

L03: Wave Equations and Plane Waves

EE3109 Applied Electromagnetics

Lecture Notes

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Jan – Apr 2023

Outline

- Conceptual Questions
- Course Matter
- Wave Equations
- Plane Waves
- Wave Propagation in Lossy Media

Announcements and Reminders

- Assignment 1 posted.
 - Assignment 1A (Q1-4): Due Feb 9
 - Assignment 1B (Q5-7): Due Feb 16
- Reminder: Video Access
 - Request and attendance (direct to TA)
 - Access through Panopto Recordings on Canvas
 - We will try to provide access within 2 days after the lecture
- Reminder: the EE3109 Office Hour starts this Friday, 1:30-2:30pm
 - Must indicate to instructor that you will come at least one day before
 - Indicate whether you will come online or in person
 - Zoom link and password has been given on Canvas
 - You can also approach the TAs for help

Tentative Course Schedule

Week #	Week of	Course Coverage
1	Jan 9	Course Introduction, Math Review
2	Jan 16	Introduction to Maxwell Equations, Materials and Boundaries
	Jan 23	<i>Happy Chinese New Year!</i>
3	Jan 30	Wave Equations, Plane Waves, Propagation in Lossy Media
4	Feb 6	Polarizations, EM Potential <i>Asst. 1a Due</i>
5	Feb 13	Reflection and Transmission <i>Asst. 1b Due</i>
6	Feb 20	Test 1 , Electromagnetics R&D
7	Feb 27	Poynting vector, Power and Velocity
8	Mar 6	Waveguides and Transmission Lines (TL)
9	Mar 13	VIP in a TL, Terminated and Lossy TLs, <i>Asst. 2 Due, Lab 1 Begins</i>
10	Mar 20	Impedance in a TL, Smith Charts,
11	Mar 27	Smith Charts Cont'd, Impedance Matching, Quarter-wave Transformers, <i>Lab 2 Begins</i>
12	Apr 3	Test 2 , Introduction to Antennas
13	Apr 10	Dipole Antennas, <i>Asst. 3 Due</i>

Review and Preview

- Last week we:
 - Reviewed electro/magneto/quasi- statics
 - Introduced Maxwell's equations
 - Examined electromagnetic material properties
 - Studied how EM fields behave at material boundaries
- This class we will look at:
 - The EM wave equation
 - The EM plane wave
 - Plane wave propagation in lossy media

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Wave Equations – Motivation

- We would like to write equations for (E, H, B, D) in a way that is **decoupled** from other quantities
- Without sources and in simple media, this can be done straightforwardly

Source-free wave equations $\vec{J} = 0, \rho = 0$

Wave is traveling in a simple non-conducting ($\sigma=0$) medium

$$\begin{aligned}\nabla \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} & \nabla \times \vec{H} &= \epsilon \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} &= 0 & \nabla \cdot \vec{H} &= 0\end{aligned}$$

Consider $\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H})$

$$= -\mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

or $\nabla \times \nabla \times \vec{E} + \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$

$$\begin{aligned}\nabla \times \nabla \times \vec{E} &= \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \\ &= -\nabla^2 \vec{E}\end{aligned}$$

$$\therefore \nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

velocity of wave propagation : $u = \frac{1}{\sqrt{\mu\epsilon}}$

Wave equation for E becomes : $\nabla^2 \vec{E} - \frac{1}{u^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$ (1)

Similarly,

$$\nabla \times \nabla \times \vec{H} + \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

$$\Rightarrow \nabla^2 \vec{H} - \frac{1}{u^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$
 (2)

Both (1) and (2) are called homogeneous vector wave equations

In rectangular coordinates, $-\nabla^2 \vec{E} = -\nabla^2 E_x \hat{x} - \nabla^2 E_y \hat{y} - \nabla^2 E_z \hat{z}$

$$\text{where } \nabla^2 E_x = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_x$$

The wave equations become

$$\nabla^2 E_x - \frac{1}{u^2} \frac{\partial^2 E_x}{\partial t^2} = 0 \qquad \nabla^2 H_x - \frac{1}{u^2} \frac{\partial^2 H_x}{\partial t^2} = 0$$

$$\nabla^2 E_y - \frac{1}{u^2} \frac{\partial^2 E_y}{\partial t^2} = 0 \qquad \nabla^2 H_y - \frac{1}{u^2} \frac{\partial^2 H_y}{\partial t^2} = 0$$

$$\nabla^2 E_z - \frac{1}{u^2} \frac{\partial^2 E_z}{\partial t^2} = 0 \qquad \nabla^2 H_z - \frac{1}{u^2} \frac{\partial^2 H_z}{\partial t^2} = 0$$

All the 6 components are decoupled.

