

EE6617: 09 Composite Hypothesis Testing

Major references:

- *Fundamentals of Statistical Processing, Volume II: Deection Theory*, 1/E, by Steven M. Kay, Prentice Hall, 1993
- *Statistical Signal Processing*, 1/E, by Louis L. Scharf, Prentice Hall, 1991

So far, we consider the scenario when the **PDF of likelihood function are completely known**.

How do we detect if some of parameters is unknown. In many of practical problems, often we try to detect a signal in noise with **unknown parameters**.

Typical examples includes

- Sine wave with unknown phase

$$s[n] = \sin(2\pi f_0 n + \underbrace{\phi}_{\text{unknown}}).$$

- DC value with unknown level

$$s[n] = \underbrace{A}_{\text{unknown}}.$$

1 Neyman-Pearson Approach: **Uniform Most Powerful (UMP) Test**

For NP test, we normally compute

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} \underset{0}{\overset{1}{\geq}} \gamma$$

for $P_{FA} = \alpha$ to maximize P_D .

However, we now have densities

$$p(\mathbf{x}; \theta_0, \mathcal{H}_0) \text{ and } p(\mathbf{x}; \theta_1, \mathcal{H}_1)$$

with **θ_0 and θ_1 are unknown**. The **$p(\mathbf{x}; \theta_i, \mathcal{H}_i)$ is the density of $\mathbf{x}$ with θ_i under $\mathcal{H}_i$** .

Now,

$$\begin{aligned} P_{FA}(\theta_0 = \theta) &= \int_{\mathcal{R}_1} p(\mathbf{x}; \theta_0 = \theta, \mathcal{H}_0) d\mathbf{x} \\ P_D(\theta_1 = \theta) &= \int_{\mathcal{R}_1} p(\mathbf{x}; \theta_1 = \theta, \mathcal{H}_1) d\mathbf{x}. \end{aligned}$$

Unknown θ_0 and θ_1 prevent us from designing robust detector.

However, the dependency on nuisance parameters would be **illusory** if the detector is a **uniformly most powerful (UMP) detector**.

Definition 1 We call a test **UMP test if the detector maximizes $P_D(\theta_1)$ for every possible θ_1 for a given $P_{FA} = \alpha$.**

Example 1: Unknown Amplitude

Consider binary hypothesis

$$\begin{aligned}\mathcal{H}_0 : \quad & \mathbf{x} = \mathbf{w} \\ \mathcal{H}_1 : \quad & \mathbf{x} = \mathbf{1} \underbrace{A}_{\text{unknown}} + \mathbf{w}\end{aligned}$$

where $\mathbf{x}, \mathbf{w} \in \mathbb{R}^N$, $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$. The amplitude A is unknown.

The ratio testing returns

$$\begin{aligned}L(\mathbf{x}) &= \exp\left(-\frac{1}{2\sigma^2} (\|\mathbf{x} - \mathbf{1}A\|^2 - \|\mathbf{x}\|^2)\right) \gtrless_0^1 \gamma \\ l(\mathbf{x}) &= -\frac{1}{2\sigma^2} (\|\mathbf{x} - \mathbf{1}A\|^2 - \|\mathbf{x}\|^2) \gtrless_0^1 \ln(\gamma).\end{aligned}$$

If we know A , we can always design NP detector. But, A is unknown.

Q: Can we still design NP detector, i.e., UMP detector?

For this particular problem, perfect knowledge of A is not necessary. **Partial knowledge of A** , either $A > 0$ or $A \leq 0$ is enough to design UMP detector.

To see this, assume $A > 0$. Then, the detector is

$$T(\mathbf{x}) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \gtrless_0^1 \gamma' \left(= \frac{\sigma^2 \ln(\gamma)}{NA} + \frac{A}{2} \right)$$

Given the statistics of $T(\mathbf{x}) \sim \mathcal{N}(\mathbf{0}, \sigma^2/N)$ under $\mathcal{H}_0$,

$$\begin{aligned}P_{FA} &= \Pr(T(\mathbf{x}) \geq \gamma'; \mathcal{H}_0) \\ &= \int_{\gamma'}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2/N}} \exp\left(-\frac{1}{2\sigma^2/N} t^2\right) dt \\ &= \int_{\frac{\gamma'}{\sqrt{\sigma^2/N}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \\ &= Q\left(\frac{\gamma'}{\sqrt{\sigma^2/N}}\right) = \alpha.\end{aligned}$$

