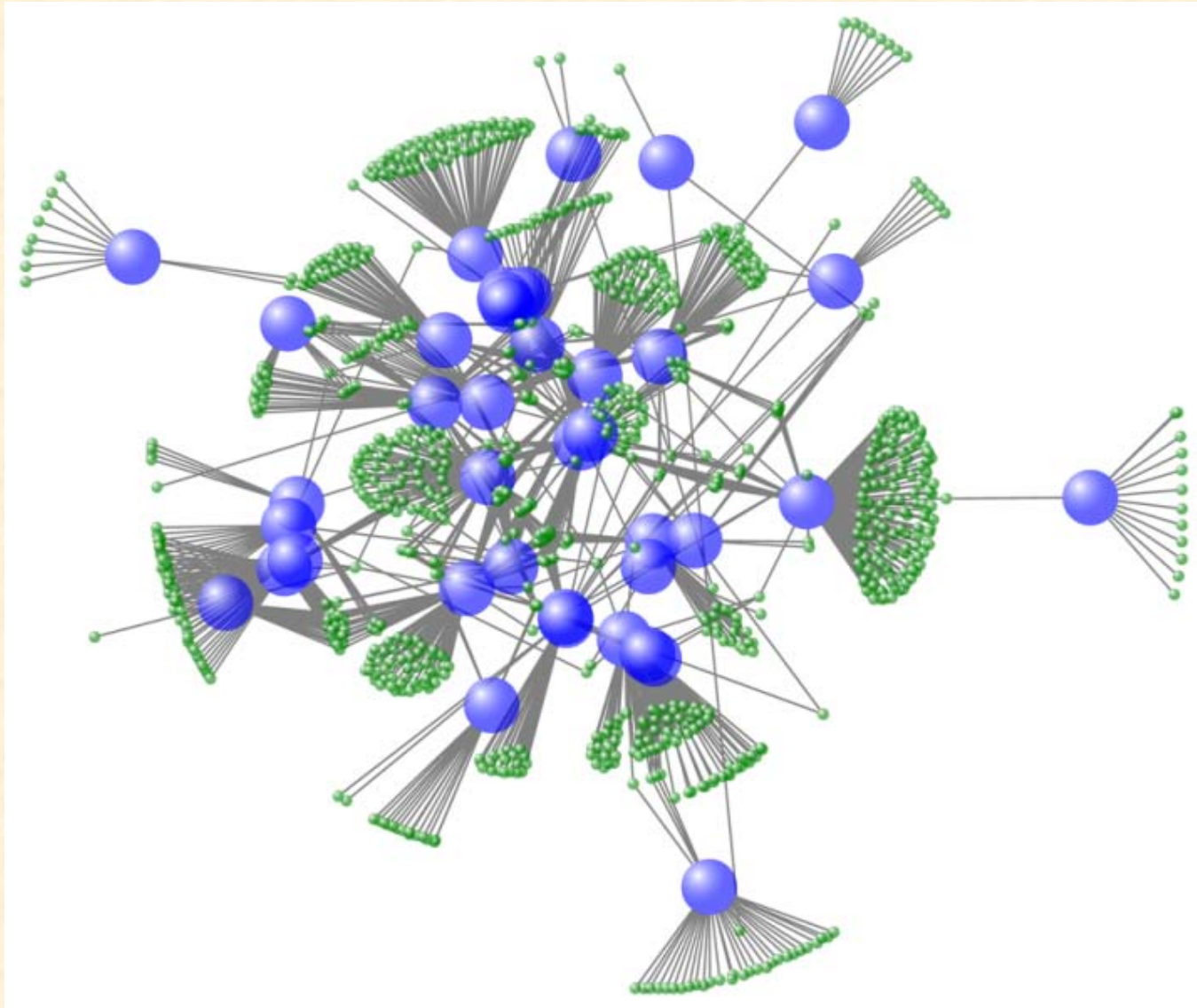
bipartite graph



Bipartite Graphs

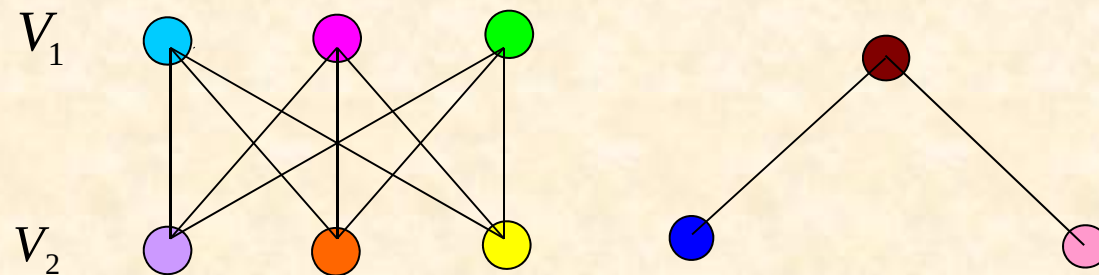


Bipartite Graphs



Circuits in Bipartite Graphs

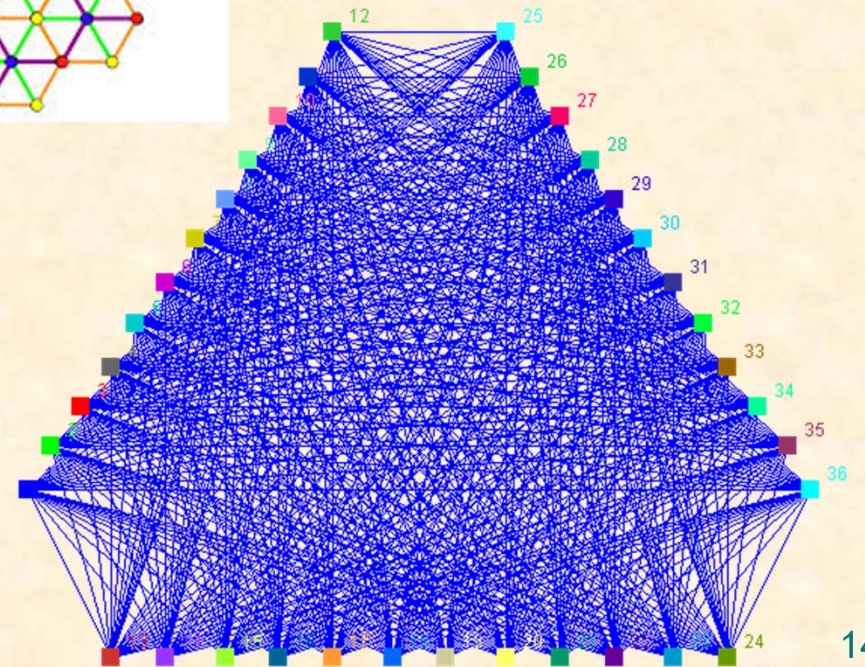
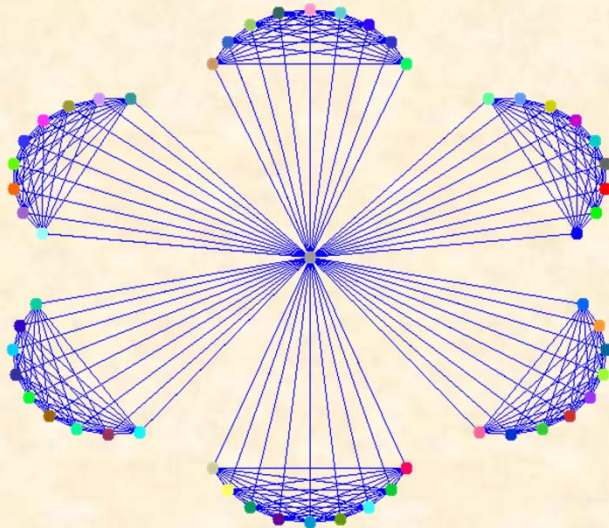
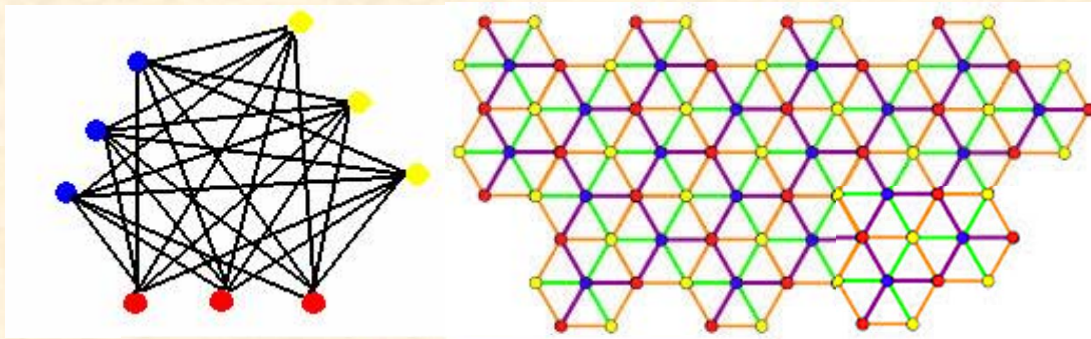
- **Theorem:** *In any bipartite graph, every loop (circuit) has an even number of edges in the path.*
- **Proof.** Let $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_n \rightarrow v_1$ be a circuit in the bipartite graph $G = G(V_1, V_2)$. Assume, without loss of generality, that $v_1 \in V_1$. Then, since G is bipartite, one must have $v_2 \in V_2, v_3 \in V_1$ etc. Finally, one must have $v_n \in V_2$ in order to form a circuit, yielding an even number of paths.



