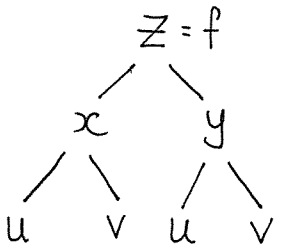
# MA2001 Solution to Take-home Ass. 2

1. $z = f(x,y)$ where $x = uv^2$, $y = u^3 v$

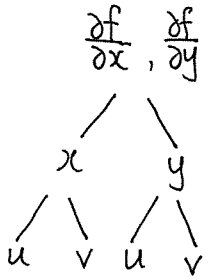



$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$
$$= v^2\frac{\partial f}{\partial x} + 3u^2v\frac{\partial f}{\partial y}$$

When $(u,v) = (1,2)$, $x = 4$ and $y = 2$.

Then $\left.\frac{\partial z}{\partial u}\right|_{(u,v)=(1,2)} = 2^2\cdot 5 + 3\cdot 1^2\cdot 2\cdot 1 = 26$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(v^2\frac{\partial f}{\partial x} + 3u^2v\frac{\partial f}{\partial y}\right)$$




$$= v^2\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) + 3u^2v\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) + 6uv\frac{\partial f}{\partial y}$$
$$= v^2\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial y}{\partial u}\right]$$
$$+ 3u^2v\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial y}{\partial u}\right]$$
$$+ 6uv\frac{\partial f}{\partial y}$$
$$= v^2\left[\frac{\partial^2 f}{\partial x^2}\cdot v^2 + \frac{\partial^2 f}{\partial y\partial x}\cdot 3u^2v\right] + 3u^2v\left[\frac{\partial^2 f}{\partial x\partial y}\cdot v^2 + \frac{\partial^2 f}{\partial y^2}\cdot 3u^2v\right]$$
$$+ 6uv\frac{\partial f}{\partial y}$$

At $(u,v) = (1,2)$,

$$\left.\frac{\partial^2 z}{\partial u^2}\right|_{(u,v)=(1,2)} = 2^2\cdot\left[(-4)\cdot 2^2 + 1\cdot 3\cdot 1^2\cdot 2\right]$$
$$+ 3\cdot 1^2\cdot 2\cdot\left[1\cdot 2^2 + (-2)\cdot 3\cdot 1^2\cdot 2\right]$$
$$+ 6\cdot 1\cdot 2\cdot 1$$
$$= 4\cdot(-10) + 6\cdot(-8) + 12$$
$$= -76$$

2. $2z^3 - 3xz + xy^2 = 3$ —— ⊛ where $z = z(x,y)$.

(a) Partially differentiate both sides of ⊛ w.r.t. $x$:

$$6z^2\frac{\partial z}{\partial x} - 3z - 3x\frac{\partial z}{\partial x} + y^2 = 0 \quad \text{——} \circledast\circledast$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{3z - y^2}{6z^2 - 3x}$$

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{(1,2,1)} = \frac{3-4}{6-3} = -\frac{1}{3}$$

Partially differentiate both sides of ⊛ w.r.t. $y$:

$$6z^2\frac{\partial z}{\partial y} - 3x\frac{\partial z}{\partial y} + 2xy = 0 \quad \text{——} \circledast\circledast\circledast$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{2xy}{3x - 6z^2}$$

$$\therefore \left.\frac{\partial z}{\partial y}\right|_{(1,2,1)} = \frac{2\cdot 2}{3-6} = -\frac{4}{3}$$

$\therefore$ The tangent plane approximation of $z(x,y)$ at $(1,2,1)$ is

$$P_1(x,y) = 1 - \frac{1}{3}(x-1) - \frac{4}{3}(y-2)$$

(b) Partially differentiate both sides of ⊛⊛ w.r.t. $x$:

$$12z\left(\frac{\partial z}{\partial x}\right)^2 + 6z^2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial x} - 3x\frac{\partial^2 z}{\partial x^2} + 0 = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{6\frac{\partial z}{\partial x} - 12z\left(\frac{\partial z}{\partial x}\right)^2}{6z^2 - 3x}$$

$$\therefore \left.\frac{\partial^2 z}{\partial x^2}\right|_{(1,2,1)} = \frac{6\left(-\frac{1}{3}\right) - 12\cdot 1\cdot\left(-\frac{1}{3}\right)^2}{6\cdot 1^2 - 3\cdot 1} = -\frac{10}{9}$$