Review: The Helmholtz Equations

Applying the phasor transformation, we have:

Wave Equations

$$\left(\nabla^2 - \frac{1}{u^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{E} = 0$$

$$\left(\nabla^2 - \frac{1}{u^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{H} = 0$$

Helmholtz Equations

$$(\nabla^2 + k^2) \mathbf{E} = 0$$

$$(\nabla^2 + k^2) \mathbf{H} = 0$$

where $k^2 = \frac{\omega^2}{u^2} = \omega^2 \mu \epsilon \Rightarrow k = \omega \sqrt{\mu \epsilon}$

This means: $(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 + k^2)F_n = 0$
where $n = x, y, \text{ or } z$, and $F = E \text{ or } H$.

Uniform Plane Waves

- A Uniform Plane Wave is characterized by **E** and **H** that are **constant** at all points across an **infinite plane**.
- For simplicity, for now we consider a co-ordinate definition where this plane is the xy-plane
- In this case, both **E** and **H** are not functions of x and y (though they may point in the x - and y - directions)
- Hence, the Helmholtz equations become: (next page)

Uniform Plane Waves

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)E_x = 0$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)H_x = 0$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)E_y = 0$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)H_y = 0$$

The z-components of **E** and **H** are zero, because
(see next page)

Uniform Plane Waves

$$\begin{aligned} E_z = \mathbf{E} \cdot \hat{\mathbf{z}} &= \left(\frac{1}{j\omega\epsilon} \nabla \times \mathbf{H} \right) \cdot \hat{\mathbf{z}} = \frac{1}{j\omega\epsilon} (\nabla \times \mathbf{H}) \cdot \hat{\mathbf{z}} \\ &= \frac{1}{j\omega\epsilon} \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \cdot \hat{\mathbf{z}} \\ &= \frac{1}{j\omega\epsilon} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) = 0 \end{aligned}$$

(and the same goes for H_z)

Uniform Plane Waves

The E_x equation again: $\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0$

The solution for E_x is:

Phasor: $(E_0^\pm = |E_0^\pm| \exp(j\phi_0^\pm))$

$$E_x(z) = E_0^+ e^{-jkz} + E_0^- e^{+jkz} = E_x^+(z) + E_x^-(z)$$

Time domain:

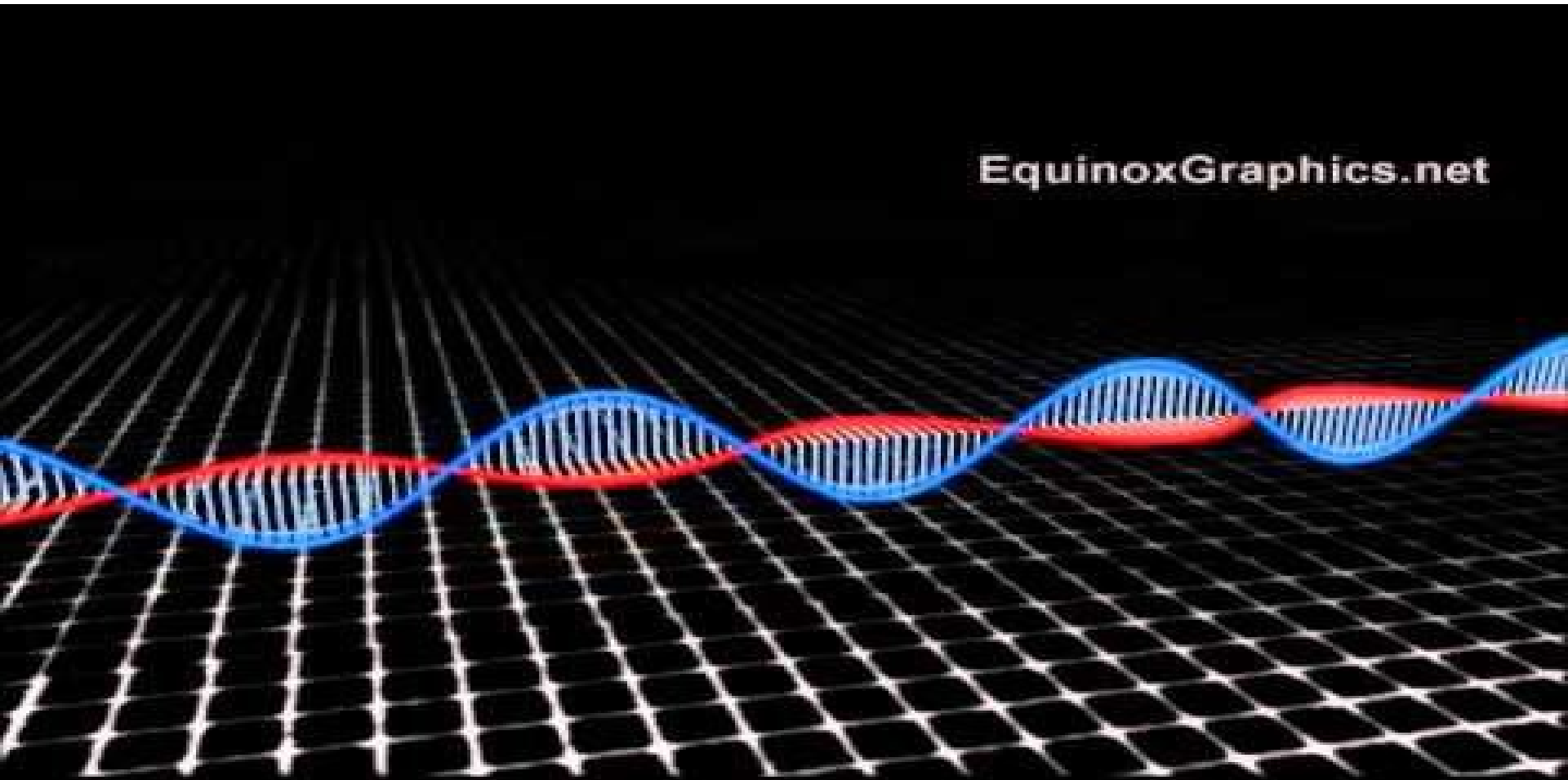
$$\begin{aligned} E_x(z, t) &= |E_0^+| \cos(\omega t - kz + \phi_0^+) + |E_0^-| \cos(\omega t + kz + \phi_0^-) \\ &= E_x^+(z, t) + E_x^-(z, t) \end{aligned}$$

A similar form applies for E_y , H_x and H_y .

Plane Waves....



Plane Wave Visualization



<https://www.youtube.com/watch?v=aCTRjVEmeC0&feature=youtu.be>

Travelling Plane Waves

Consider $E_x^+(z)$ (the same applies to E_y , H_x and H_y):

$$E_x^+(z) = E_0^+ e^{-jkz} \Rightarrow E_x^+(z, t) = |E_0^+| \cos(\omega t - kz + \phi_0^+)$$

Consider the phase of this travelling wave ($\varphi = \omega t - kz + \phi_0^+$):

The wavelength (λ) is the distance through which the wave changes in phase by 2π rad:

$$\lambda \equiv |\Delta z| \text{ s.t. } \Delta\varphi = 2\pi \Rightarrow \lambda = \frac{2\pi}{k}$$

The phase velocity (v_p) of the wave is the rate at which the constant phase point travels:

Rearrange the phase expression to get $z = \frac{\omega}{k}t + \underbrace{\frac{(\phi_0^+ - \varphi)}{k}}_{\text{Constant w.r.t. time}}$

$$\Rightarrow v_p = \frac{dz}{dt} = \frac{\omega}{k} = f\lambda$$

Exercise: Find the velocity of $E_x^-(z)$

Example 1: Relating E to H

Let us consider a forward travelling plane wave, with x denoting the direction of the E-field. i.e. $\mathbf{E} = E_x \hat{\mathbf{x}} = E_0 e^{-jkz} \hat{\mathbf{x}}$. What is the corresponding $\mathbf{H}$?