Note: The converse is also true (counterexample: right picture above)

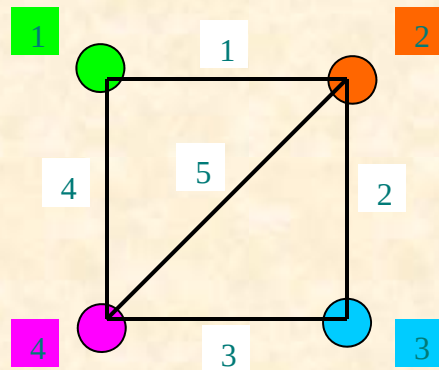
Tripartite Graphs

Tripartite graph: All nodes are partitioned into three sets in such a way that no two vertices contained in any one of the three parts are adjacent



Adjacency and Incidence Matrices

- ❖ For a graph G with nodes $N(G) = \{1, 2, \dots, n\}$, its **adjacency matrix** A is defined to be the $n \times n$ constant matrix whose ij th entry is 1 if node i connects node j ; or 0 otherwise. (connectivity matrix)
- ❖ Further, let G have edges $M(E) = \{1, 2, \dots, m\}$. Then, its **incidence matrix** M is defined to be the $n \times m$ constant matrix whose ij th entry is 1 if node i connects edge j ; or 0 otherwise.
- ❖ **Example:**



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

(always
square and
symmetrical)

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(usually
non-square)

Laplacian Matrix

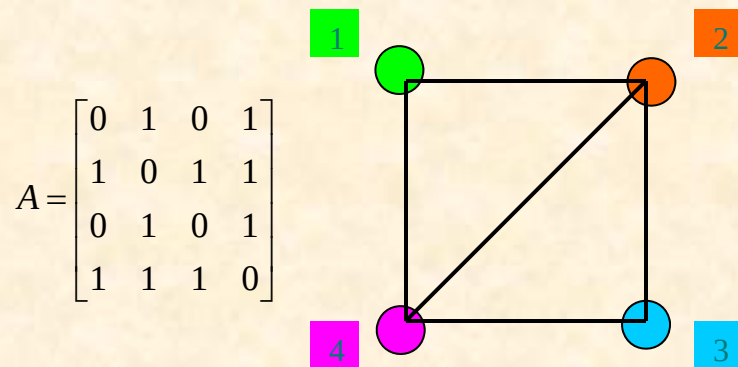
Definition: Laplacian matrix (or, admittance matrix or Kirchhoff matrix), denoted $L = [L_{ij}]$, is defined as

$$L_{ij} = \begin{cases} k(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j, \quad v_i \text{ adjacent } v_j \\ 0 & \text{otherwise} \end{cases}$$

where $k(v_i)$ is the degree of node v_i

It is always semi-positive definite; with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_n$

Example:



$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - A$$

Eigenvalues of Laplacian Matrix

For a connected undirected and unweighted graph G of size N , its Laplacian matrix L is symmetrical and semi-positive definite, with zero row-sums (hence, zero column-sums). Let the eigenvalues of L be $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$, which are real and nonnegative. Then

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$$

If G has $m < N$ components (in the above, $m = 1$), then

$$0 = \lambda_1 = \lambda_2 = \dots = \lambda_m < \lambda_{m+1} \leq \dots \leq \lambda_N$$

In particular, if $m = N$ then

$$0 = \lambda_1 = \lambda_2 = \dots = \lambda_N$$

This corresponds to a set of N isolated nodes for which $L = 0$.

Verification

To verify the algebraic connectivity $\lambda_2 > 0$ for a connected network, let x be a nonzero eigenvector associated with the eigenvalue $\lambda_1 = 0$ of its Laplacian matrix L .

Then, $Lx = 0$, so that $x^T Lx = \sum_{(u,v) \in E} (x_u - x_v)^2 = 0$.

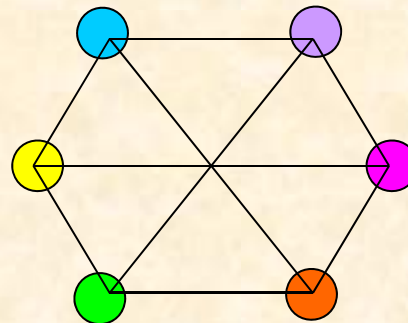
Consequently, $x_u = x_v$ for every pair of nodes (u,v) in the network since the network is connected.

This implies that $x = a[1,1,\dots,1]^T$ for some constant $a \neq 0$, namely, the eigenvalue $\lambda_1 = 0$ has multiplicity 1; therefore $\lambda_2 \neq 0$.

By the semi-positiveness of the Laplacian matrix L , one has $\lambda_2 > 0$.

Regular Graphs

- ❖ A graph in which every node has the same degree is called a regular graph; if every node has degree r then the graph is called a regular graph of degree r (or, r -regular)
- ❖ **Theorem:** A regular graph of degree r with N nodes has $rN/2$ edges.
- ❖ **Proof.** Since every node connects with r edges, there are rN connecting edges. However, each edge has been doubly counted, so it should be divided by two.
- ❖ **Example:**
A regular graph of degree 3, with 6 nodes, which has $3 \times 6 / 2 = 9$ edges.



Eigenvalues of Regular Graphs

Let G be a connected r -regular graph with N nodes, with eigenvalues $\{\mu_1, \mu_2, \dots, \mu_{N-1}, \mu_N\}$ of its adjacency matrix A

(i) $-r \leq \mu_1 \leq \dots \leq \mu_{N-1} < \mu_N = r$

(ii) G is bipartite if and only if the above eigenvalues are symmetrical about 0; and if and only if $\mu_1 = -r$

$$-r = \mu_1 < \mu_2 \leq \dots \leq \mu_{N-1} < \mu_N = r$$

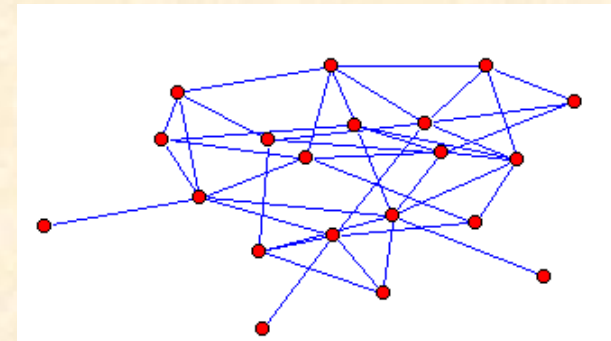
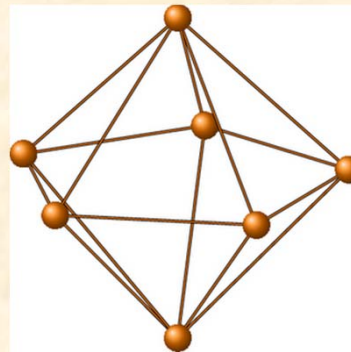
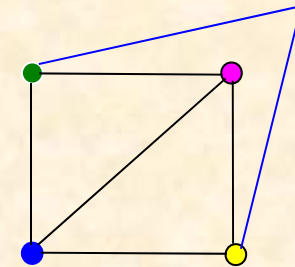
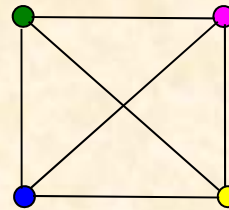


$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$-1 = \mu_1 < \mu_2 = 1$$

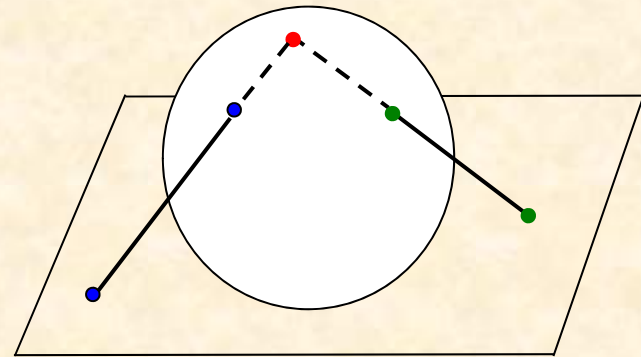
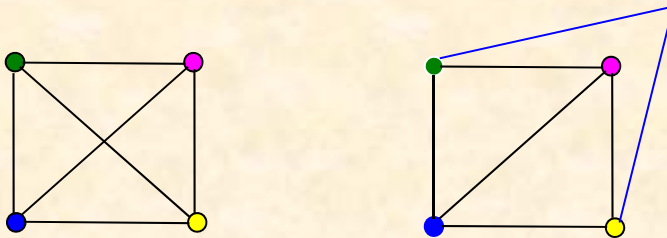
Graph Embedding