Partially differentiate both sides of ⊛⊛⊛ w.r.t. $y$:

$$12z\left(\frac{\partial z}{\partial y}\right)^2 + 6z^2\frac{\partial^2 z}{\partial y^2} - 3x\frac{\partial^2 z}{\partial y^2} + 2x = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = \frac{12z\left(\frac{\partial z}{\partial y}\right)^2 + 2x}{3x - 6z^2}$$

$$\therefore \left.\frac{\partial^2 z}{\partial y^2}\right|_{(1,2,1)} = \frac{12\cdot 1\cdot\left(-\frac{4}{3}\right)^2 + 2\cdot 1}{3-6} = -\frac{70}{9}$$

Partially differentiate both sides of (**) w.r.t. $y$:

$$12z\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial x}+6z^2\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)-3\frac{\partial z}{\partial y}-3x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)+2y=0$$

$$\Rightarrow \frac{\partial^2 z}{\partial y\partial x}=\frac{-12z\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right)+3\frac{\partial z}{\partial y}-2y}{6z^2-3x}$$

$$\therefore \left.\frac{\partial^2 z}{\partial y\partial x}\right|_{(1,2,1)}=\frac{-12\cdot 1\cdot\left(-\frac{4}{3}\right)\left(-\frac{1}{3}\right)+3\cdot\left(-\frac{4}{3}\right)-2\cdot 2}{6-3}=-\frac{40}{9}$$

$\therefore$ The quadratic surface approximation of $z(x,y)$ at $(1,2,1)$ is

$$P_2(x,y)=1-\tfrac{1}{3}(x-1)-\tfrac{4}{3}(y-2)+\frac{1}{2!}\left[-\tfrac{10}{9}(x-1)^2-2\times\tfrac{40}{9}(x-1)(y-2)-\tfrac{70}{9}(y-2)^2\right]$$

3. $f(x,y)=e^{3x}\cos(2y)$

(a) $f_x=3e^{3x}\cos(2y)$, $f_x(0,0)=3$

$f_y=-2e^{3x}\sin(2y)$, $f_y(0,0)=0$

$f(0,0)=1$

$\therefore$ Linear approximation of $f(x,y)$ at $(0,0)$ is

$$P_1(x,y)=1+3x$$

(b) $f_{xx}=9e^{3x}\cos(2y)$

$f_{xy}=-6e^{3x}\sin(2y)$

$f_{yy}=-4e^{3x}\cos(2y)$

$$E(x,y)=\frac{1}{2}\left[x^2 f_{xx}(\theta x,\theta y)+2xy\,f_{xy}(\theta x,\theta y)+y^2 f_{yy}(\theta x,\theta y)\right]$$

where $0<\theta<1$.

$$\therefore |E(x,y)|\le\frac{1}{2}\left[|x|^2|f_{xx}(\theta x,\theta y)|+2|x||y||f_{xy}(\theta x,\theta y)|+|y|^2|f_{yy}(\theta x,\theta y)|\right]$$

$$= \frac{1}{2}\left[9\cdot|x|^2\,|e^{30x}|\,|\cos(20y)| + 2\cdot|x|\,|y|\cdot 6\cdot|e^{30x}|\,|\sin(20y)| + 4\,|y|^2\,|e^{30x}|\,|\cos(20y)|\right]$$

$$< \frac{1}{2}\left[9\cdot(0.1)^2\cdot e^{0.3}\cdot 1 + 2\cdot(0.1)^2\cdot 6\cdot e^{0.3}\cdot 1 + 4\cdot(0.1)^2\cdot e^{0.3}\cdot 1\right]$$

(since $|x|, |y| \le 0.1$, $e^{30x} < e^{0.3}$ for $0<\theta<1$, $|\sin(20y)|, |\cos(20y)| \le 1$)

$$= \frac{25}{2}\cdot(0.1)^2\cdot e^{0.3}$$

$$= \frac{1}{8}e^{0.3}$$

4. $f(x,y) = xy\,e^{-(x^2+y^2)}$

$$f_x = y\,e^{-(x^2+y^2)} - 2x^2y\,e^{-(x^2+y^2)} = y(1-2x^2)\,e^{-(x^2+y^2)}$$

$$f_y = x\,e^{-(x^2+y^2)} - 2xy^2e^{-(x^2+y^2)} = x(1-2y^2)\,e^{-(x^2+y^2)}$$

Set $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} y(1-2x^2)\,e^{-(x^2+y^2)} = 0 \\ x(1-2y^2)\,e^{-(x^2+y^2)} = 0 \end{cases} \Rightarrow \begin{cases} y=0 \text{ or } x=\pm\frac{1}{\sqrt{2}} \\ x=0 \text{ or } y=\pm\frac{1}{\sqrt{2}} \end{cases}$

