

Test 1 Solution

9 Oct 2023: 6:00pm~6:45pm (YEUNG LT-15)

- As a closed-book test, you are not allowed to use any written materials online/offline.
- Write your answers on the question papers directly.
- Only approved calculators are allowed. Laptops/phones must be switched off.
- Unless otherwise specified, SI units are assumed in the questions and 3 significant figures should be used in numerical answers.
- The total score is 100. There are a total of **5 pages**.

The following formulas may or may not be useful.

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \mathbf{E}(x, y, z) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{a}}_R & d\mathbf{B} &= \frac{\mu_0}{4\pi} \frac{Id\mathbf{L} \times \hat{\mathbf{a}}_R}{R^2} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} & \oint_S \mathbf{D} \cdot d\mathbf{S} &= Q & \oint_L \mathbf{H} \cdot d\mathbf{L} &= I \\ \nabla \cdot \mathbf{D} &= \rho_v & \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon_r \mathbf{E} & \mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 \mu_r \mathbf{H} \\ \nabla \cdot \mathbf{B} &= 0 & \epsilon_0 &= 8.854 \times 10^{-12} \text{ (F/m)} & \mu_0 &= 4\pi \times 10^{-7} \text{ (H/m)}\end{aligned}$$

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \hat{\mathbf{a}}_\rho \quad \mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{\mathbf{a}}_r + \sin\theta \hat{\mathbf{a}}_\theta) \quad \mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\mathbf{a}}_\phi \quad \mathbf{B} = \mu_0 n I \hat{\mathbf{a}}_z$$

$$\begin{aligned}V_{AB} &= -\int_B^A \mathbf{E} \cdot d\mathbf{L} & \nabla^2 V &= -\frac{\rho_v}{\epsilon} & V &= -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \\ \nabla^2 &= \nabla \cdot \nabla\end{aligned}$$

$$\begin{aligned}\nabla &= \frac{\partial}{\partial x} \hat{\mathbf{a}}_x + \frac{\partial}{\partial y} \hat{\mathbf{a}}_y + \frac{\partial}{\partial z} \hat{\mathbf{a}}_z & \nabla \times \mathbf{F} &= \frac{1}{\rho} \begin{vmatrix} \hat{\mathbf{a}}_\rho & \rho \hat{\mathbf{a}}_\phi & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} \\ \nabla f &= \frac{\partial f}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial f}{\partial z} \hat{\mathbf{a}}_z & \nabla \times \mathbf{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin\theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin\theta F_\phi \end{vmatrix} \\ \nabla f &= \frac{\partial f}{\partial r} \hat{\mathbf{a}}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\mathbf{a}}_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi & \nabla \times \mathbf{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin\theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin\theta F_\phi \end{vmatrix}\end{aligned}$$

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{\partial}{\partial \phi} F_\phi + \frac{\partial}{\partial z} (\rho F_z) \right] \\ \nabla \cdot \mathbf{F} &= \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} (r^2 \sin\theta F_r) + \frac{\partial}{\partial \theta} (r \sin\theta F_\theta) + \frac{\partial}{\partial \phi} (r F_\phi) \right]\end{aligned}$$

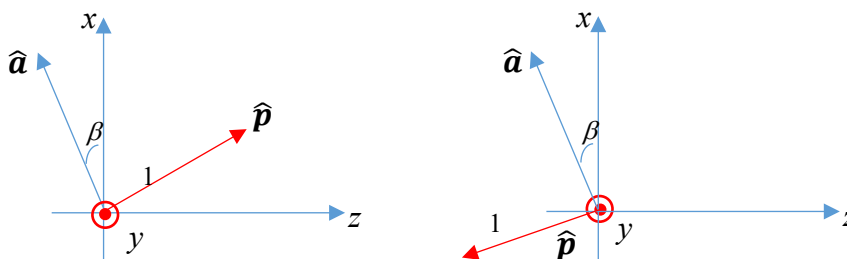
Question 1 (16 points)

Fill in the blanks.