Apply Faraday's Law:

$$\begin{aligned} H(z) &= -\frac{1}{j\omega\mu} \nabla \times \mathbf{E} = -\frac{1}{j\omega\mu} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (E_0 e^{-jkz}, 0, 0) \\ &= -\frac{1}{j\omega\mu} \frac{\partial}{\partial z} (E_0 e^{-jkz}) \hat{\mathbf{y}} = -E_0 \frac{-jk}{j\omega\mu} e^{-jkz} \hat{\mathbf{y}} = \frac{k}{\omega\mu} E_x \hat{\mathbf{y}} \end{aligned}$$

Recall: $k = \omega\sqrt{\mu\varepsilon}$. Define the **Intrinsic Impedance**:

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Rearrange the above to get:

$$H(z) = \frac{1}{\eta} E_x \hat{\mathbf{y}} = \frac{1}{\eta} E_0 e^{-jkz} \hat{\mathbf{y}}$$

Remarks:

1. **Phase:** E and H travel in phase.
2. **Amplitude:** E and H differ in amplitude by the factor η . i.e.

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{|\mathbf{E}|}{|\mathbf{H}|} \equiv \text{Intrinsic Impedance } [\Omega]$$

3. **Direction:** Define $\mathbf{k} \equiv \text{Wave Vector } [\text{m}^{-1}]$

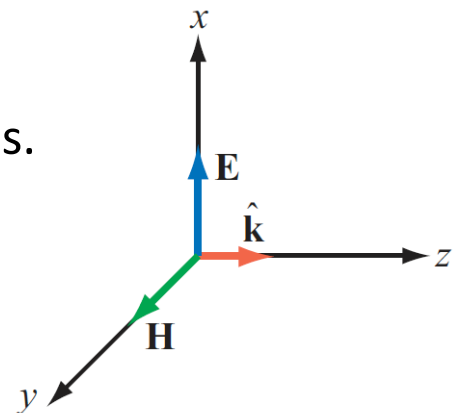
Such that $|\mathbf{k}| = k$; $\hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}$ points in the direction of phase progression.

(E, H, k) form a right-handed triplet: $\hat{\mathbf{E}} \times \hat{\mathbf{H}} = \hat{\mathbf{k}}$

These are true when μ and ε both have positive real parts.

Hence, for plane waves in this case:

$$\begin{aligned}\mathbf{E} &= \eta \mathbf{H} \times \hat{\mathbf{k}} \\ \mathbf{H} &= \frac{1}{\eta} \hat{\mathbf{k}} \times \mathbf{E}\end{aligned}$$



Example 2 (Ulaby Example 7-1)

The electric field of a 1-MHz plane wave travelling in the +z-direction in air points along the x-direction. If this field reaches a peak value of 1.2π [mV/m] at $t = 0$ and $z = 50$ [m], obtain an expression for $E(z,t)$ and $H(z,t)$, then plot them as a function of z at $t=0$.

Solution: At $f = 1$ MHz, the wavelength in air is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1 \times 10^6} = 300 \text{ m},$$

and the corresponding wavenumber is $k = (2\pi/300)$ (rad/m). The general expression for an x -directed electric field traveling in the +z-direction is given by Eq. (7.34a) as

$$\begin{aligned} \mathbf{E}(z, t) &= \hat{\mathbf{x}} |E_{x0}^+| \cos(\omega t - kz + \phi^+) \\ &= \hat{\mathbf{x}} 1.2\pi \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \phi^+\right) \text{ (mV/m)}. \end{aligned}$$

The field $\mathbf{E}(z, t)$ is maximum when the argument of the cosine function equals zero or a multiple of 2π . At $t = 0$ and $z = 50$ m, this condition yields

$$-\frac{2\pi \times 50}{300} + \phi^+ = 0 \quad \text{or} \quad \phi^+ = \frac{\pi}{3}.$$

Example 7-1 cont.