∴ The stationary points are :

$$(0,0),\ \left(-\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}}\right),\ \left(-\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}\right),\ \left(\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}}\right),\ \left(\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}\right)$$

$$f_{xx} = y(-4x)\,e^{-(x^2+y^2)} + y(1-2x^2)\cdot(-2x)\,e^{-(x^2+y^2)}$$

$$= -2xy(3-2x^2)\,e^{-(x^2+y^2)}$$

$$f_{yy} = x(-4y)\,e^{-(x^2+y^2)} + x(1-2y^2)\cdot(-2y)\,e^{-(x^2+y^2)}$$

$$= -2xy(3-2y^2)\,e^{-(x^2+y^2)}$$

$$f_{xy} = (1-2x^2)\,e^{-(x^2+y^2)} - 2y^2(1-2x^2)\,e^{-(x^2+y^2)}$$

$$= (1-2x^2)(1-2y^2)\,e^{-(x^2+y^2)}$$

| Stationary point $(x_0, y_0)$ | $f_{xx}(x_0, y_0)$ | $f_{yy}(x_0, y_0)$ | $f_{xy}(x_0, y_0)$ | $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$ | Nature |
|---|---|---|---|---|---|
| $(0, 0)$ | $0$ | $0$ | $1$ | $-1 \quad (<0)$ | Saddle point |
| $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ | $-2e^{-1}$ $(<0)$ | $-2e^{-1}$ | $0$ | $4e^{-2} \quad (>0)$ | local maximum |
| $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ | $2e^{-1}$ $(>0)$ | $2e^{-1}$ | $0$ | $4e^{-2} \quad (>0)$ | local minimum |
| $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ | $2e^{-1}$ $(>0)$ | $2e^{-1}$ | $0$ | $4e^{-2} \quad (>0)$ | local minimum |
| $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ | $-2e^{-1}$ $(<0)$ | $-2e^{-1}$ | $0$ | $4e^{-2} \quad (>0)$ | local maximum |

5. $f(x,y) = x^2 - 2xy + y^2 - y$

$\nabla f = (2x-2y)\vec{i} + (-2x+2y-1)\vec{j}$

$\nabla f(2,1) = 2\vec{i} - 3\vec{j}$

Let $\vec{u} = a\vec{i} + b\vec{j}$ be a unit vector, i.e. $\boxed{a^2+b^2=1}$ —①

$D_{\vec{u}} f(2,1) = \nabla f(2,1) \cdot \vec{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 2a - 3b$

(a) $D_{\vec{u}} f(2,1)$ is the highest when $\vec{u}$ is in the direction of $\nabla f(2,1)$.

$\therefore \vec{u} = \frac{\nabla f(2,1)}{|\nabla f(2,1)|} = \frac{2}{\sqrt{13}}\vec{i} - \frac{3}{\sqrt{13}}\vec{j}$

& $D_{\vec{u}} f(2,1) = |\nabla f(2,1)| = \sqrt{13}$

(b) $D_{\vec{u}} f(2,1)$ is the smallest when $\vec{u}$ is in the opposite direction of $\nabla f(2,1)$.

$\therefore \vec{u} = \frac{-\nabla f(2,1)}{|\nabla f(2,1)|} = -\frac{2}{\sqrt{13}}\vec{i} + \frac{3}{\sqrt{13}}\vec{j}$

& $D_{\vec{u}} f(2,1) = -|\nabla f(2,1)| = -\sqrt{13}$

5(c) $D_{\vec{u}}f(2,1) = 0$ when $\vec{u}$ is orthogonal to $\nabla f(2,1)$.

$$\therefore \begin{cases} 2a - 3b = 0 \\ a^2 + b^2 = 1 \end{cases} \Rightarrow a^2 + \left(\frac{2a}{3}\right)^2 = 1 \Rightarrow a^2 = \frac{9}{13} \Rightarrow a = \pm\frac{3}{\sqrt{13}}$$

When $a = \frac{3}{\sqrt{13}}$, $b = \frac{2}{3}\left(\frac{3}{\sqrt{13}}\right) = \frac{2}{\sqrt{13}}$.

When $a = -\frac{3}{\sqrt{13}}$, $b = \frac{2}{3}\left(-\frac{3}{\sqrt{13}}\right) = -\frac{2}{\sqrt{13}}$.

$$\therefore \vec{u} = \frac{3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j} \quad \text{or} \quad \vec{u} = -\frac{3}{\sqrt{13}}\vec{i} - \frac{2}{\sqrt{13}}\vec{j}$$

