

Write your answers on **paper** using the **following format**:

Answer to Question X

Scan your answers **or** take **photos** of them.

Submit **scanned** answers **or** their **photos** in a **single file** to Canvas **before 4:40 pm**.

Name your file in following format: **Quiz2_SID_LastName_FirstName**

Part 1: Short questions:

40 marks

Write down all the steps in your calculations.

Q1) Compute the first derivative of $f(x) = x^3 + \log(x + 1) - 15$ at $x_i = 1$ with $h = 0.1$ using the low-accuracy forward approximation.

Answer for natural logarithm: 3.7979

Answer for logarithm base 10: 3.5219

Q2) Compute the absolute percentage of the relative true error of your answer in Q1.

natural logarithm $f'(x_i) = 3.5$

logarithm base 10: $f'(x_i) = 3.2171$

Answer for natural logarithm: Error= 8.5115

Answer for logarithm base 10: Error 9.4725

Q3) Compute the first derivative of $f(x) = x^3 + \log(x + 1) - 15$ at $x_i = 1$ with $h = 0.1$ using the high-accuracy forward approximation.

Answer for natural logarithm: 3.4793

Answer for logarithm base 10: 3.1968

Q4) Compute the absolute percentage of the relative true error of your answer in Q3.

Answer for natural logarithm: Error: 0.5928

Answer for logarithm base 10: Error: 0.6318

Q5) Compute the second derivative of $f(x) = x^3 + \log(x + 1) - 15$ at $x_i = 1$ with $h = 0.1$ using the low-accuracy forward approximation.

Answer for natural logarithm: 6.373

Answer for logarithm base 10: 6.5014

Q6) Compute the absolute percentage of the relative true error of your answer in Q5.

Natural logarithm $f''(x_i) = 5.75$

Logarithm in base 10 $f''(x_i) = 5.8914$

Answer for natural logarithm: Error: 10.8345

Answer for logarithm base 10: Error: 10.3537

Q7) Compute the second derivative of $f(x) = x^3 + \log(x + 1) - 15$ at $x_i = 1$ with $h = 0.1$ using the high-accuracy forward approximation.

Answer for natural logarithm: 5.7528

Answer for logarithm base 10: 5.8926

Q8) Compute the absolute percentage of the relative true error of your answer in Q7.

Answer for natural logarithm: Error: 0.0486

Answer for logarithm base 10: Error: 0.0206

Q9) Consider the function $f(x) = x^3 - \log(2x + 1) - x$. Suppose that we would like to find the root of $f(x)$ using the false-position method. Write down the formula for computing x_r for $f(x)$.

Answer:
$$x_r = x_u - \frac{(x_u^3 - \log(2x_u + 1) - x_u)(x_l - x_u)}{x_l^3 - \log(2x_l + 1) - x_l - (x_u^3 - \log(2x_u + 1) - x_u)}$$

Q10) Consider the function $f(x) = (5 - x^2)e^{x^2} - 5$. Suppose that we would like to find the root of $f(x)$ using the Newton-Raphson method. Write down the formula for computing x_{k+1} .

Answer:
$$x_{k+1} = x_k - \frac{(5 - x_k^2)e^{x_k^2} - 5}{-2x_k e^{x_k^2} + 2x_k(5 - x_k^2)e^{x_k^2}}$$

Part 2: Long questions:

Q1)

30 marks

a) Fit Newton's interpolation model to the data in the table below. (18 marks)

x_i	y_i
1	2.2
2	3.22
3	3.89
4	4.39

$$b_0 = 2.2$$

$$b_1 = f[x_1, x_0] = 1.02$$

$$f[x_2, x_1] = 0.67$$

$$f[x_3, x_2] = 0.5$$

$$b_2 = f[x_2, x_1, x_0] = -0.175$$

$$f[x_3, x_2, x_1] = -0.085$$

$$b_3 = f[x_3, x_2, x_1, x_0] = 0.03$$

b) Evaluate your model for the data points in the table below. (3 marks)

x_i	y_i
5	
6	
7	

x_i	y_i
5	4.9
6	5.6
7	6.67

c) The datapoints in Part a are from the function $f(x) = 2\ln(2x + 1)$. Use this information to obtain the percentage of the true relative error for each data point in Part b. (3 marks)

x_i	y_i	Error
5	4.9	2.17
6	5.6	9.16
7	6.67	23.15

d) Fit the Lagrange's interpolation model to the data in the table below. (6 marks)

$$L_0(x) = \frac{(x-2)(x-3)(x-4)}{1-2} \frac{1-3}{1-3} \frac{1-4}{1-4}$$

$$L_1(x) = \frac{(x-1)(x-3)(x-4)}{2-1} \frac{2-3}{2-3} \frac{2-4}{2-4}$$

$$L_2(x) = \frac{(x-1)(x-2)(x-4)}{3-1} \frac{3-2}{3-2} \frac{3-4}{3-4}$$

$$L_3(x) = \frac{(x-1)(x-2)(x-3)}{4-1} \frac{4-2}{4-2} \frac{4-3}{4-3}$$

x_i	y_i
1	2.2
2	3.22
3	3.89
4	4.39

Q2) Table below is obtained using the false-position method for a function $f(x)$. Some values in the table are missing.

Find the missing values using the steps of the false position method.

The last column in the table below is the percentage of the error ($\epsilon_k = \frac{x_r^{new} - x_{k-1}^{old}}{x_k^{new}} \times 100$) (not the absolute percentage of the error).