- (a) Light is an ___ electromagnetic ___ wave that is governed by the Maxwell's equations.
- (b) A voltage of 0.47 V is equivalent to ___ 470 ___ (47) ___ mV.
- (c) The dot product between $\hat{x}$ and $-0.5\hat{y}$ is ___ 0 ____.
- (d) For vectors $\mathbf{A} = 2\hat{x} + 2\hat{y}$ and $\mathbf{B} = \hat{y}$, there is an angle of ___ $\frac{\pi}{4}$ ___ or 0.785 ($\frac{3\pi}{4}$)(rad).
- (e) Consider vector $\mathbf{E} = 2\mathbf{C} - \mathbf{D}$ for $\mathbf{C} = \hat{x} + \hat{y}$ and $\mathbf{D} = \hat{y}$, we have $E = \sqrt{5}$ ($\sqrt{10}$)
- (f) A point with rectangular coordinates $(x, y, z) = (1, 1, 1)$ has spherical coordinates of:
- radius $r = \sqrt{3}$ (3)___
 - polar angle $\theta = 0.955$ or 54.7° (48.2 or 0.841)___
 - azimuthal angle $\phi = \frac{\pi}{4}$ or 45° (63.4 or 1.107)___

Question 2 (14 points)

On the xz -plane, there is an angle of $\beta = 20^\circ$ between unit vector $\hat{a}$ and the x -axis.



- (a) Label $\hat{y}$ on the above diagram.
- (b) Express $\hat{a}$ in terms of $\hat{x}$ and $\hat{z}$. Please use 3 significant figures.
- $\hat{a} = 0.940\hat{x} - 0.342\hat{z}$ ($\hat{a} = 0.906\hat{x} - 0.423\hat{z}$)
- (c) Draw $\hat{p} = \hat{a} \times \hat{y}$ on the above diagram accurately.
- (d) Find $\hat{p}$ in terms of $\hat{x}$ and $\hat{z}$. Please use 3 significant figures.

$\hat{p} = 0.342\hat{x} + 0.940\hat{z}$ ($\hat{a} = -0.423\hat{x} - 0.906\hat{z}$)

Question 3 (15 points)

A field is given as:

$$\mathbf{G}(x, y, z) = \exp\left(-\frac{y}{a}\right) \hat{\mathbf{x}} + \frac{xy}{a^2} \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

Find $\text{div } \mathbf{G}$.

Solution:

$$\text{div } \mathbf{G} = \nabla \cdot \mathbf{G}$$

$$= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot \mathbf{G}$$

$$= \frac{\partial}{\partial x} \left(\exp\left(-\frac{y}{a}\right) \right) + \frac{xy}{a^2} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} 1$$

$$= \frac{x}{a^2} \text{ (or } \frac{x}{a^3} \text{ in different versions)}$$

Question 4 (15 points)

A disc of radius $a = 3$ m is placed on the xy -plane. It is centered at the origin.

Expressed in the cylindrical coordinates (ρ, ϕ) , the disc has charges with a density of:

$$K = Q_s \cos \frac{\pi \rho^2}{2a^2} \sin \frac{\phi}{2}$$

where $Q_s = 23 \mu\text{C}/\text{m}^2$. Find the total charge of the disc.

Solution:

$$\text{Differential charge: } dQ = K dA = Q_s \cos \frac{\pi \rho^2}{2a^2} \sin \frac{\phi}{2} \rho d\rho d\phi$$

$$\text{Total charge } Q = \int_{\text{disc}} dQ$$

$$= Q_s \int_{\rho=0}^a \cos \frac{\pi \rho^2}{2a^2} \rho d\rho \int_{\phi=0}^{2\pi} \sin \frac{\phi}{2} d\phi$$

$$= Q_s \frac{a^2}{\pi} \left(\sin \frac{\pi \rho^2}{2a^2} \right) \Big|_0^3 \left(-2 \cos \frac{\phi}{2} \right) \Big|_0^{2\pi}$$

$$= 264 \mu\text{C}$$

$$\text{Differential charge: } dQ = K dA = Q_s \cos \frac{\pi \rho^2}{2a^2} \exp\left(\frac{\phi}{2}\right) \rho d\rho d\phi$$

$$\text{Total charge } Q = \int_{\text{disc}} dQ$$

$$= Q_s \int_{\rho=0}^a \cos \frac{\pi \rho^2}{2a^2} \rho d\rho \int_{\phi=0}^{2\pi} \exp \frac{\phi}{2} d\phi$$