Free-space Characteristic Impedance:

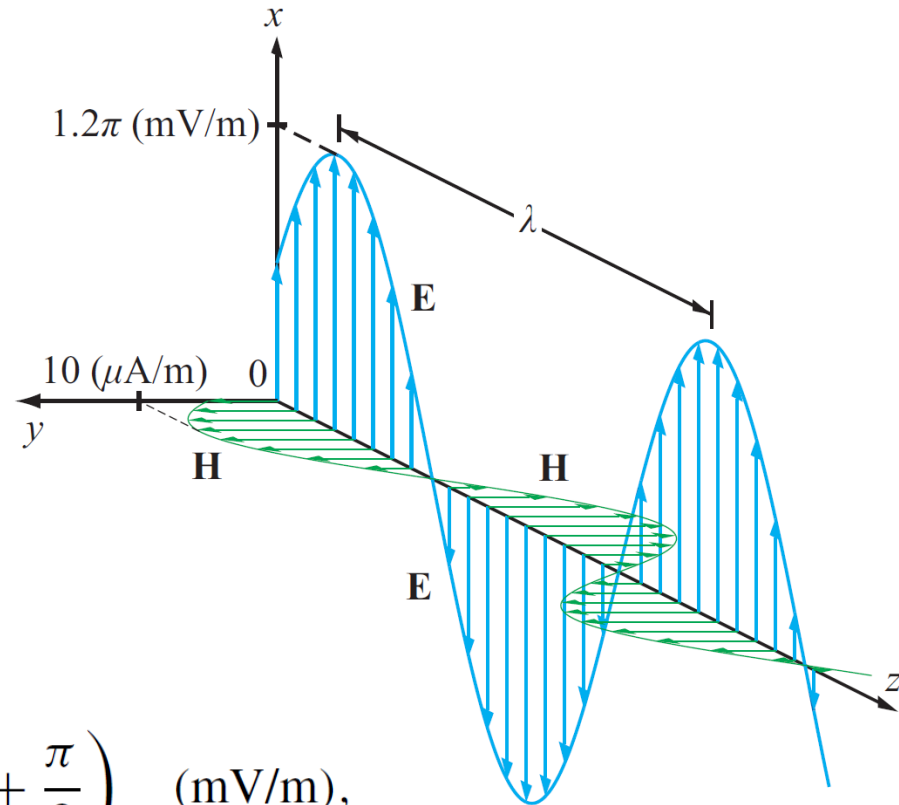
$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 377 \, \Omega \cong 120\pi \, \Omega$$

Hence,

$$\mathbf{E}(z, t) = \hat{\mathbf{x}} 1.2\pi \cos \left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \frac{\pi}{3} \right) \quad (\text{mV/m}),$$

and from Eq. (7.34b) we have

$$\begin{aligned} \mathbf{H}(z, t) &= \hat{\mathbf{y}} \frac{E(z, t)}{\eta_0} \\ &= \hat{\mathbf{y}} 10 \cos \left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \frac{\pi}{3} \right) \quad (\mu\text{A/m}), \end{aligned}$$



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Revisit: Uniform Plane Waves

Helmholtz equation for E_x : $\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0$

A similar form applies for E_y , H_x and H_y .

Phasor Solution:

$$E_x(z) = E_0^+ e^{-jkz} + E_0^- e^{+jkz} = E_x^+(z) + E_x^-(z)$$

What happens if the medium is lossy?

$$k \rightarrow k_c = \omega \sqrt{\mu_0 \epsilon_c}$$

The wave number becomes complex!

Generalize the formulation:

$$\begin{aligned} \text{Propagation Constant} &\equiv \\ \gamma &= \alpha + j\beta = jk_c = j\omega \sqrt{\mu_0 \epsilon_c} \end{aligned}$$

What is α and β in terms of the material parameters?

$$\begin{aligned}(\alpha + j\beta)^2 &= (j\omega\sqrt{\mu\epsilon_c})^2 = -\omega^2\mu\epsilon_c \\(\alpha^2 - \beta^2) + j2\alpha\beta &= -\omega^2\mu\epsilon' + j\omega^2\mu\epsilon''\end{aligned}$$

