L04b: Electromagnetic Potentials

EE3109 Applied Electromagnetics

Lecture Notes

Dr. Alex M. H. Wong

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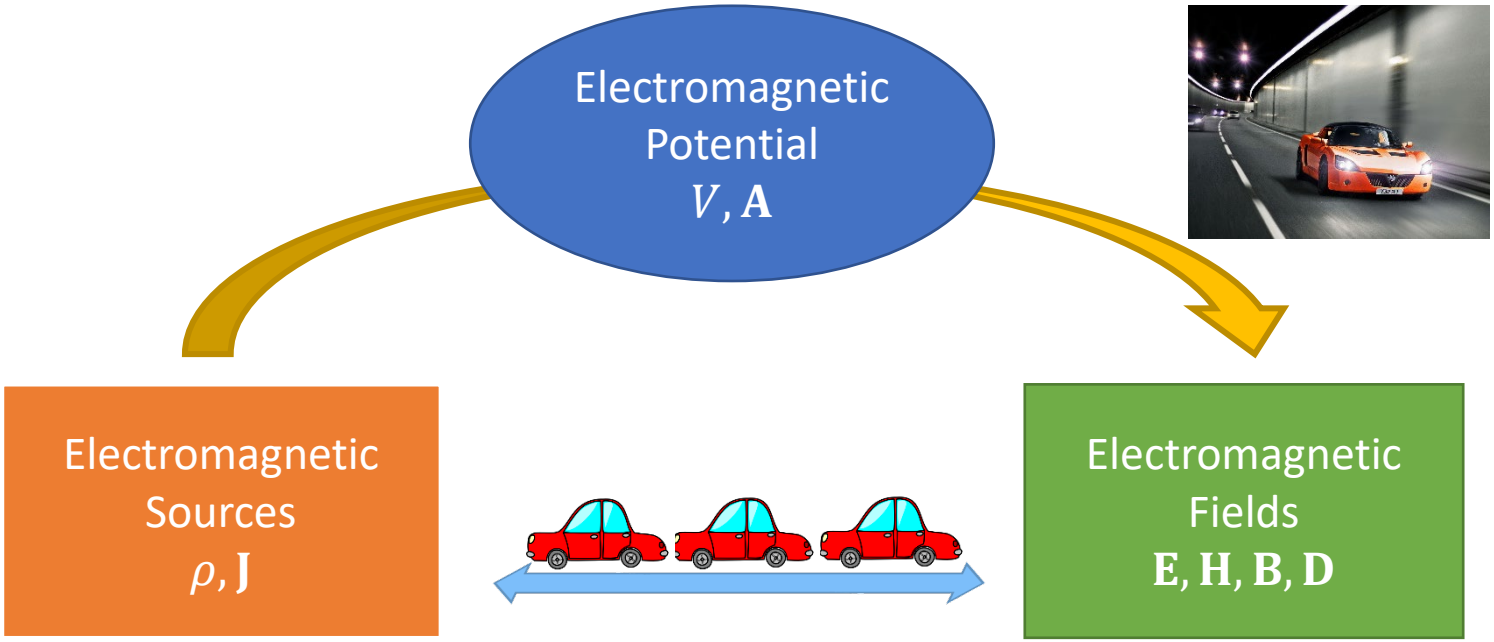
Overview

- In this note we explain the motivation and demonstrate the application of electromagnetic potentials
- We cover the following major aspects
  - Defining electromagnetic (EM) potentials
  - Wave equations for EM potentials
  - Solving for EM fields with EM potentials
  - Example: Deriving the Biot-Savart Law from Maxwell's Equations

Electromagnetic Potential - Concept

- Electromagnetic potentials lead to more efficient calculations of the effect of sources on electromagnetic fields



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Electromagnetic Potentials – Definition

Consider $\nabla \cdot \mathbf{B} = 0$

Recalling the identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, we may define a Magnetic Vector Potential $\mathbf{A}$ such that

$$\boxed{\nabla \times \mathbf{A} = \mathbf{B}}$$

Substitute into the Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial (\nabla \times \mathbf{A})}{\partial t} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}$$

$$\Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

Now recall the identity $\nabla \times (\nabla V) = 0$, we may define a scalar potential V such that

$$-\nabla V = \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$$

Note that, in the static limit, this definition becomes $-\nabla V = \mathbf{E}$, which is consistent with the electrostatic case)

Rearranging, we have

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

Now we apply the electromagnetic potentials to Ampere's Law: (next page…)

$$\frac{1}{\mu}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J} + \mu\varepsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right)$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \nabla\left(\mu\varepsilon \frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t}\right)$$