- **Jordan curve** on the plane: A continuous curve with no self-crossings.
- A graph G can be **embedded** in the Euclidean 3d space **if** it is **isomorphic** to a graph in the space with Jordan curve edges.
- **Theorem:** *Every simple graph can be embedded in the Euclidean 3d space.*

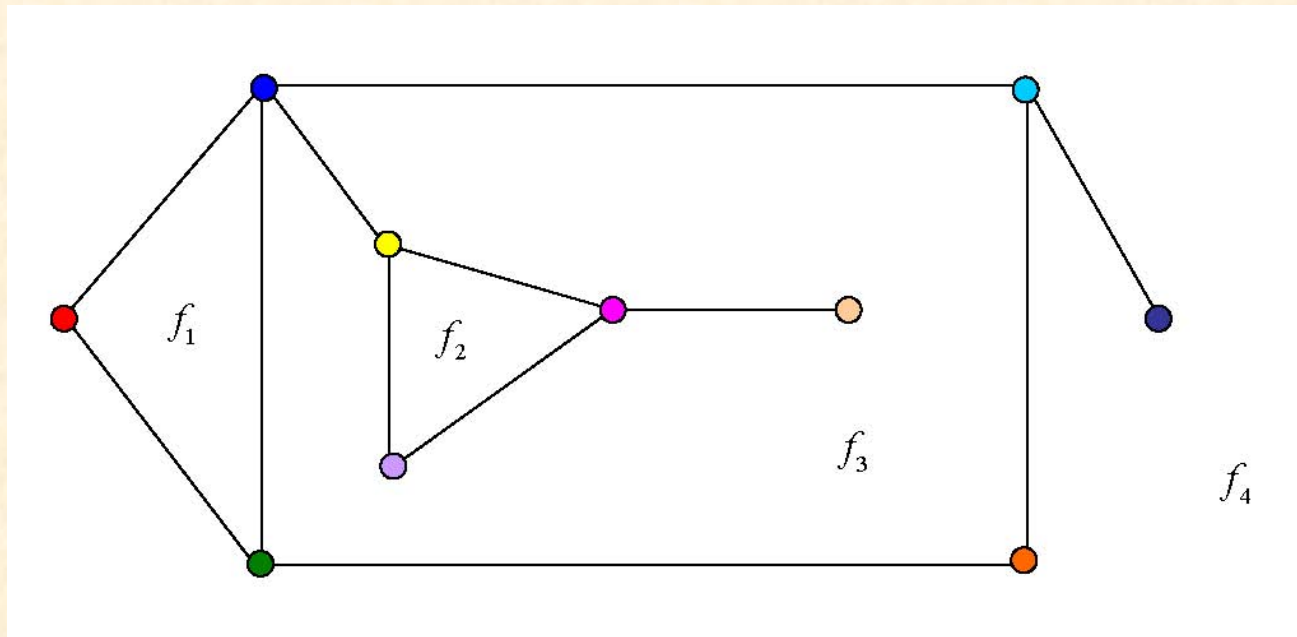


Planar Graphs

- **Plane graph** is one that can be drawn on the plane without crossing edges
- **Planar graph** is one that is isomorphic to a plane graph.
- Every planar graph can be embedded on a plane (within the Euclidean 3d space)
- **Theorem:** *A graph is planar if and only if it can be embedded on the surface of a sphere.*



Euler Formula for Plane Graphs



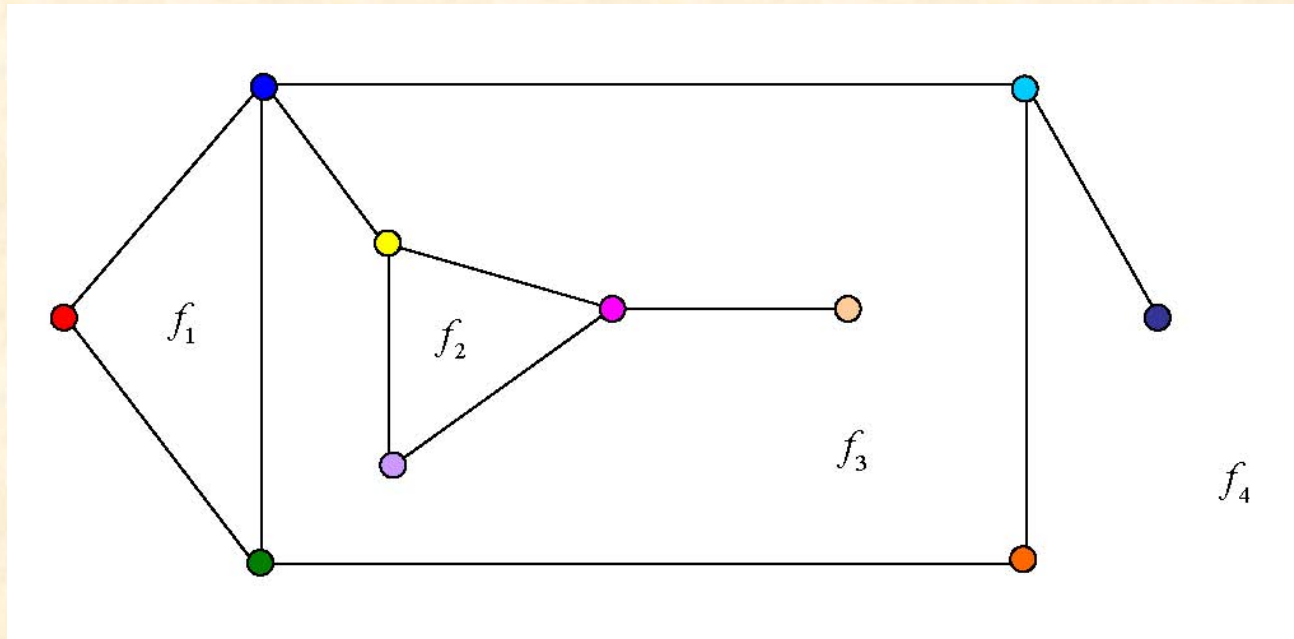
There are four faces: f_1 , f_2 , f_3 and f_4 (infinite face)

Euler Formula for Plane Graphs

- **Theorem** (Euler, 1750) *Let n , e and f be the number of nodes, edges and faces of a connected plane graph, respectively. Then*

$$n - e + f = 2$$

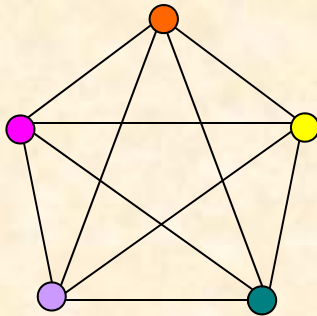
Proof. Apply induction on the number of edges of the graph.



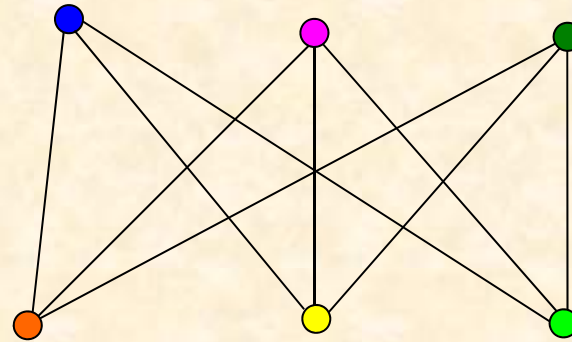
$$n = 10, e = 12, f = 4 \rightarrow n - e + f = 2$$

Non-Planar Graphs

- Two special yet important non-planar graphs:



Graph K_5



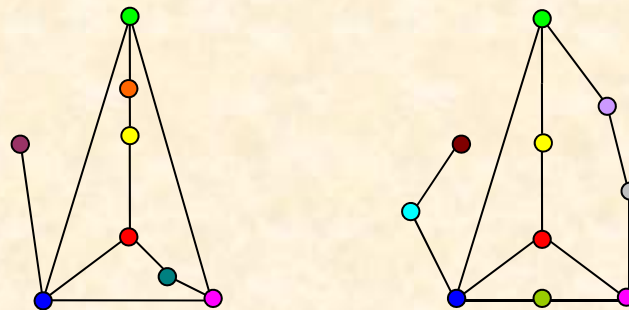
Graph $K_{3,3}$



K. Kuratowski (1896 -1980)

Homeomorphism

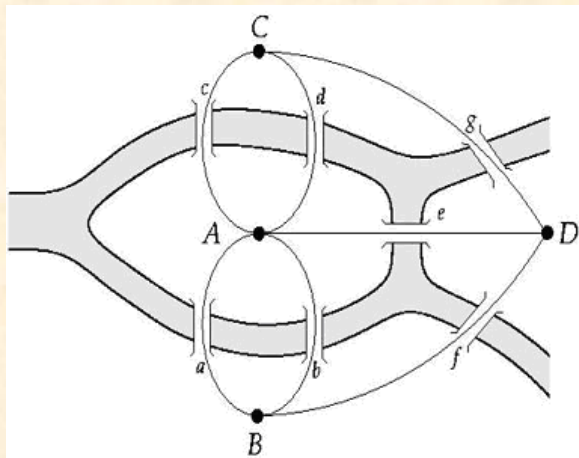
- Two graphs are said to be **homeomorphic**, if they can both be obtained from the same graph by inserting new nodes of degree two into edges.
(namely, identical to within nodes of degree two)



- Theorem** (Kuratowski Theorem) *A graph is planar if and only if it contains no subgraphs homeomorphic to K_5 or $K_{3,3}$*

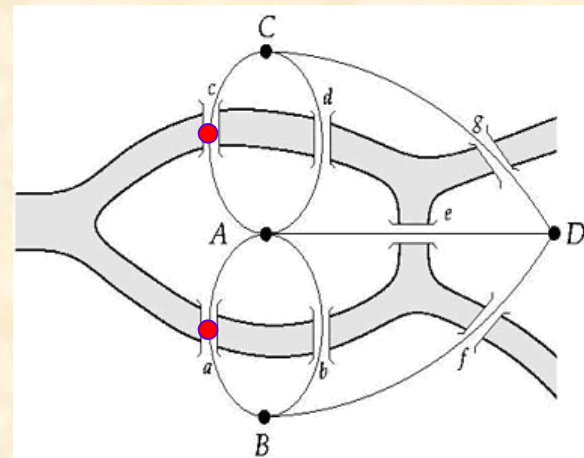
An Application Example

Recall: The seven-bridge problem of Königsburg



With multiple edges

Homeomorphic



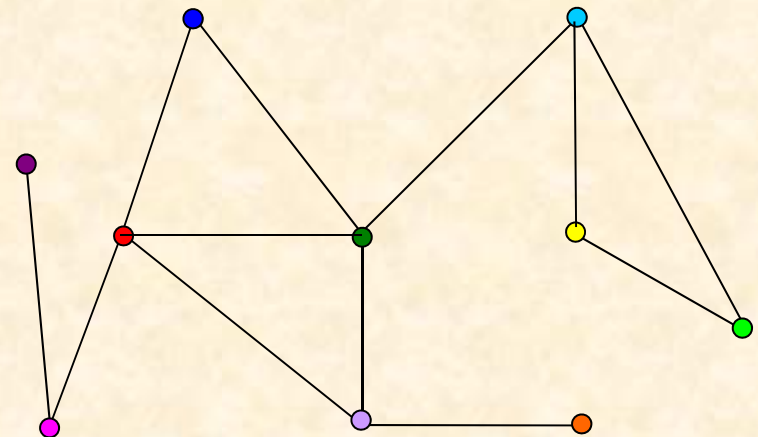
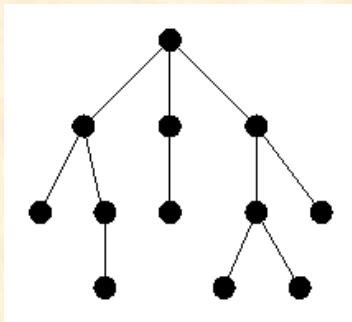
A simple graph

BREAK

10 minutes

More Concepts

- **Walk:** A finite sequence of edges, one after another, in the form of $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$ where $N(G) = \{v_1, v_2, \dots, v_n\}$ are nodes.
- A walk is denoted by $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_n$ and the number of edges in a walk is called its **length**.
- **Trail:** A walk in which all edges are distinct.
- **Path:** A trail in which all nodes are distinct, except perhaps $v_1 = v_n$ which is called a **closed path**, often called a **circuit** (or, sometimes, a **loop** or a **cycle**).
- **Tree:** A graph with no circuits.

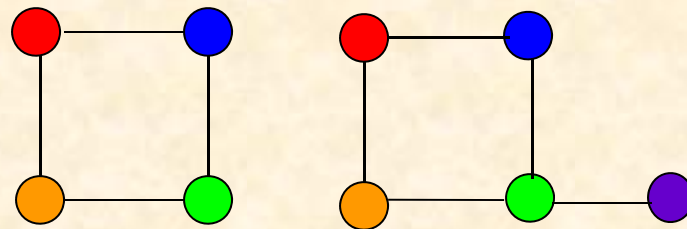


- **Lemma:** *If every node in a graph has degree $r \geq 2$ then this graph contains a circuit (loop).*

- **Proof.**

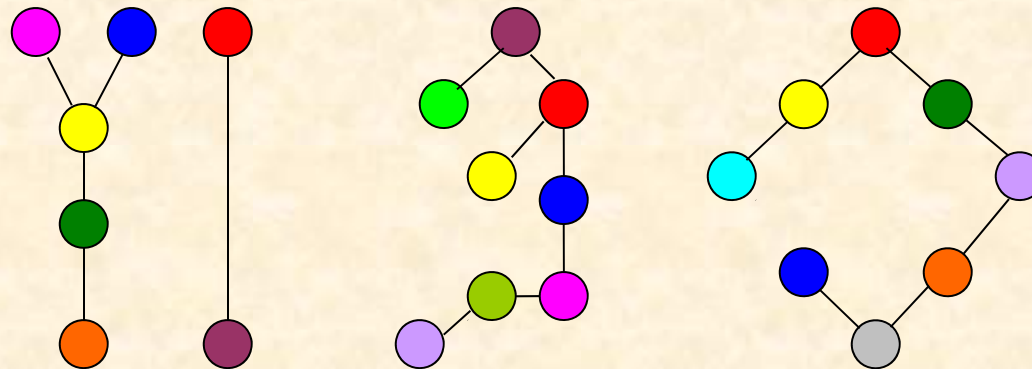
Consider a simple graph. Starting from any node v_0 , construct a walk $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots$ in such a way that v_1 is any adjacent node of v_0 and, for $i = 1, 2, \dots$, node v_{i+1} is any (except v_{i-1}) adjacent node of v_i . Since every node has degree $r \geq 2$, such a node v_{i+1} exists. Because the graph has finitely many nodes, the walk eventually connects to a node that has been chosen before. This walk yields a circuit in the graph.

The converse may not be true:



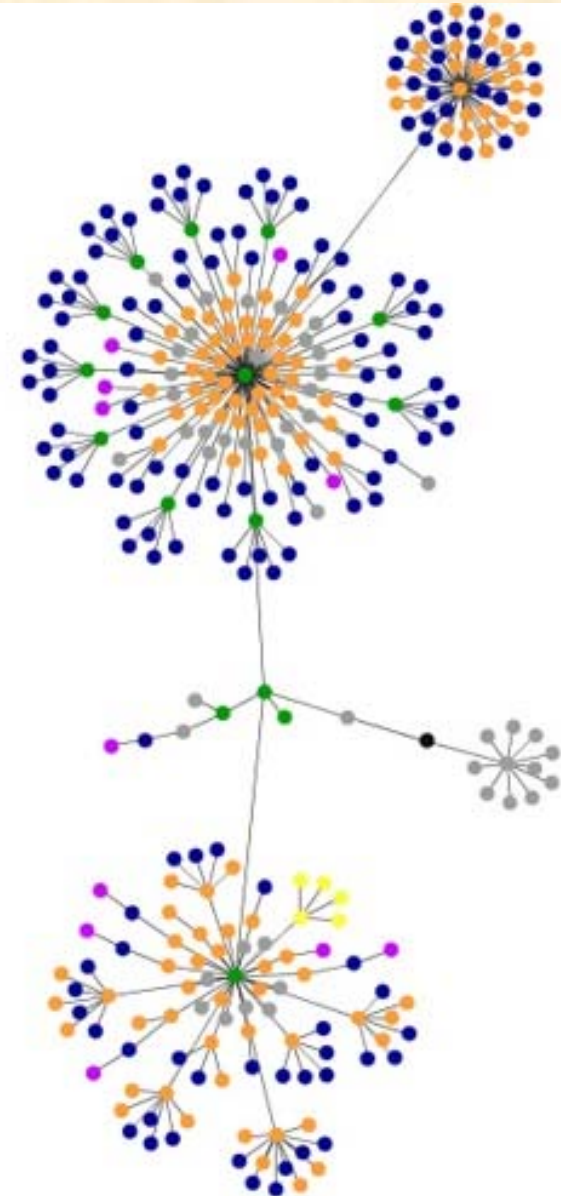
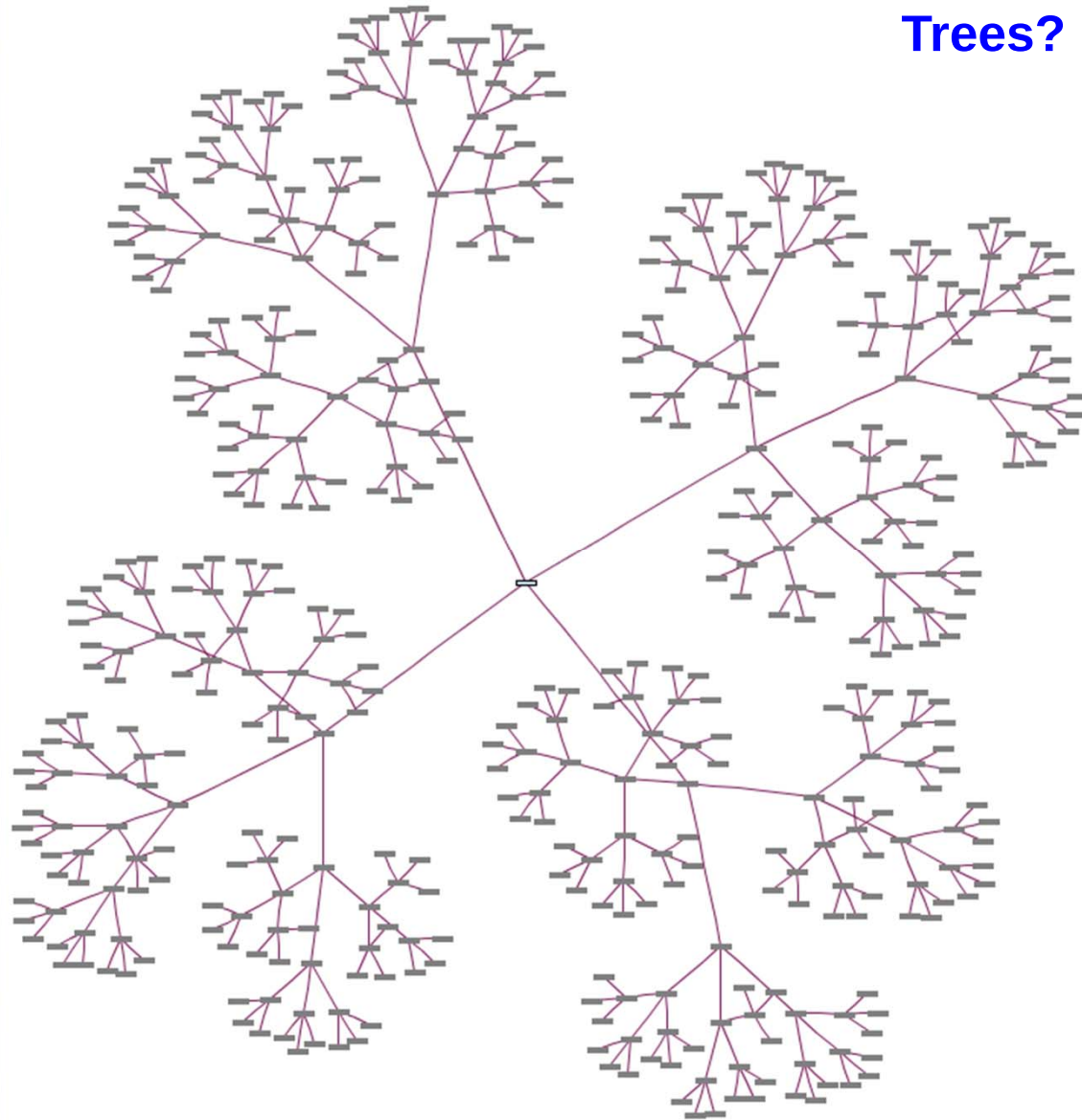
Trees

- **Tree:** A connected graph without circuits
- **Forest:** A family of unconnected trees



- A tree with N nodes has $N - 1$ edges
- Sum of node degrees in a tree
 $= 2 \times (\text{number of edges}) = 2 (N - 1)$

Trees?



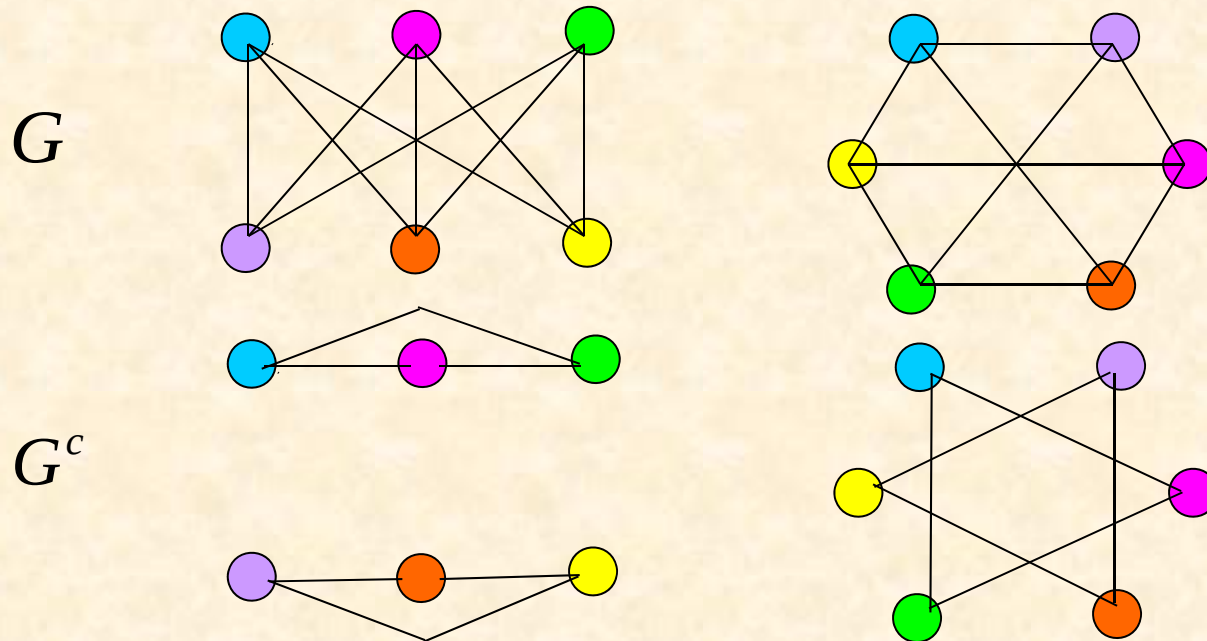
Some Basic Results

Theorem: Let T be a graph with N nodes. Then, the following statements are equivalent:

- 1) T is a tree;
- 2) T has $N - 1$ edges but contains no circuits;
- 3) T has $N - 1$ edges and is connected;
- 4) T is connected, but the removal of any edge will disconnect the graph;
- 5) every pair of nodes of T are connected by exactly one path;
- 6) T contains no circuits, but the addition of any new edge creates exactly one circuit.

Complementary Graph

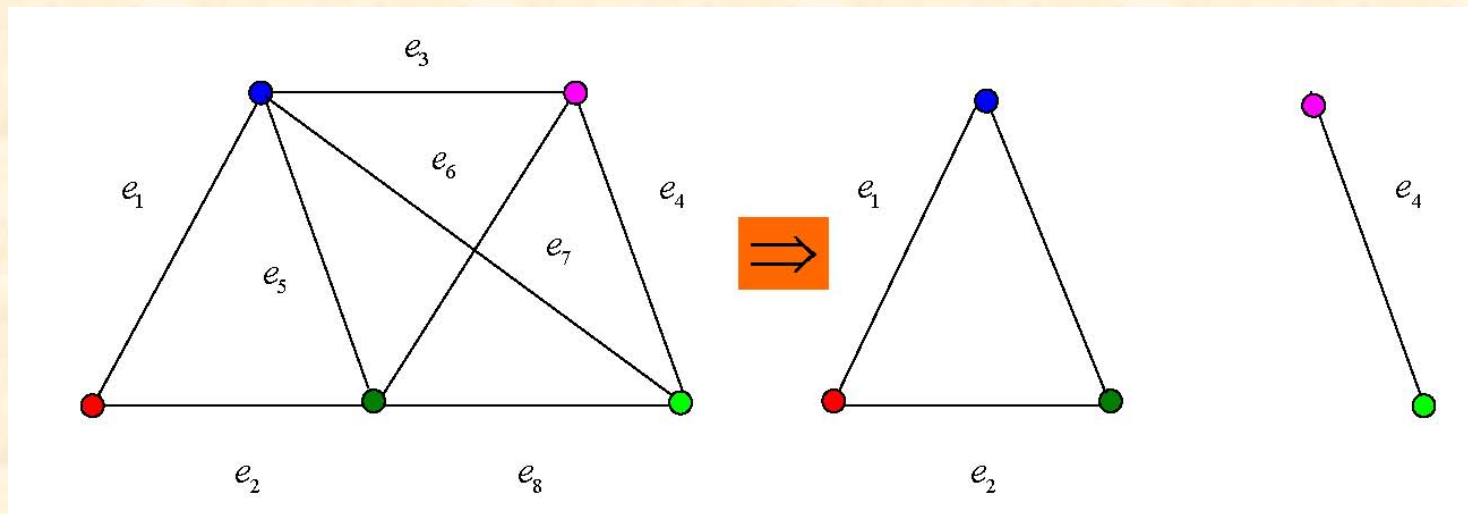
For a given graph G , its *complementary graph* G^c is the graph containing all the nodes of G and all the edges that are not in G