Equating the real and imaginary parts gives:

$$(\alpha^2 - \beta^2) = -\omega^2\mu\epsilon'; \quad 2\alpha\beta = \omega^2\mu\epsilon''$$

One can easily write α in terms of β (or vice versa) with the second equation. Substitution into the first one allows one to find the following:

$$\text{Attenuation Constant} \equiv \alpha = \omega \left\{ \frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2} - 1 \right] \right\}^{1/2} \quad [\text{Np/m}]$$

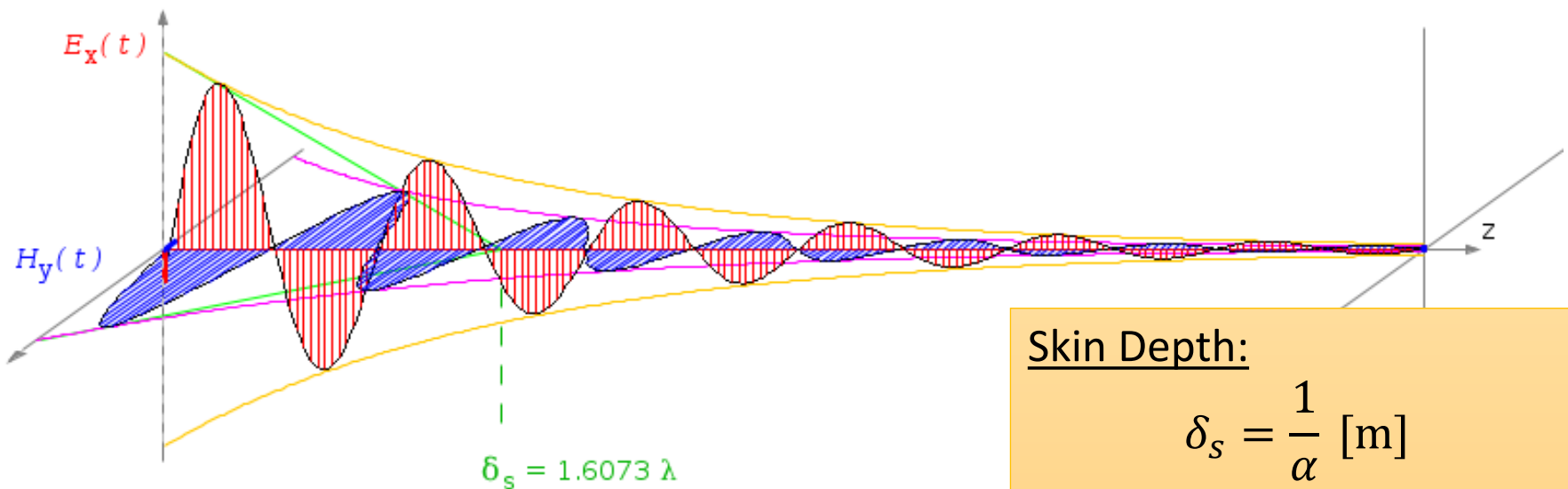
$$\text{Phase Constant} \equiv \beta = \omega \left\{ \frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2} + 1 \right] \right\}^{1/2} \quad [\text{rad/m}]$$

Hence the (positively travelling) electric field phasor can be written as

$$E_x = E_0^+ e^{-(jk_c)z} = E_0^+ e^{-(\alpha + j\beta)z} = E_0^+ e^{-\alpha z} e^{-j\beta z}$$

and the corresponding time-domain E-field is:

$$E_x(z, t) = |E_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \angle E_0^+)$$



Skin Depth:

$$\delta_s = \frac{1}{\alpha} \text{ [m]}$$

Distance after which the wave decays to $e^{-1} \cong 0.37$ of its original value.