We only defined $\nabla \times \mathbf{A} = \mathbf{B}$. We are at liberty to define $\nabla \cdot \mathbf{A}$ without affecting the value of the EM fields. ***We can define $\nabla \cdot \mathbf{A}$ in a way that simplifies our problem!***

<u>Lorentz Gauge</u>: we define $\boxed{\nabla \cdot \mathbf{A} = -\mu\varepsilon \frac{\partial V}{\partial t}}$

The above equation becomes

$$\boxed{\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}}$$

(Non-homogeneous wave equation for $\mathbf{A}$)

Now look at Gauss' Law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$$

$$-\nabla \cdot \left(\nabla V + \frac{\partial \mathbf{A}}{\partial t} \right) = \frac{\rho}{\varepsilon}$$

$$\nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon}$$

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}}$$

(Non-homogeneous wave equation for V)

Remarks:

1. The two wave equations connect sources $(\rho, \mathbf{J})$ with potentials $(V, \mathbf{A})$
2. The waves have a speed of $v_p = 1/\sqrt{\mu\varepsilon}$

Solving the differential equations:

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

Phasor forms: $(k = \omega\sqrt{\mu\varepsilon})$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$

$$\nabla^2 V + k^2 V = -\frac{\rho}{\varepsilon}$$

(Separable) ↓

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$
$$\nabla^2 A_y + k^2 A_y = -\mu J_y$$
$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

All are non-homogeneous wave equations and should have the same solution form.

Solution to the Spherical Wave Equation:

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$$\nabla^2 V + k^2 V = -\frac{\rho}{\varepsilon}$$

Since the equation is "linear and space-invariant", we apply a Green's function (impulse response) method. First assume the source ρ is a delta function (point charge) at location $\mathbf{r}'$ and find the answer, then construct the answer for a general $\rho(\mathbf{r})$ as an integral of individual delta-function sources.

Solution to the Spherical Wave Equation with delta source:

$$\nabla^2 V + k^2 V = -\frac{\delta^3(\mathbf{R})}{\varepsilon}$$

(defined $\mathbf{R} = \mathbf{r} - \mathbf{r}'$)

Solutions should be of the form $V = C\frac{e^{jkR}}{R}$, where $R = |\mathbf{R}|$, and C is a constant.

Verifying the Solution to the Spherical Wave Equation:

1. Calculate $\nabla^2 V + k^2 V$ for $\mathbf{R} \neq 0$:

$$\begin{aligned}
\nabla^2 V + k^2 V &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + k^2 V \\
&= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R}\left(C\frac{e^{-jkR}}{R}\right)\right) + k^2 C\frac{e^{-jkR}}{R} \\
&= \frac{C}{R^2}\frac{\partial}{\partial R}\left(R^2\left(-jk\frac{e^{-jkR}}{R} - \frac{e^{-jkR}}{R^2}\right)\right) + k^2 C\frac{e^{-jkR}}{R} \\
&= -\frac{C}{R^2}\frac{\partial}{\partial R}\left(e^{-jkR}(jkR+1)\right) + k^2 C\frac{e^{-jkR}}{R} \\
&= -\frac{C}{R^2}\left(-jke^{-jkR}(jkR+1) + jke^{-jkR}\right) + k^2 C\frac{e^{-jkR}}{R} \\
&= -\frac{C}{R}\left(k^2 e^{-jkR}\right) + k^2 C\frac{e^{-jkR}}{R} = 0
\end{aligned}$$

(As required)

Verifying the Solution to the Spherical Wave Equation (Cont'd):

2. We cannot evaluate either side of the function at $R = 0$, but we can take its **volumetric integral** over a **sphere of radius a, centered at $R = 0$**, and then take the **limit as $a \to 0$**.

Here we use v (lower case) to represent the volume

(integration along θ and ϕ leads to this)

$$\int_{sph.} k^2 V \, dv = \int_{sph.} Ck^2 \frac{e^{-jkR}}{R} dv = Ck^2 \int_0^a \frac{e^{-jkR}}{R} (4\pi R^2) \, dR$$

$$= 4\pi Ck^2 \int_0^a R e^{-jkR} \, dR = 4\pi Ck^2 \left\{ \left[\frac{jR}{k} e^{-jkR} \right]_0^a - \int_0^a \frac{j}{k} e^{-jkR} \, dR \right\}$$

(integration by parts)

$$= j4\pi Cka e^{-jka} + 4\pi Ck^2 \left[\frac{1}{k^2} e^{-jka} \right]_0^a = j4\pi Cka e^{-jka} + 4\pi C\left(e^{-jka} - 1\right)$$

Here both terms approach zero as $a \to 0$.