Graph Connectivity

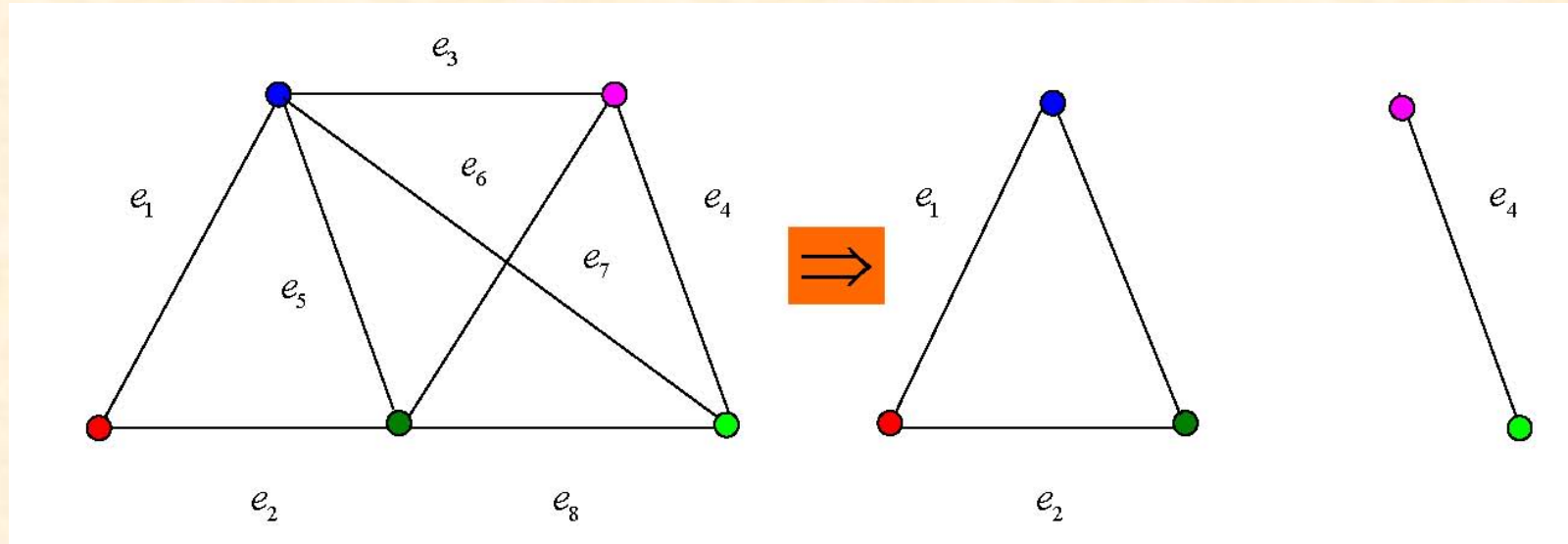
- **Q:** How many edges or nodes must be removed in order to disconnect an originally connected graph?
- **Note:** If a node is removed, then all edges joining it will also be removed; but the converse is not so.

Example:



Disconnecting Sets and Cut-Sets

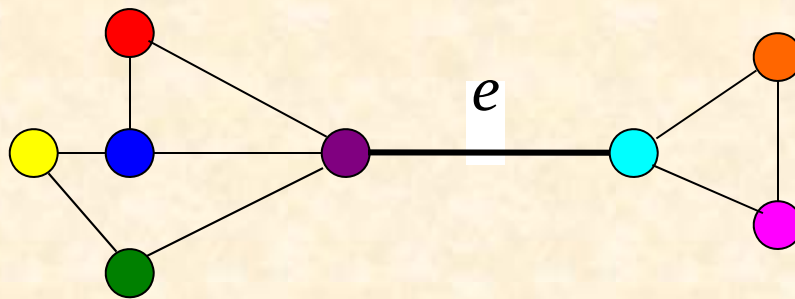
- **Disconnecting set:** A set of edges, $E_0(G)$, after it is being removed, the graph G will become unconnected.
- **Cut-Set:** The smallest disconnecting set, i.e., no proper subset of which is a disconnecting set.
- **Example:** $E_0^1(G) = \{e_1, e_2\}$ $E_0^2(G) = \{e_1, e_2, e_5\}$ $E_0^3(G) = \{e_3, e_6, e_7, e_8\}$
are disconnecting sets, in which both $E_0^1(G)$ and $E_0^3(G)$ are cut-sets.



Bridges

Bridge: A cut-set with only one edge

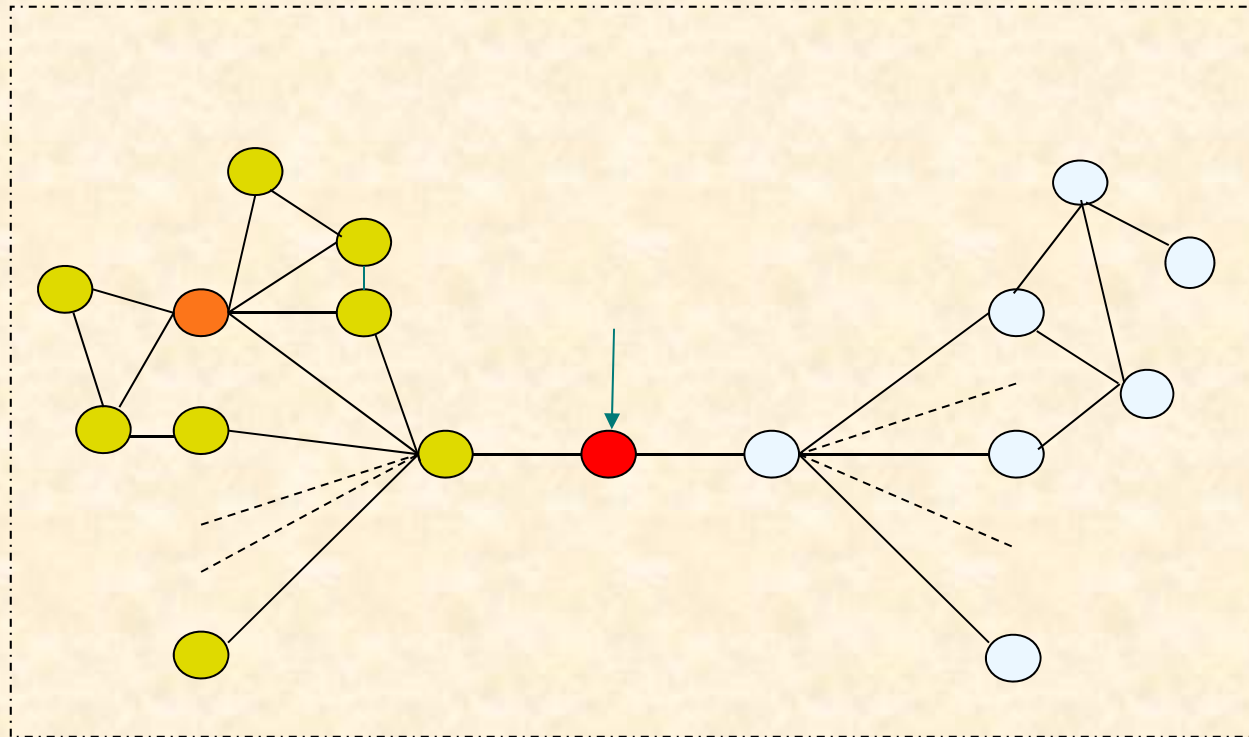
Example: cut-set $\{e\}$ below is a bridge



Bridge is also called Edge Connectivity

Importance of bridges

In a network, a node of low degree may be more important than a node of high degree. For example, on a **bridge**:

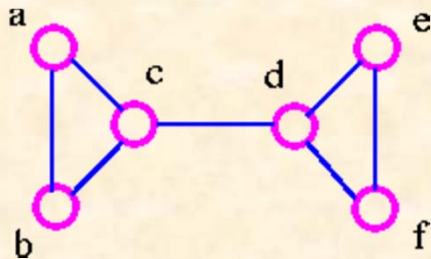


Node Connectivity

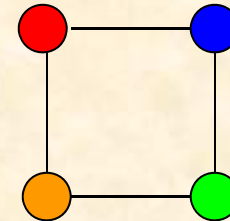
Node Connectivity, $NC(G)$, of a connected graph G is the minimum number of nodes whose removal disconnects G