Complex Characteristic Impedance

The complex characteristic impedance can also be written in terms of material parameters:

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon' - j\epsilon''}} = \sqrt{\frac{\mu}{\epsilon'}} \left(1 - \frac{j\epsilon''}{\epsilon'}\right)^{-\frac{1}{2}}$$

With this definition, the corresponding magnetic field phasor (for E_x^+) is written as:

$$\mathbf{H} = -\frac{1}{\eta_c} \mathbf{E} \times \hat{\mathbf{k}} = -H_0^+ e^{-\alpha z} e^{-j\beta z} \hat{\mathbf{y}}, \text{ where } H_0^+ = \frac{E_0^+}{\eta_c}$$

And the corresponding time domain H field is:

$$\mathbf{H}(z, t) = |H_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \angle H_0^+ + \pi) \hat{\mathbf{y}}$$

Note: $\mathbf{H}$ is no longer in phase with $\mathbf{E}$, because of η_c

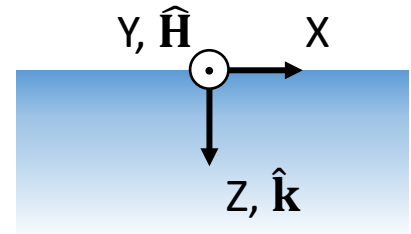
Simplified Equations for Lossless, Low-loss and Conductive Media

Table 7-1: Expressions for α , β , η_c , u_p , and λ for various types of media.

	Any Medium	Lossless Medium ($\sigma = 0$)	Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)	Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)	Units
$\alpha =$	$\omega \left[\frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'} \right)^2} - 1 \right] \right]^{1/2}$	0	$\frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\pi f \mu \sigma}$	(Np/m)
$\beta =$	$\omega \left[\frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'} \right)^2} + 1 \right] \right]^{1/2}$	$\omega \sqrt{\mu\varepsilon}$	$\omega \sqrt{\mu\varepsilon}$	$\sqrt{\pi f \mu \sigma}$	(rad/m)
$\eta_c =$	$\sqrt{\frac{\mu}{\varepsilon'}} \left(1 - j \frac{\varepsilon''}{\varepsilon'} \right)^{-1/2}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$(1 + j) \frac{\alpha}{\sigma}$	(Ω)
$u_p =$	ω / β	$1 / \sqrt{\mu\varepsilon}$	$1 / \sqrt{\mu\varepsilon}$	$\sqrt{4\pi f / \mu \sigma}$	(m/s)
$\lambda =$	$2\pi / \beta = u_p / f$	u_p / f	u_p / f	u_p / f	(m)
Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma/\omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon < 0.01$ and a good conducting medium if $\varepsilon''/\varepsilon' > 100$					

Example: Plane Wave in Seawater

A uniform plane wave is traveling in seawater. Assume that the x - y plane resides just below the sea surface and the wave travels in the $+z$ -direction into the water. The constitutive parameters of seawater are $\epsilon_r = 80$, $\mu_r = 1$, and $\sigma = 4$ S/m. If the magnetic field at $z = 0$ is $\mathbf{H}(0, t) = \hat{\mathbf{y}} 100 \cos(2\pi \times 10^3 t + 15^\circ)$ (mA/m),



- (a) obtain expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$, and
- (b) determine the depth at which the magnitude of $\mathbf{E}$ is 1% of its value at $z = 0$.

Solution: (a) Since $\mathbf{H}$ is along $\hat{\mathbf{y}}$ and the propagation direction is $\hat{\mathbf{z}}$, $\mathbf{E}$ must be along $\hat{\mathbf{x}}$. Hence, the general expressions for the phasor fields are

$$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_{x0} e^{-\alpha z} e^{-j\beta z}, \quad (7.78a)$$

$$\tilde{\mathbf{H}}(z) = \hat{\mathbf{y}} \frac{E_{x0}}{\eta_c} e^{-\alpha z} e^{-j\beta z}. \quad (7.78b)$$

Example Cont'd

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{4}{2\pi \times 10^3 \times 80 \times (10^{-9}/36\pi)} \\ = 9 \times 10^5.$$

This qualifies seawater as a good conductor at 1 kHz and allows us to use the good-conductor expressions given in Table 7-1:

$$\alpha = \sqrt{\pi f \mu \sigma} \\ = \sqrt{\pi \times 10^3 \times 4\pi \times 10^{-7} \times 4} \\ = 0.126 \quad (\text{Np/m}), \quad (7.79a)$$

$$\beta = \alpha = 0.126 \quad (\text{rad/m}), \quad (7.79b)$$

$$\eta_c = (1 + j) \frac{\alpha}{\sigma} \\ = (\sqrt{2} e^{j\pi/4}) \frac{0.126}{4} = 0.044 e^{j\pi/4} \quad (\Omega). \quad (7.79c)$$