Verifying the Solution to the Spherical Wave Equation (Cont'd):

We define $\mathbf{P} = -\nabla V$, which means $\nabla \cdot \mathbf{P} = -\nabla \cdot \nabla V = -\nabla^2 V$

Now $\mathbf{P} = -\nabla V = -\nabla\left(C\frac{e^{-jkR}}{R}\right) = C\frac{\partial}{\partial R}\left(\frac{e^{-jkR}}{R}\right)\hat{\mathbf{R}} = C\frac{e^{-jkR}}{R}\left(\frac{1}{R} + jk\right)\hat{\mathbf{R}}$

$$\int_{sph.} \nabla^2 V \, dv = -\int_{sph.} \nabla \cdot \mathbf{P} \, dv = -\oint_{surf} \mathbf{P} \cdot d\mathbf{s}$$

(Divergence theorem)

Since V is radially symmetric and decays with $\mathbf{R}$, $\mathbf{P}$ always points in the direction of $\hat{\mathbf{R}}$.

$$\ldots = -\oint_{surf} \mathbf{P} \cdot d\mathbf{s} = 4\pi a^2 |\mathbf{P}(R = a)| = 4\pi a^2 \left[C\frac{e^{-jka}}{a}\left(\frac{1}{a} + jk\right)\right]$$

$$= 4\pi C a e^{-jka}\left(\frac{1}{a} + jk\right) = 4\pi C e^{-jka} + j4\pi C k a e^{-jka}$$

So $\lim_{a\to 0} \int_{sph.} \nabla^2 V \, dv = 4\pi C$ due to contribution from the first term.

Verifying the Solution to the Spherical Wave Equation (Cont'd):

Combining the final results from Slides 10 – 12, we see that on the LHS,

$$\int_{sph.} \nabla^2 V + k^2 V \, dv = 4\pi C$$

Now for the RHS: $\int_{sph.} \frac{\delta^3(\mathbf{R})}{\varepsilon_0} \, dv = 1/\varepsilon_0$ (property of the delta function).

Equating both sides, we see that $4\pi C = \frac{1}{\varepsilon_0} \Rightarrow C = \frac{1}{4\pi\varepsilon_0}$, i.e.

$$\nabla^2 V + k^2 V = -\frac{\delta^3(\mathbf{R})}{\varepsilon_0} \Rightarrow \; V = \frac{1}{4\pi\varepsilon_0}\left(\frac{e^{-jkR}}{R}\right)$$

Similar mathematics apply for the A equations:

$$\nabla^2 A_n + k^2 A_n = -\mu_0 \delta^3(\mathbf{R}) \Rightarrow \; A_n = \frac{\mu_0}{4\pi}\left(\frac{e^{-jkR}}{R}\right)$$

Where $n = x, y$ or z.

Solution to the Spherical Wave Equation:

3. Now that we have obtained the "Green's functions" to both differential equations, we can write the general answers to the differential equations by convolving the Green's functions with the current/charge profile:

$$\nabla^2 V + k^2 V = \frac{\rho(\mathbf{r})}{\varepsilon} \quad \Rightarrow \quad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{e^{-jkR}}{R} \rho(\mathbf{r}')\, d\mathbf{r}'$$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}(\mathbf{r}) \quad \Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{v'} \frac{e^{-jkR}}{R} \mathbf{J}(\mathbf{r}')\, d\mathbf{r}'$$

Where again, $R = |\mathbf{r} - \mathbf{r}'|$, and $\mathbf{r}'$ and v' denote the (current/voltage) source regions.

A(r) and V(r) are called the **retarded magnetic and electric potentials** respectively.

Electromagnetic Potential - Summary

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

Electromagnetic Potential
$V, \mathbf{A}$

Electromagnetic Sources
$\rho, \mathbf{J}$

$$\nabla \times \mathbf{A} = \mathbf{B}$$

$$\nabla \cdot \boldsymbol{A} = -\mu\varepsilon \frac{\partial V}{\partial t}$$

$$-\nabla V = \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$$

Electromagnetic Fields
$\mathbf{E}, \mathbf{H}, \mathbf{B}, \mathbf{D}$

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Obtaining EM Potentials from Sources

1. Potential Equations in Time Domain

$$\nabla^2 \mathbf{A}(\mathbf{r},t) - \mu\varepsilon \frac{\partial^2 \mathbf{A}(\mathbf{r},t)}{\partial t^2} = -\mu \mathbf{J}(\mathbf{r})$$

$$\nabla^2 V(\mathbf{r},t) - \mu\varepsilon \frac{\partial^2 V(\mathbf{r},t)}{\partial t^2} = -\frac{\rho(\mathbf{r})}{\varepsilon}$$