When $NC(G) \geq k$, the graph is said to be k -connected

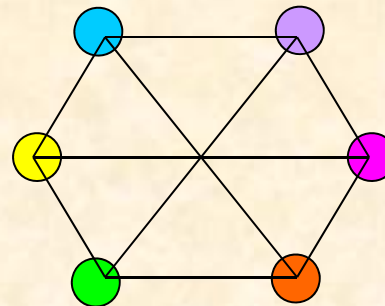
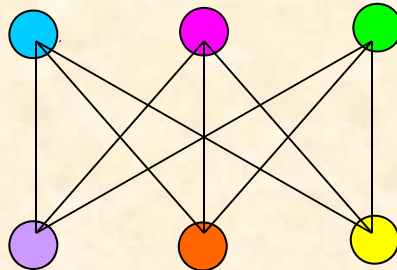
$NC(G) = 1$



$NC(G) = 2$

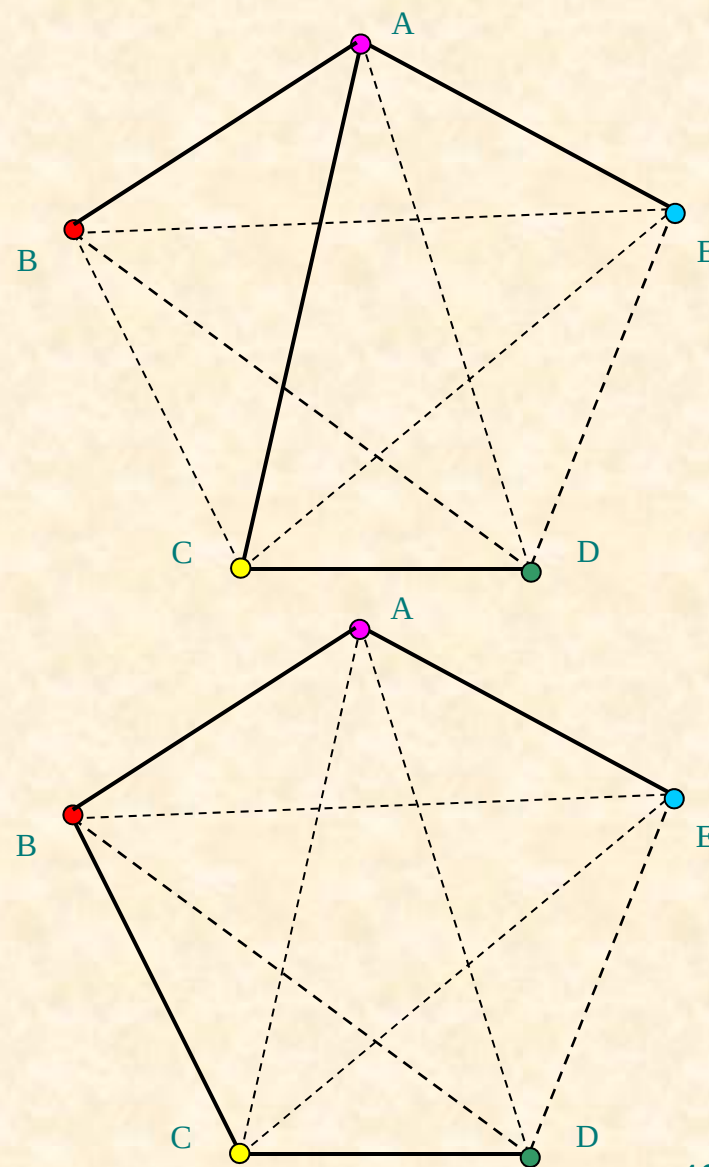


$NC(G) = ?$



More about Trees

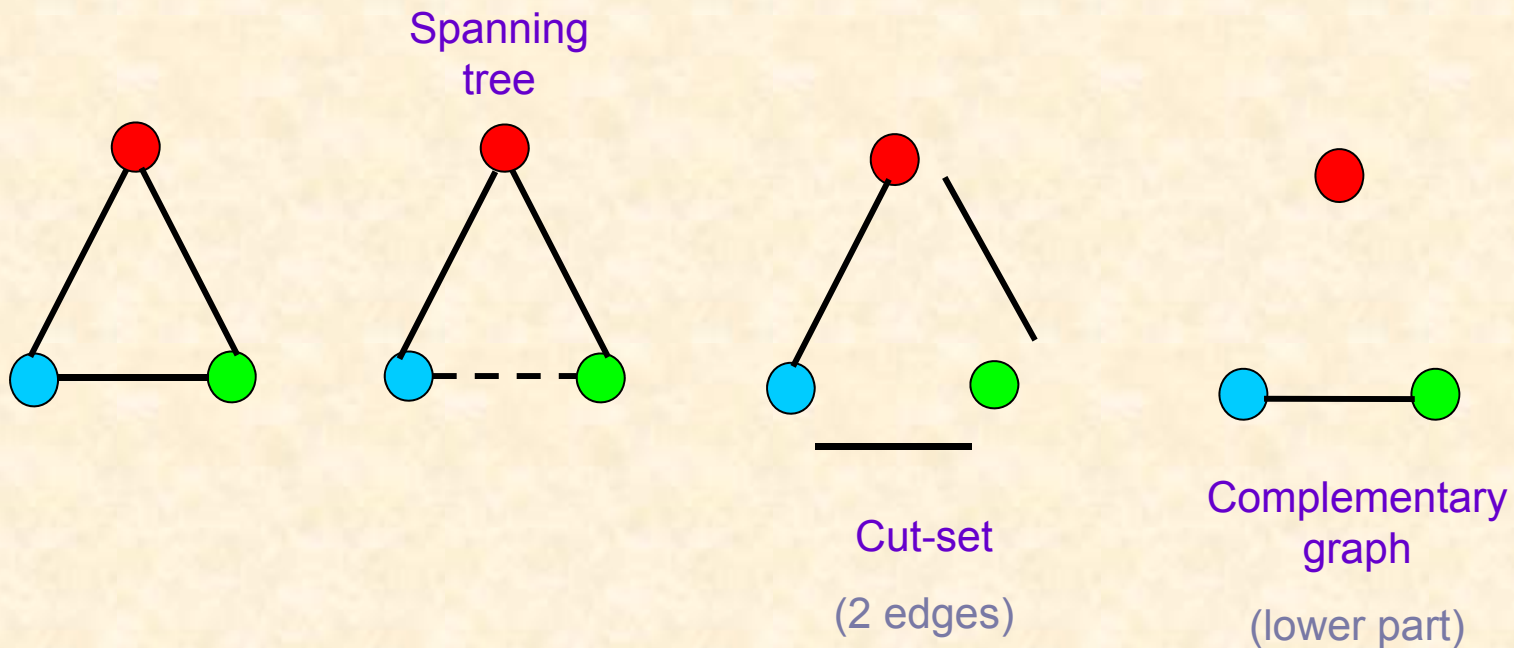
- Starting from a given graph G , if it has a circuit, then remove one edge from the circuit (Clearly, the resulting graph remains to be connected.)
- Repeat this procedure until no circuits are left out
- The final resulting graph is a **tree**. This tree is called a **spanning tree** of graph G
- Spanning tree usually is not unique



Some Results

Theorem: Let T be any spanning tree of graph G . Then

- 1) every cut-set of G has an edge in common with T
- 2) every circuit of G has an edge in common with the complementary graph of T



Eigenvalues of Spanning Trees

Let L be the Laplacian matrix of a connected graph G of size N , with eigenvalues

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$$

Let $L(i, j)$ be the matrix obtained from L by deleting its i row and j column. Then, the number n of spanning trees in G satisfies

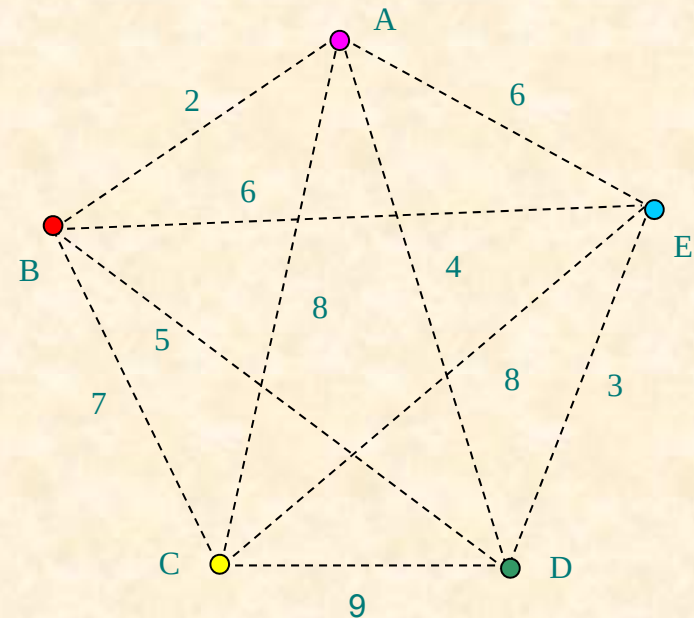
$$n = \left| \det[L(i, j)] \right| = \frac{\lambda_2 \lambda_3 \dots \lambda_N}{N}$$

Example:


$$L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \lambda_1 = 0 \quad \lambda_2 = 2 \quad n = 1$$

Minimum Connector Problem

- **Minimum connector problem:** Suppose that one wants to build a highway network connecting N given cities, in such a way that a car can travel from any city to any other city, but the total mileage of the highways is minimum.
- Clearly, the graph formed by taking the N cities as nodes and the connecting highways as edges must be a tree.
- The problem is to find an efficient algorithm to decide which tree connecting these cities reaches the minimum total mileage.