Example Cont'd

$$\begin{aligned}\mathbf{E}(z, t) &= \Re \left[\hat{\mathbf{x}} |E_{x0}| e^{j\phi_0} e^{-\alpha z} e^{-j\beta z} e^{j\omega t} \right] \\ &= \hat{\mathbf{x}} |E_{x0}| e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + \phi_0) \\ &\quad (\text{V/m}),\end{aligned}\tag{7.80a}$$

$$\begin{aligned}\mathbf{H}(z, t) &= \Re \left[\hat{\mathbf{y}} \frac{|E_{x0}| e^{j\phi_0}}{0.044 e^{j\pi/4}} e^{-\alpha z} e^{-j\beta z} e^{j\omega t} \right] \\ &= \hat{\mathbf{y}} 22.5 |E_{x0}| e^{-0.126z} \cos(2\pi \times 10^3 t \\ &\quad - 0.126z + \phi_0 - 45^\circ) \quad (\text{A/m}).\end{aligned}\tag{7.80b}$$

Example Cont'd

At $z = 0$,

$$\mathbf{H}(0, t) = \hat{\mathbf{y}} 22.5 |E_{x0}| \cos(2\pi \times 10^3 t + \phi_0 - 45^\circ) \quad (\text{A/m}). \quad (7.81)$$

By comparing Eq. (7.81) with the expression given in the problem statement,

$$\mathbf{H}(0, t) = \hat{\mathbf{y}} 100 \cos(2\pi \times 10^3 t + 15^\circ) \quad (\text{mA/m}),$$

we deduce that

$$\begin{aligned} 22.5 |E_{x0}| &= 100 \times 10^{-3} \\ |E_{x0}| &= 4.44 \quad (\text{mV/m}), \end{aligned}$$

$$\phi_0 - 45^\circ = 15^\circ \quad \text{or} \quad \phi_0 = 60^\circ.$$

Example Cont'd

Hence, the final expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ are

$$\mathbf{E}(z, t) = \hat{\mathbf{x}} 4.44 e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + 60^\circ) \quad (\text{mV/m}), \quad (7.82a)$$

$$\mathbf{H}(z, t) = \hat{\mathbf{y}} 100 e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + 15^\circ) \quad (\text{mA/m}). \quad (7.82b)$$

(b) The depth at which the amplitude of $\mathbf{E}$ has decreased to 1% of its initial value at $z = 0$ is obtained from

$$0.01 = e^{-0.126z}$$

or

$$z = \frac{\ln(0.01)}{-0.126} = 36.55 \text{ m}$$

On Nepers and dB

- α is usually [Np/m], where Np, or Neper, is unitless
 - Eg. 1 [Np/m] means the E/H-field amplitude decays to e^{-1} of its original value after 1 [m].
- It is useful to also quote the decay in [dB/m]
 - Eg. 1 [dB/m] means the plane wave's power decays to 1/10 of its original value after 10 [m].
- (Will be shown in Lect. 6: A plane wave's power density is proportional to the square of the E-field.)

Converting Nepers <---> dB

-1 Neper is the logarithmic measure when the wave amplitude decays to e^{-1} of its original value; **-10 dB** is logarithmic measure when the wave power decays to **1/10** of its original value.

Converting -1 Np into dB: Consider $E(z_1) = E(z_0)e^{-1}$ (i.e. the E-field decays by 1 Np from z_0 to z_1). The decay in dB is:

$$20 \log \left(\frac{E(z_1)}{E(z_0)} \right) = 20 \log(e^{-1}) = -\frac{20 \ln e}{\ln 10} = -\frac{20}{\ln 10} \text{ [dB]} \\ \cong -8.686 \text{ [dB]}$$

Hence **1 Np = 8.686 dB** or **1 dB = 0.1151 Np**

Summary

- In this class we looked at:
 - The EM wave equation
 - The EM plane wave
 - Relationship between E and H fields in a plane wave
 - Wave decay and propagation in lossy media