

EE 3008. Random Process Tutorial Solutions

Q1) All possible outcomes are listed below,

HHH HHT HTH HTT THH THT TTH TTT

There three events with two heads and one tail, i.e., HHT HTH THH

Therefore the probability is 3/8.

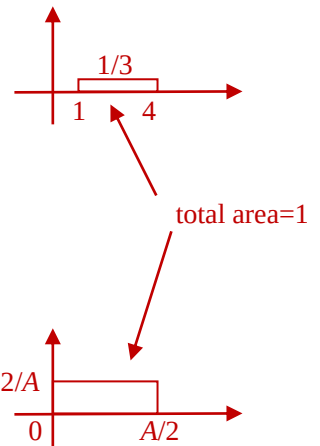
$$\text{Q2)} \quad \Pr\{0.5 < x < 3/4\} = \int_{0.5}^{3/4} f(x)dx = \int_{0.5}^{3/4} 2x dx = \left(\frac{3}{4}\right)^2 - 0.5^2 = \frac{5}{16}$$

$$\Pr\{-0.5 < x < 0.5\} = \int_{-0.5}^{0.5} f(x)dx = \int_{-0.5}^{0.5} 2x dx = (0.5)^2 - 0 = 0.25$$

Q3) Note that the range for X is 4-1=3. Therefore the PDF value should be 1/3 within [1, 4] so as to make the total area = 1.

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{3} \int_1^4 x dx = \frac{1}{3} \frac{4^2 - 1^2}{2} = 2.5$$

$$\text{var}(X) = E(X^2) - (E(x))^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - 2.5^2 = \frac{1}{3} \int_1^4 x^2 dx - 2.5^2 = 0.75$$

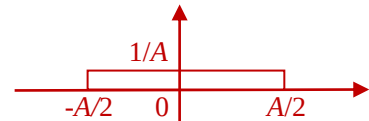


$$\text{Q4)} \quad \bar{\varepsilon} = \frac{2}{A} \int_0^{A/2} \varepsilon d\varepsilon = \frac{1}{A} (\varepsilon^2) \Big|_0^{A/2} = \frac{A}{4}$$

$$\overline{\varepsilon^2} = \frac{2}{A} \int_0^{A/2} \varepsilon^2 d\varepsilon = \frac{2}{A} \left(\frac{\varepsilon^3}{3}\right) \Big|_0^{A/2} = \frac{A^2}{12}$$

$$\text{Q5)} \quad \bar{\varepsilon} = \frac{1}{A} \int_{-A/2}^{A/2} \varepsilon d\varepsilon = \frac{1}{2A} (\varepsilon^2) \Big|_{-A/2}^{A/2} = 0$$

$$\overline{\varepsilon^2} = \frac{1}{A} \int_{-A/2}^{A/2} \varepsilon^2 d\varepsilon = \frac{1}{A} \left(\frac{\varepsilon^3}{3}\right) \Big|_{-A/2}^{A/2} = \frac{A^2}{12}$$



Q6) (a) 0.477, (b) 0.818 and (c) 0.998.

Q7) (a) Compare with a Gaussian PDF $p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$, we can see that

mean $m = -1$ and variance $\sigma^2 = 9/2$ (i.e., $\sigma = 2.12$).

(b) Since mean = -1, we have $\Pr\{x \geq -1\} = 0.5$.

(c) Refer to the distribution plot:



$$\Pr(x \leq -4) = \Pr(x \geq 2) = Q\left(\frac{(2 - (-1))}{2.12}\right) = Q(1.4151) \approx 0.0785$$

$$\Pr(x \leq 0.8) = 1 - \Pr(x \geq 0.8) = 1 - Q\left(\frac{(0.8 - (-1))}{2.12}\right) = 1 - Q(0.8491) \approx 1 - 0.1979 = 0.8021$$

$$\Pr\{-4 \leq x \leq 0.8\} = \Pr\{x \leq 0.8\} - \Pr\{x \leq -4\} = 0.8021 - 0.0785 = 0.7236.$$

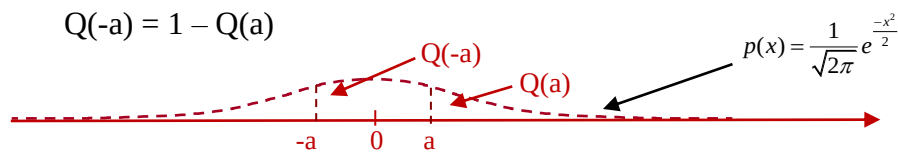
Q8) (i) $\Pr\{X>177\} = Q((177-165)/5) = Q(12/5) = Q(2.4) = 0.0082$.

(ii) We can do the question in the following way.

$$\Pr\{X<162\} = 1 - \Pr\{X>162\} = 1 - Q((162-165)/5) = 1 - Q(-0.6).$$

However, the table of $Q(x)$ only is only given for $x \geq 0$. We use the following useful equation:

$$Q(-a) = 1 - Q(a)$$



Thus $\Pr\{X<162\} = 1 - Q(-0.6) = Q(0.6) = 0.2742$.

(iii) $\Pr\{X>162\} = 1 - \Pr\{X<162\} = 1 - 0.2742 = 0.7258$.

Q9) Now $\tau=0.01$. From the lecture note, η is a Gaussian random variable with $m=0$ and $\sigma_\eta^2 = \tau\sigma^2 = 0.04$ (i.e., $\sigma_\eta=0.2$). $\Pr\{\eta>1\} = Q((1-m)/\sigma_\eta) = Q(5) = 2.9 \times 10^{-7}$.