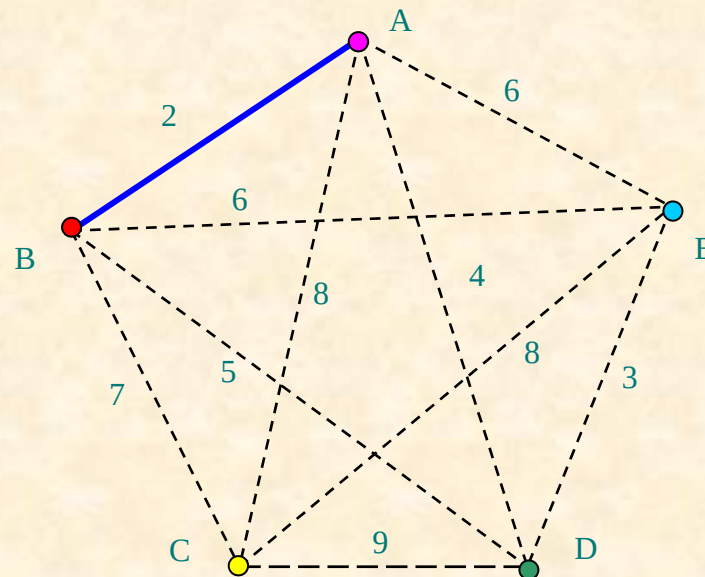


Greedy Algorithm

- **Theorem (Kruskal Greedy Algorithm)** *Let G be a connected graph with N nodes. Then, the following constructive scheme yields a solution to the minimum connector problem:*
- *let e_1 be an edge of G with the smallest weight;*
- *choose $e_2, \dots, e_{N-1}$ one by one, by choosing an edge e_i (not previously chosen) with a smallest weight, subject to the condition that it forms no circuit with all the previous edges $\{e_1, \dots, e_{i-1}\}$;*
- *repeat this procedure until no more edge can be chosen*
- *the resulting graph is a spanning tree, i.e., the subgraph of G with edges $\{e_1, \dots, e_{N-1}\}$*

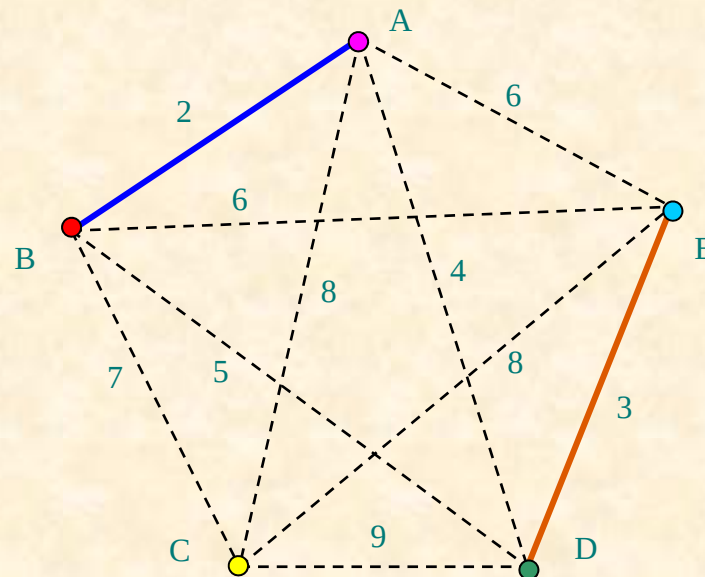
Example

Apply the greedy algorithm →



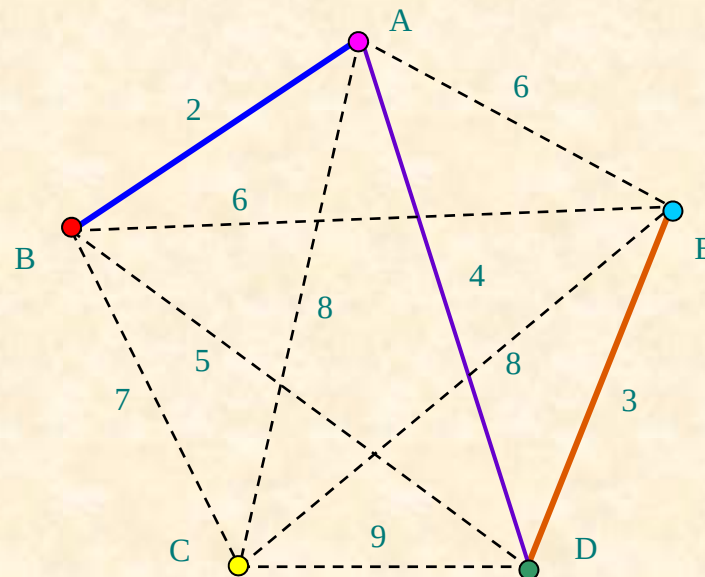
Example

Continue the greedy algorithm →



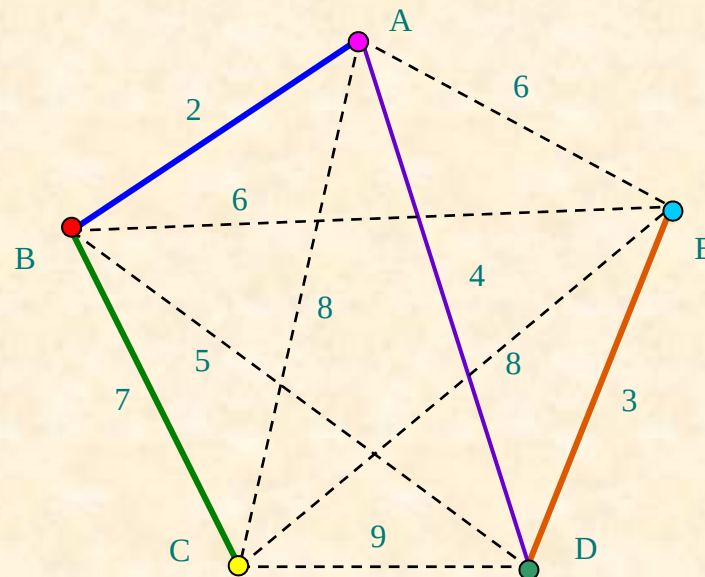
Example

Continue the greedy algorithm →



Result of the Example

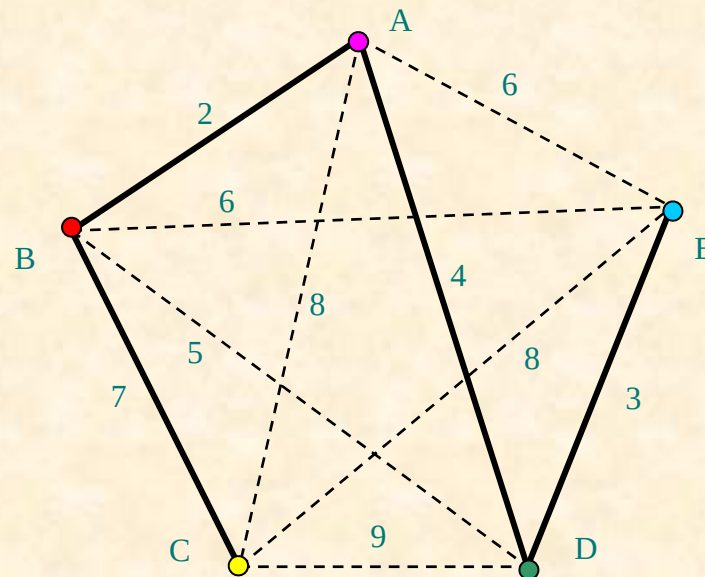
Continue the greedy algorithm →



Because $BD = 5$ or
 $AE = 6$ or $BE = 6$
will form a circuit

Result of the Example

The greedy algorithm finally yields:



No more edge
can be added

End

