

MA3160: Probability and Stochastic Processes
Homework assignment #2 – solutions

1. We throw two balanced dice simultaneously, and we denote by $X, Y \in \{1, 2, 3, 4, 5, 6\}$ the numbers that appear. These two random variables X and Y are assumed to be independent.

- (i) Compute $\mathbb{E}[X + Y]$, and then $\text{Var}(X + Y)$.

We have

$$\mathbb{E}[X] = \mathbb{E}[Y] = \sum_{i=1}^6 \frac{1}{6} \cdot i = \frac{7}{2}.$$

Hence, using linearity of expectation,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = \frac{7}{2} + \frac{7}{2} = 7.$$

We also have

$$\mathbb{E}[X^2] = \mathbb{E}[Y^2] = \sum_{i=1}^6 \frac{1}{6} \cdot i^2 = \frac{91}{6}.$$

Hence,

$$\text{Var}(X) = \text{Var}(Y) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

Since X and Y are independent,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) = \frac{35}{12} + \frac{35}{12} = \frac{35}{6}.$$

- (ii) We now repeat this random experiment 300 times: the numbers obtained are denoted by $(X_1, Y_1), \dots, (X_{300}, Y_{300})$ (all these random variables are still assumed to be independent). Let $S = \sum_{i=1}^{300} (X_i + Y_i)$ be the sum of all the numbers obtained: compute $\mathbb{E}[S]$ and $\text{Var}(S)$.

By linearity,

$$\mathbb{E}[S] = \mathbb{E}\left[\sum_{i=1}^{300} (X_i + Y_i)\right] = \sum_{i=1}^{300} \mathbb{E}[X_i + Y_i] = 300 \cdot \mathbb{E}[X_1 + Y_1] = 300 \cdot 7 = 2100.$$

Using independence of successive throws,

$$\text{Var}(S) = \text{Var}\left(\sum_{i=1}^{300} (X_i + Y_i)\right) = \sum_{i=1}^{300} \text{Var}(X_i + Y_i) = 300 \cdot \text{Var}(X_1 + Y_1) = 300 \cdot \frac{35}{6} = 1750.$$

- (iii) Using the Central Limit Theorem, give an approximation of $\mathbb{P}(S \geq 2200)$ in terms of the cumulative distribution function $\phi(\cdot)$ of a random variable with standard normal distribution (i.e. $\mathcal{N}(0, 1)$).

Let us rewrite $\mathbb{P}(S \geq 2200)$ in order to be able to use the Central Limit Theorem, using the values for $\mathbb{E}[S]$ and $\text{Var}(S)$ that we have found.

$$\mathbb{P}(S \geq 2200) = \mathbb{P}\left(\frac{S - \mathbb{E}[S]}{\sqrt{\text{Var}(S)}} \geq \frac{2200 - \mathbb{E}[S]}{\sqrt{\text{Var}(S)}}\right) \simeq \mathbb{P}\left(\frac{S - \mathbb{E}[S]}{\sqrt{\text{Var}(S)}} \geq 2.39\right) \simeq \phi(-2.39),$$

where the last equality follows from the Central Limit Theorem.

2. The arrival times of customers in a travel agency can be modeled by a Poisson process $(N_t)_{t \geq 0}$ with some rate $\lambda > 0$. The average time between two successive customers is 40 minutes.

- (i) Find the value of λ .

We know that in a Poisson process with rate λ , the interarrival times have exponential distribution with parameter λ , so they have expectation $\frac{1}{\lambda}$. Here, this implies that $\lambda = 1.5$ customers per hour.

- (ii) Evaluate the following probabilities:

- (a) the probability that the first customer arrives within 1 hour,

This probability is equal to

$$\mathbb{P}(N_1 \geq 1) = 1 - \mathbb{P}(N_1 = 0) = 1 - e^{-\lambda} = 1 - e^{-1.5}$$

(since N_1 has Poisson distribution with parameter $\lambda = 1.5$).

- (b) the probability that after 3 hours, at least 5 customers have arrived,

This probability is equal to

$$\mathbb{P}(N_3 \geq 5) = 1 - \mathbb{P}(N_3 \leq 4)$$

and we have

$$\begin{aligned} \mathbb{P}(N_3 \leq 4) &= \mathbb{P}(N_3 = 0) + \mathbb{P}(N_3 = 1) + \mathbb{P}(N_3 = 2) + \mathbb{P}(N_3 = 3) + \mathbb{P}(N_3 = 4) \\ &= e^{-4.5} + e^{-4.5} \cdot \frac{4.5}{1!} + e^{-4.5} \cdot \frac{(4.5)^2}{2!} + e^{-4.5} \cdot \frac{(4.5)^3}{3!} + e^{-4.5} \cdot \frac{(4.5)^4}{4!} \end{aligned}$$

(since N_3 has Poisson distribution with parameter $3\lambda = 4.5$).

- (c) the probability that no customer arrives in the first 2 hours, but at least 3 customers arrive in the first 4 hours.