(d) $D_{\vec{u}}f(2,1) = -2$ when

$$\begin{cases} 2a - 3b = -2 \\ a^2 + b^2 = 1 \end{cases} \Rightarrow a^2 + \left(\frac{2a+2}{3}\right)^2 = 1 \Rightarrow 13a^2 + 8a - 5 = 0$$

$$\Rightarrow (a+1)(13a-5) = 0$$

$$\Rightarrow a = -1 \text{ or } a = \frac{5}{13}$$

When $a = -1$, $b = \frac{1}{3}[2(-1)+2] = 0$

When $a = \frac{5}{13}$, $b = \frac{1}{3}\left[2\left(\frac{5}{13}\right)+2\right] = \frac{12}{13}$

$$\therefore \vec{u} = -\vec{i} \quad \text{or} \quad \vec{u} = \frac{5}{13}\vec{i} + \frac{12}{13}\vec{j}$$

6. $S = [6 - a(-2)^2 - b(-2) - c]^2 + [2 - a(-1)^2 - b(-1) - c]^2 + [1 - a(1)^2 - b(1) - c]^2$

$+ [4.5 - a(2)^2 - b(2) - c]^2$

$= (6 - 4a + 2b - c)^2 + (2 - a + b - c)^2 + (1 - a - b - c)^2 + (4.5 - 4a - 2b - c)^2$

$S_a = 2(6-4a+2b-c)(-4) + 2(2-a+b-c)(-1) + 2(1-a-b-c)(-1)$

$+ 2(4.5-4a-2b-c)(-4)$

$= -90 + 68a + 20c$

$$S_b = 2(6-4a+2b-c)\cdot(2) + 2(2-a+b-c)(1) + 2(1-a-b-c)(-1) + 2(4.5-4a-2b-c)(-2)$$

$$= 8 + 20b$$

$$S_c = 2(6-4a+2b-c)(-1) + 2(2-a+b-c)(-1) + 2(1-a-b-c)(-1) + 2(4.5-4a-2b-c)(-1)$$

$$= -27 + 20a + 8c$$

Set $S_a = S_b = S_c = 0 \Rightarrow \begin{cases} -90+68a+20c = 0 \\ 8+20b = 0 \\ -27+20a+8c = 0 \end{cases}$

$$\Rightarrow \quad b = -\tfrac{2}{5} \quad \& \quad \begin{cases} 68a+20c = 90 \\ 20a+8c = 27 \end{cases}$$

$$\Rightarrow \quad b = -\tfrac{2}{5}, \quad a = \tfrac{5}{4}, \quad c = \tfrac{1}{4}$$

$\therefore$ The stationary point is $(a,b,c) = (\tfrac{5}{4}, -\tfrac{2}{5}, \tfrac{1}{4})$.

(b) $S_{aa} = 68$, $S_{bb} = 20$, $S_{cc} = 8$,

$S_{ab} = S_{ba} = 0$, $S_{ac} = S_{ca} = 20$, $S_{bc} = S_{cb} = 0$

$\therefore S(a,b,c) - S(\tfrac{5}{4}, -\tfrac{2}{5}, \tfrac{1}{4})$

$$= \tfrac{1}{2}\left[68(a-\tfrac{5}{4})^2 + 20(b+\tfrac{2}{5})^2 + 8(c-\tfrac{1}{4})^2 + 2\cdot 20(a-\tfrac{5}{4})(c-\tfrac{1}{4})\right]$$

$$= \tfrac{1}{2}\begin{bmatrix} a-\frac{5}{4} & b+\frac{2}{5} & c-\frac{1}{4} \end{bmatrix} \begin{bmatrix} 68 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 8 \end{bmatrix} \begin{bmatrix} a-\frac{5}{4} \\ b+\frac{2}{5} \\ c-\frac{1}{4} \end{bmatrix}$$

6(c) $A = \begin{bmatrix} 68 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 8 \end{bmatrix}$

Leading principal minors are

$68 > 0$,

$\begin{vmatrix} 68 & 0 \\ 0 & 20 \end{vmatrix} = 1360 > 0$

and $|A| = 68 \begin{vmatrix} 20 & 0 \\ 0 & 8 \end{vmatrix} + 20 \begin{vmatrix} 0 & 20 \\ 20 & 0 \end{vmatrix} = 68 \times 20 \times 8 - 20^3$

$= 2880 > 0$

Since all leading principal minors of A are positive,

A is positive definite.

$\therefore S(a,b,c) - S(\frac{5}{4}, -\frac{2}{5}, \frac{1}{4}) \geq 0$ for all $a, b, c$.

(Note: $S(a,b,c) - S(\frac{5}{4}, -\frac{2}{5}, \frac{1}{4}) = 0$ only when $a = \frac{5}{4}$, $b = -\frac{2}{5}$, $c = \frac{1}{4}$.)

$\therefore$ S has a global minimum at $(\frac{5}{4}, -\frac{2}{5}, \frac{1}{4})$.