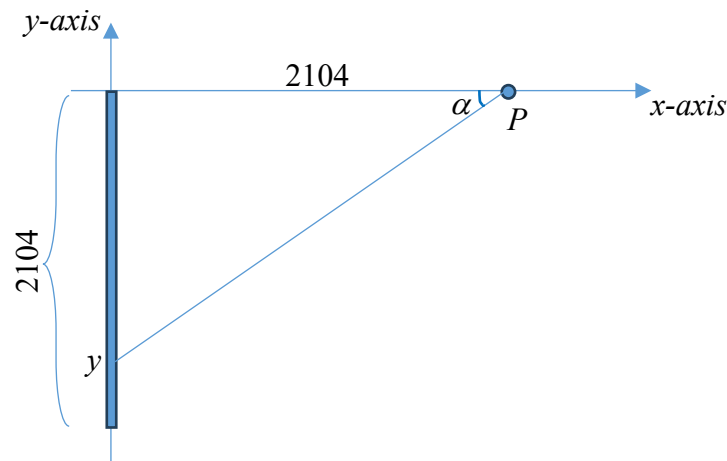
$$= Q_s \frac{a^2}{\pi} \left(\sin \frac{\pi \rho^2}{2a^2} \right) \Big|_0^3 \cdot 2 \exp\left(\frac{\phi}{2}\right) \Big|_0^{2\pi} = -2.92 \text{ mC}$$

Question 5 (40 points)

Consider a line charge placed on the y -axis from $y = -2104$ (meters) to $y = 0$.

It carries a constant charge density of 2023 (pC/m).

Measurement of the electric field is made at point P at $(x, y) = (2104, 0)$ in meters.



(a) On the line charge, what is the amount of differential charge dQ contained in a segment from y to $y + dy$?

$$\underline{\hspace{2cm}} 2023 \times 10^{-12} \, dy \underline{\hspace{2cm}}$$

(b) For the above segment, an angle α is defined in reference to the x -axis (as shown in the diagram). Write down each of the following in terms of α .

- The coordinate y of the segment.

$$\underline{\hspace{2cm}} -2104 \tan \alpha \underline{\hspace{2cm}}$$

- The unit vector $\hat{\mathbf{a}}$ pointing from the segment to P .

$$\underline{\hspace{2cm}} \cos \alpha \hat{\mathbf{x}} + \sin \alpha \hat{\mathbf{y}} \underline{\hspace{2cm}}$$

- The distance L from the segment to P .

$$\underline{\hspace{2cm}} 2104 \sec \alpha \underline{\hspace{2cm}}$$

- The segment length dy .

$$\underline{\hspace{2cm}} -2104 \sec^2 \alpha \, d\alpha \underline{\hspace{2cm}}$$

(c) What is the differential electric field due to the segment dy acting on P ? Simplify the answer by expressing everything in terms of α as much as possible.

Solution:

$$\begin{aligned} \text{Differential field } d\mathbf{E} &= \frac{1}{4\pi\epsilon_0} 2023 \, dy \frac{\hat{\mathbf{a}}}{L^2} \\ &= \frac{2023}{4\pi\epsilon_0} \frac{-2104 \sec^2 \alpha \, d\alpha}{2104^2 \sec^2 \alpha} (\cos \alpha \hat{\mathbf{x}} + \sin \alpha \hat{\mathbf{y}}) \\ &= \frac{2023}{8416\pi\epsilon_0} (-d\alpha) (\cos \alpha \hat{\mathbf{x}} + \sin \alpha \hat{\mathbf{y}}) \end{aligned}$$

(d) Find the magnitude of the electric field at P that is caused by the whole line charge.

Solution:

$$\begin{aligned}\text{Total } \mathbf{E} &= \int_{y=-2104}^0 d\mathbf{E} \\ &= \frac{-2023}{8416\pi\epsilon_0} \int_{45^\circ}^0 (\cos\alpha \hat{x} + \sin\alpha \hat{y}) d\alpha \\ &= \frac{2023}{8416\pi\epsilon_0} [\sin\alpha - \cos\alpha]_0^{45^\circ} \\ &= \frac{2023}{8416\pi\epsilon_0} \left[\frac{1}{\sqrt{2}} \hat{x} - \left(\frac{1}{\sqrt{2}} - 1 \right) \hat{y} \right]\end{aligned}$$

$$\begin{aligned}\text{The magnitude } E &= \frac{2023}{8416\pi\epsilon_0} \sqrt{\frac{1}{2} + \left(\frac{1}{\sqrt{2}} - 1 \right)^2} \\ &= 6.61 \text{mV/m}\end{aligned}$$

~END~