This probability is

$$\mathbb{P}(N_2 = 0, N_4 \geq 3) = \mathbb{P}(N_2 = 0, N_4 - N_2 \geq 3) = \mathbb{P}(N_2 = 0) \cdot \mathbb{P}(N_4 - N_2 \geq 3)$$

(using independence of increments). On the one hand, N_2 is Poisson($2\lambda = 3$) so

$$\mathbb{P}(N_2 = 0) = e^{-2\lambda} = e^{-3}.$$

On the other hand, $N_4 - N_2$ is also Poisson($2\lambda = 3$), which gives

$$\mathbb{P}(N_4 - N_2 \geq 3) = 1 - \mathbb{P}(N_4 - N_2 \leq 2) = 1 - \left(e^{-3} + e^{-3} \cdot \frac{3}{1!} + e^{-3} \cdot \frac{3^2}{2!} \right).$$

(iii) For $t \geq 0$, compute $\text{Cov}(N_{2t}, N_{t+4})$.

We distinguish the two cases:

- If $t \leq 4$, then $2t \leq t + 4$, so

$$\text{Cov}(N_{2t}, N_{t+4}) = \text{Cov}(N_{2t}, N_{2t}) + \text{Cov}(N_{2t}, N_{t+4} - N_{2t}) = \text{Var}(N_{2t}) + 0$$

($\text{Cov}(N_{2t}, N_{t+4} - N_{2t}) = 0$ since the increments N_{2t} and $N_{t+4} - N_{2t}$ are independent).
Hence,

$$\text{Cov}(N_{2t}, N_{t+4}) = 2t \cdot \lambda = 3t.$$

- If $t > 4$, then $2t > t + 4$, so

$$\text{Cov}(N_{2t}, N_{t+4}) = \text{Cov}(N_{t+4}, N_{t+4}) + \text{Cov}(N_{2t} - N_{t+4}, N_{t+4}) = \text{Var}(N_{t+4}) + 0$$

(for similar reasons as before, $\text{Cov}(N_{2t} - N_{t+4}, N_{t+4}) = 0$). We deduce

$$\text{Cov}(N_{2t}, N_{t+4}) = (t + 4) \cdot \lambda = (1.5) \cdot (t + 4).$$

(iv) We now assume that each customer decides to book a trip to Europe with probability 0.3, independently of the other customers, and we denote by $(N'_t)_{t \geq 0}$ the corresponding counting process (i.e. of customers who decide to visit Europe). What is the distribution of $(N'_t)_{t \geq 0}$?

We use the theorem seen in class about “decomposing” Poisson processes (here, “type I” would mean “booking a trip to Europe”, and the corresponding probability is $p_I = 0.3$). This result implies that $(N'_t)_{t \geq 0}$ is a Poisson process, with rate $\lambda' = \lambda \cdot (0.3)$.

3. Consider the example known as gambler’s ruin, discussed in class, with $N = 3$: this is a Markov chain with state space $E = \{0, 1, 2, 3\}$, and at each step, the gambler has a probability $p \in (0, 1)$ to win 1, and $1 - p$ to lose 1, until his fortune reaches 0 or 3.

(i) Write explicitly all the transition probabilities $(p_{x,y})_{x,y \in E}$.

We have $p_{0,0} = p_{3,3} = 1$, $p_{1,2} = p_{2,3} = p$, and $p_{1,0} = p_{2,1} = 1 - p$. All other transition probabilities $p_{x,y}$ ($x, y \in E$) are equal to 0. Note that of course, we could also answer this question by writing explicitly the (4 by 4) transition matrix.

- (ii) *For the Markov Chain started at 1, i.e. with initial distribution δ_1 , compute $\mathbb{P}_{\delta_1}(X_n = 1)$ for all $n \geq 0$.*

We observe that there is only one possible “path” for the chain to come back to 1 in n steps: it has to “go back and forth” between 1 and 2 (since as soon as the chain hits 0 or 3, it gets “stuck” there, so it cannot go back to 1). Hence, n needs to be even, and if $n = 2k$,

$$\mathbb{P}_{\delta_1}(X_{2k} = 1) = p_{1,2} \cdot p_{2,1} \cdot \dots \cdot p_{1,2} \cdot p_{2,1} = (p(1-p))^k.$$

If n is odd, $\mathbb{P}_{\delta_1}(X_n = 1) = 0$.

- (iii) *Compute $\mathbb{P}_{\delta_1}(X_n = 1, X_{n+1} = 0)$ for all $n \geq 0$.*

We have

$$\mathbb{P}_{\delta_1}(X_n = 1, X_{n+1} = 0) = \mathbb{P}_{\delta_1}(X_{n+1} = 0 | X_n = 1) \cdot \mathbb{P}_{\delta_1}(X_n = 1).$$

On the one hand, this probability is equal to 0 if n is odd (from (ii)). On the other hand, if n is even, i.e. of the form $n = 2k$,

$$\begin{aligned} \mathbb{P}_{\delta_1}(X_{2k} = 1, X_{2k+1} = 0) &= \mathbb{P}_{\delta_1}(X_{2k+1} = 0 | X_{2k} = 1) \cdot \mathbb{P}_{\delta_1}(X_{2k} = 1) \\ &= p_{1,0} \cdot (p(1-p))^k \\ &= (1-p) \cdot (p(1-p))^k. \end{aligned}$$

- (iv) *Compute the probability that the gambler ultimately gets ruined, i.e. that there exists $n_0 \geq 0$ such that $X_{n_0} = 0$ (and so $X_n = 0$ for all $n \geq n_0$).*

In this question, we still assume that the gambler starts at 1 (a similar reasoning could be followed when the gambler starts at 2).