5 marks

Write down all your steps.

Iteration	x_l	x_r	x_u	$f(x_l)$	$f(x_r)$	$f(x_u)$	ϵ
1	1	1.42	2	-0.5634	Missing	Missing	-
2	Missing	Missing	2	Missing	-0.4628	Missing	19.2207
3	Missing	Missing	2	Missing	-0.089	Missing	Missing
4	Missing	Missing	2	Missing	-0.0143	Missing	Missing

Answer:

Iteration	x_l	x_r	x_u	$f(x_l)$	$f(x_r)$	$f(x_u)$	ϵ
1	1	1.42	2	-0.5634	-1.0858	0.7781	0
2	1.42	1.7579	2	-1.0858	-0.4628	0.7781	19.2207
3	1.7579	1.8482	2	-0.4628	-0.089	0.7781	4.8865
4	1.8482	1.8638	2	-0.089	-0.0143	0.7781	0.8358
5	1.8638	1.8662	2	-0.0143	-0.0022	0.7781	0.1313

First iteration: $f(x_u)$ can be found using $x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$ where the only unknown is $f(x_u)$. Since x_u does not change, $f(x_u)$ remains the same in the rest of the iterations.

Second iteration: x_l in the second iteration is the same as x_r in the first iteration. So, $f(x_l)$ in the second equation is the same as $f(x_r)$ in the first equation. Therefore, the only unknown in the second iteration is x_r , which can be computed using the error value.

The rest of the unknowns can be found similarly.

Q3) Table below is obtained using the Newton-Raphson method for a function $f(x)$. Some values in the table are missing.

Find the missing values using the steps of the Newton-Raphson method.

Note that the last column is the percentage of the error ($\epsilon_k = \frac{x_k - x_{k-1}}{x_k} \times 100$) (not the absolute percentage of the error).

5 marks

	x_k	$f'(x_k)$	$f(x_k)$	ϵ
$k = 0$	6	59	Missing	-
$k = 1$	Missing	31.0721	Missing	-23.3449
$k = 2$	Missing	Missing	2.8675	-12.329
$k = 3$	Missing	18.1725	Missing	-3.3185
$k = 4$	Missing	Missing	0.0008	-0.2254
$k = 5$	Missing	18.0104	0	-0.001

Answer:

	x_k	$f'(x_k)$	$f(x_k)$	ϵ
$k = 0$	6	59	67	-
$k = 1$	4.8644	31.0721	16.5896	-23.3449
$k = 2$	4.3305	20.6157	2.8675	-12.329
$k = 3$	4.1914	18.1725	0.1713	-3.3185
$k = 4$	4.182	18.0111	0.0008	-0.2254
$k = 5$	4.1819	18.0104	0	-0.001

x_1 can be found using the error equation $\epsilon_1 = \frac{x_1 - x_0}{x_0} \times 100$ as the only unknown is x_1 .

$f(x_0)$ can be found using the equation $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ as the only unknown is $f(x_0)$.

The rest of missing values can be found using similar steps.

Q4) Consider the function $f(x) = \cos(ax)$.

20 marks

- a) Suppose that we approximate first derivative of $f(x)$ at x_i using the forward different method. Use Taylor's expansion to drive an expression for the approximation error.
- b) Suppose that we approximate first derivative of $f(x)$ at x_i using the centred different method. Use Taylor's expansion to drive an expression for the approximation error.
- c) Based on your answers in parts a and b, which approximate is more accurate? Why?

Answer:

a) Using Taylor's expansion, we have

$$f(x_{i+1}) = f(x_i) + hf^1(x_i) + \frac{h^2 f^2(x_i)}{2!} + \frac{h^3 f^3(x_i)}{3!} + \dots$$

Thus, we have

$$f^1(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{h f^2(x_i)}{2!} - \frac{h^2 f^3(x_i)}{3!} + \dots$$

Therefore, the error is $\frac{h f^2(x_i)}{2!} + \frac{h^2 f^3(x_i)}{3!} + \dots$

b)

Using Taylor's expansion, we have

$$f(x_{i+1}) = f(x_i) + hf^1(x_i) + \frac{h^2 f^2(x_i)}{2!} + \frac{h^3 f^3(x_i)}{3!} + \dots$$

$$f(x_{i-1}) = f(x_i) - hf^1(x_i) + \frac{h^2 f^2(x_i)}{2!} - \frac{h^3 f^3(x_i)}{3!} + \dots$$

Subtracting the second equation from the first equation, we have

$$f(x_{i+1}) - f(x_{i-1}) = 2hf^1(x_i) + 2\frac{h^3 f^3(x_i)}{3!} + 2\frac{h^5 f^5(x_i)}{5!}$$

Thus, we have

$$f^1(x_i) = \frac{f(x_{i+1}) - f(x_{i-1}))}{2h} - \frac{h^2 f^3(x_i)}{3!} - \frac{h^4 f^5(x_i)}{5!} + \dots$$

Thus, the error is $\frac{h^2 f^3(x_i)}{3!} + \frac{h^4 f^5(x_i)}{5!} + \dots$

c) The centred approximation is more accurate because the errors terms with odd orders of h (that is $\frac{h^1 f^2(x_i)}{2!}, \frac{h^3 f^4(x_i)}{4!}, \dots$) will cancel out.