Q10) Now $\tau=0.01$. From the lecture note, η is a Gaussian random variable with $m=0$ and $\sigma_\eta^2 = \tau\sigma^2 = 0.09$. Thus $\sigma_\eta=0.3$. $\Pr\{\eta>1\} = Q((1-m)/0.3) = Q(3.3) = 4.8 \times 10^{-4}$.

Comparing Q9 and Q10, we can see that a larger σ results in a larger tail area.



Q11) Now $\tau=0.01$. The integration of the constant -6 is $-6\tau = -0.06$. Thus $z = -0.06 + \eta$ where η is a Gaussian random variable with $m = 0$ and $\sigma_\eta^2 = \tau\sigma^2 = 0.04$ (i.e., $\sigma_\eta=0.2$). Then z is a random Gaussian variable with $m = -0.06$ and $\sigma_z = \sigma_\eta = 0.2$.



We then have $\Pr\{z>0\} = Q((0 - (-0.06))/\sigma_z) = Q(0.06/0.2) = Q(0.3) = 0.382$.

Q12) Now $\tau=0.01$. The integration of the constant -50 is $-50\tau = -0.5$. Thus $z = -0.5 + \eta$ where η is a Gaussian random variable with $m = 0$ and $\sigma_\eta^2 = \tau\sigma^2 = 0.04$ (i.e., $\sigma_\eta=0.2$). Then z is a random Gaussian variable with $m = -0.5$ and $\sigma_z = \sigma_\eta = 0.2$. We then have

$$\Pr\{z>0\} = Q((0 - (-0.5))/\sigma_z) = Q(0.5/0.2) = Q(2.5) = 0.0062.$$



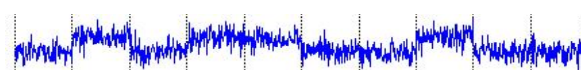
The effect is that the negative constant offset in z becomes larger. Thus the probability of $\Pr\{z>0\}$ becomes smaller.

Q13) Now $\tau=0.04$. The integration of the constant -50 is $-50\tau = -2$. Thus $z = -2 + \eta$ where η is a Gaussian random variable with $m = 0$ and $\sigma_\eta^2 = \tau\sigma^2 = 0.16$ (i.e., $\sigma_\eta=0.4$). Then z is a random Gaussian variable with $m = -2$ and $\sigma_z = \sigma_\eta = 0.4$. We then have

$$\Pr\{z>0\} = Q((0 - (-2))/0.4) = Q(2/0.4) = Q(5) = 2.9 \times 10^{-7}.$$



Note: Compared with Q12, the negative constant offset is increased by 4 times and noise deviation σ_η is increased by only 2 times. The effect of increasing integration period is that $\Pr\{z>0\}$ becomes smaller.



$$\text{Q14)} \quad \Pr\{1 \rightarrow -1\} = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-1)^2}{2\sigma^2}} dx \stackrel{y=\frac{x-1}{\sigma}}{=} \int_{-\infty}^{-1/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \int_{1/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = Q(1/\sigma)$$

$\Pr\{-1 \rightarrow 1\} = \Pr\{1 \rightarrow -1\}$. Overall error probability = $Q(1/\sigma)$.

Q15) (a) All events are listed below with probabilities (conditional on $x=-1$ or $+1$).

$$\xi = \begin{cases} -2, & \text{with Pr}=0.2 \\ 0, & \text{with Pr}=0.6 \\ +2, & \text{with Pr}=0.2 \end{cases} \quad \text{Lead to errors.}$$

	$y=-3$	$y=-1$	$y=+1$	$y=+3$
$x=-1$	Pr=0.1	Pr=0.3	Pr=0.1	Pr=0
$x=+1$	Pr=0	Pr=0.1	Pr=0.3	Pr=0.1

For example, the probability of the joint event of $y=+1$ and $x=-1$ is

$$\Pr = 0.5 \times 0.2 = 0.1$$

since $\Pr=0.5$ for $x=-1$ and $\Pr=0.2$ for $\xi=+2$.

Similarly, the probability of the joint event of $y=-1$ and $x=+1$ is

$$\Pr = 0.5 \times 0.2 = 0.1.$$

The above two situations lead to errors. The total error probability is
error probability = $0.1 + 0.1 = 0.2$.

(b) Now

$$\xi = \begin{cases} -2, & \text{with Pr}=0.4 \\ 0, & \text{with Pr}=0.2 \\ +2, & \text{with Pr}=0.4 \end{cases} \quad \text{Lead to errors.}$$

	$y=-3$	$y=-1$	$y=+1$	$y=+3$
$x=-1$	Pr=0.2	Pr=0.1	Pr=0.2	Pr=0
$x=+1$	Pr=0	Pr=0.2	Pr=0.1	Pr=0.2

For example, the probability of the joint event of $y=+1$ and $x=-1$ is

$$\Pr = 0.5 \times 0.4 = 0.2.$$

Similarly, the probability of the joint event of $y=-1$ and $x=+1$ is

$$\Pr = 0.5 \times 0.4 = 0.2.$$

The above two situations lead to errors. The total error probability is
error probability = $0.2 + 0.2 = 0.4$.

(c) We can do better with the following detection strategy.

$$\hat{x} = \begin{cases} -1, & \text{if } y=-3 \text{ or } +1 \\ +1, & \text{if } y=+3 \text{ or } -1 \end{cases} \quad \text{Lead to errors.}$$

The error events are highlighted below.

	$y=-3$	$y=-1$	$y=+1$	$y=+3$
$x=-1$	Pr=0.2	Pr=0.1	Pr=0.2	Pr=0
$x=+1$	Pr=0	Pr=0.2	Pr=0.1	Pr=0.2

The total error probability is now $0.1+0.1=0.2$. This is smaller than that for b).

This example shows that threshold detection using a middle point (such as 0) is not necessarily optimal for a binary system.