2. Corresponding Phasors

$$\nabla^2 \mathbf{A}(\mathbf{r}) + k^2 \mathbf{A}(\mathbf{r}) = -\mu \mathbf{J}(\mathbf{r})$$

$$\nabla^2 V(\mathbf{r}) + k^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon}$$

$$(k = \omega\sqrt{\mu\varepsilon})$$

3. Solutions to Differential Equations

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dV'$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dV'$$

4. Back to Electromagnetic Field (Phasors)

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla V - j\omega \mathbf{A}$$

Example 1

6.29 A Hertzian dipole is a short conducting wire carrying an approximately constant current over its length l. If such a dipole is placed along the z-axis with its midpoint at the origin, and if the current flowing through it is $i(t) = I_0 \cos \omega t$, find

(a) The retarded vector potential $\mathbf{A}$

(b) Find the corresponding $\mathbf{B}$ field

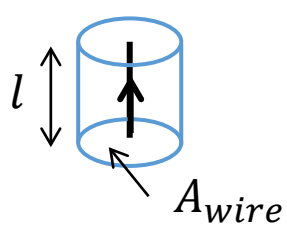
Solution:

First find the current density phasor: $I(\mathbf{r}) = I_0$

Then, write $\mathbf{J}(\mathbf{r})$: In this question the current density concentrates near the origin, so $\mathbf{J}(\mathbf{r}) \cong C_0 \delta^3(\mathbf{r})\, \hat{\mathbf{z}}$.

$$\Rightarrow \mathbf{A}(\mathbf{r}) \cong \frac{\mu}{4\pi} \int_{V'} \frac{C_0 \delta^3(\mathbf{r}') e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \hat{\mathbf{z}}\, dV' = \frac{\mu C_0}{4\pi} \frac{e^{-jk|\mathbf{r}|}}{|\mathbf{r}|} \hat{\mathbf{z}}$$

But what is the value of C_0? It represents the value obtained by integrating the current density over the (infinitesimally) small volume where it is non-zero (i.e. the conducting wire): $C_0 = J \times A_{wire} \times l = I_0 l$. Substitution into the above gives:



$$\Rightarrow \mathbf{A}(\mathbf{r}) \cong \frac{\mu I_0 l}{4\pi r} e^{-jk \cdot r}\, \hat{\mathbf{z}} \qquad \text{(where } r = |\mathbf{r}|\text{)}$$

Example 1 Cont'd

(b) Find the corresponding **B** field

Solution:

First find the current density phasor: $I(\mathbf{r}) = I_0$

Then, write $\mathbf{J}(\mathbf{r})$: In this question the current density concentrates near the origin, so $\mathbf{J}(\mathbf{r}) \cong C_0 \delta^3(\mathbf{r})\, \hat{\mathbf{z}}$.

$$\Rightarrow \mathbf{A}(\mathbf{r}) \cong \frac{\mu}{4\pi} \int_{V'} \frac{C_0 \delta^3(\mathbf{r}') e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \hat{\mathbf{z}}\, dV' = \frac{\mu C_0}{4\pi} \frac{e^{-jk|\mathbf{r}|}}{|\mathbf{r}|} \hat{\mathbf{z}}$$

But what is the value of C_0? It represents the value obtained by integrating the current density over the (infinitesimally) small volume where it is non-zero (i.e. the conducting wire): $C_0 = J \times A_{wire} \times l = I_0 l$. Substitution into the above gives:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \cong \nabla \times \frac{\mu I_0 l}{4\pi r} e^{-jkr}\, \hat{\mathbf{z}} = \frac{\partial}{\partial y}\left(\frac{\mu I_0 l}{4\pi r} e^{-jkr}\right) \hat{\mathbf{x}} - \frac{\partial}{\partial x}\left(\frac{\mu I_0 l}{4\pi r} e^{-jkr}\right) \hat{\mathbf{y}}$$

Notice that $\mathbf{B}(\mathbf{r})$ points orthogonal to $\mathbf{A}(\mathbf{r})$ and $\mathbf{r}$. For a general case:

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi} \int_{C'} \frac{I(\mathbf{r}')\, d\boldsymbol{l} \times \hat{\mathbf{R}}}{R^2} e^{-jkR}}$$

(Time-varying form of Biot-Savart Law)

Summary

- In this note we
  - Defined the EM potentials (Lorentz Gauge)
  - Derived wave equations for EM potentials
  - Found EM fields from known sources using EM potentials
  - Derived the Biot-Savart Law from Maxwell's Equations