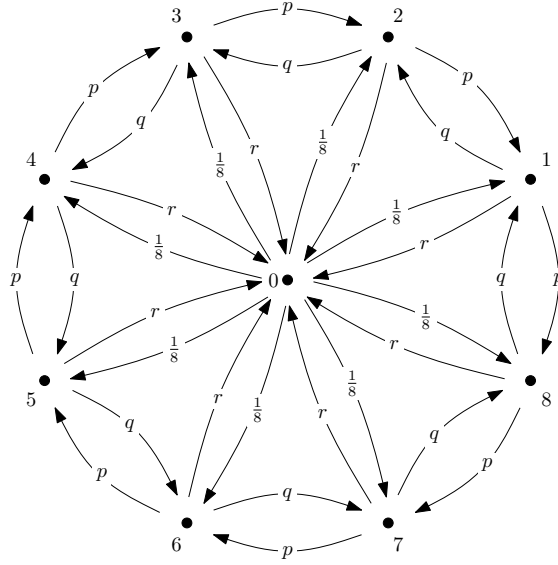
We decompose the event $\{\exists m \geq 0 : X_m = 0\}$ according to the first time τ at which the chain hits 0 (i.e. at which the gambler gets ruined). Note that for the chain starting in 1, $\{\tau = n\} = \{X_n = 0 \text{ and } X_{n-1} = 1\}$, so:

$$\begin{aligned} \mathbb{P}_{\delta_1}(\exists m \geq 0 : X_m = 0) &= \sum_{n=0}^{\infty} \mathbb{P}_{\delta_1}(\tau = n) \\ &= \sum_{n=1}^{\infty} \mathbb{P}_{\delta_1}(X_n = 0, X_{n-1} = 1) \\ &= \sum_{k=0}^{\infty} \mathbb{P}_{\delta_1}(X_{2k} = 1, X_{2k+1} = 0) \\ &= \sum_{k=0}^{\infty} (1-p) \cdot (p(1-p))^k \\ &= \frac{1-p}{1-p(1-p)}, \end{aligned}$$

where we used the previous question for the third and fourth equalities (we also used the usual formula for the sum of a geometric series).

4. We consider the Markov chain $(X_n)_{n \geq 0}$ on the “network” depicted below: there is one central node (denoted 0), and eight external nodes (denoted $1, \dots, 8$). We denote the state space by $E = \{0, 1, \dots, 8\}$. For some given $p, q, r \in (0, 1)$ with $p + q + r = 1$, we adopt the following transition rules (see the figure below).

- When the chain is at the central node, it moves to one of the external nodes chosen uniformly at random (so with probability $\frac{1}{8}$ each).
- When the chain is at one of the external nodes, it moves to one of its neighboring external nodes, clockwise (with probability p) or counter-clockwise (with probability q), or to the central node (with probability r).



It starts at the central node, i.e. the initial distribution is δ_0 .

- (i) Write explicitly the transition probabilities $(p_{x,y})_{x,y \in E}$. Is this chain irreducible? aperiodic?

We have $p_{0,i} = \frac{1}{8}$ and $p_{i,0} = r$ ($1 \leq i \leq 8$), $p_{1,2} = p_{2,3} = \dots = p_{7,8} = p_{8,1} = q$, and $p_{2,1} = p_{3,2} = \dots = p_{8,7} = p_{1,8} = p$. All other transition probabilities are equal to 0 (again, we could also write explicitly the transition matrix).

This Markov chain is irreducible, since any two states $x, y \in E$ communicate.

This chain is also aperiodic. Indeed, for every $x \in E$, we have: starting from x , it is possible to come back to x in 2 or 3 steps, so the period of x is 1 (since $\gcd(2, 3) = 1$).

- (ii) How many invariant distributions does this Markov chain possess? Find such an invariant distribution π .

Since this Markov chain is irreducible, and E is finite, it possesses only one invariant distribution π (using the theorem seen in class).

Using the symmetry, we look for such a π of the form $\pi(0) = a$ and $\pi(i) = b$ ($i = 1, \dots, 8$), for some $a, b \in (0, 1)$ that we need to determine. It follows from the relation “ $\pi = \pi \cdot P$ ” that we should have $a = 8r \cdot b$. Hence, we find

$$\pi(0) = \frac{r}{1+r} \quad \text{and} \quad \pi(i) = \frac{1}{8(1+r)} \quad (i = 1, \dots, 8).$$

(iii) *Is the distribution found in (ii) reversible?*

We need to check whether the invariant distribution π found in (ii) satisfies the detailed-balance condition. From this relation, we find that π is reversible if and only if $p = q$ (and the value of r does not matter).