Then, we decide γ' such that $\frac{\gamma'}{\sqrt{\sigma^2/N}} = Q^{-1}(\alpha)$ and $\gamma' = \sqrt{\sigma^2/N}Q^{-1}(\alpha)$. **Notice that the test statistic $T(\mathbf{x})$ is completely independent of A .**

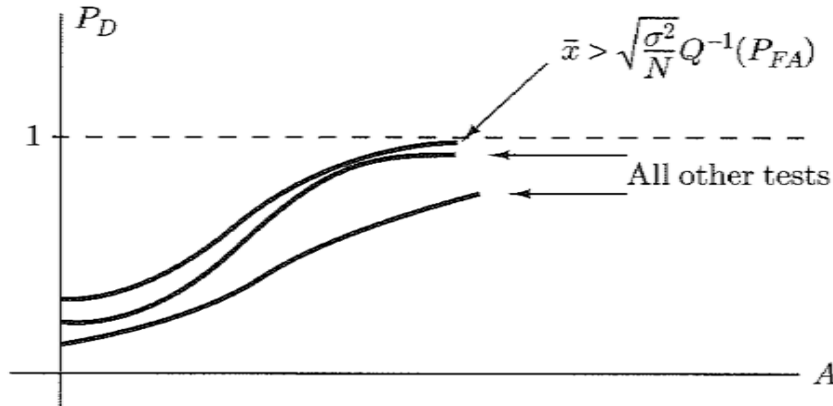
Using γ' and $T(\mathbf{x}) \sim \mathcal{N}(1A, \sigma^2/N)$ under $\mathcal{H}_1$,

$$\begin{aligned} P_D &= \Pr(T(\mathbf{x}) \geq \gamma'; \mathcal{H}_1) \\ &= \int_{\gamma'}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2/N}} \exp\left(-\frac{1}{2\sigma^2/N}(t - A)^2\right) dt \\ &= Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2/N}}\right) = Q\left(Q^{-1}(\alpha) - \frac{A}{\sqrt{\sigma^2/N}}\right). \end{aligned}$$

The NP theorem tells us that the optimal detector is

$$\frac{1}{N} \sum_{n=0}^{N-1} x[n] \geq_0^1 \sqrt{\sigma^2/N}Q^{-1}(\alpha). \quad (1)$$

Q: Is the detector in (1) is the UMP test? ■



The UMP detector **exists** and is equivalent to the NP detector (**Karlin-Rubin's Theorem**),

- if the test is **one-sided** (i.e., if the test statistic $T(\mathbf{x})$ is a nondecreasing function of observations) and
- if the $T(\mathbf{x})$ under $\mathcal{H}_0$ is **independent of the unknown parameters**.

2 Generalized Likelihood Ratio Test (GLRT)

Unfortunately, most problems can not be solved with a UMP test. In this case, how can we design detector? There is a **robust detector** design criterion.

Alternatively, consider a test

$$L_G(\mathbf{x}) = \frac{\max_{\theta_1} p(\mathbf{x}; \theta_1, \mathcal{H}_1)}{\max_{\theta_0} p(\mathbf{x}; \theta_0, \mathcal{H}_0)} \underset{0}{\overset{1}{\geq}} \gamma.$$

This approach is called **generalized likelihood ration test (GLRT)**.

Unlike UMP, **we can not argue optimality**, but in the absence of straightforward framework, this provides a **robust detector**.

Example 2: BPSK without Prior Distribution

Consider detecting a noisy DC signal is positive or negative

$$\mathbf{x} = \mathbf{1} \underbrace{A}_{\text{unknown}} + \mathbf{w}, \quad \mathbf{x}, \mathbf{w} \in \mathbb{R}^N, \quad \mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}).$$

The hypotheses are

$$\begin{aligned} \mathcal{H}_0 : & \quad A \leq 0, \quad \theta_0 = A \in (-\infty, 0] \\ \mathcal{H}_1 : & \quad A \geq 0, \quad \theta_1 = A \in [0, \infty) \end{aligned}.$$

Thus,

$$\begin{aligned} p(\mathbf{x}; \theta_0, \mathcal{H}_0) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{x} - \mathbf{1}\theta_0\|^2\right) \\ p(\mathbf{x}; \theta_1, \mathcal{H}_1) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{x} - \mathbf{1}\theta_1\|^2\right). \end{aligned}$$

Solve for

$$\begin{aligned} \hat{\theta}_0 &= \underset{\theta_0 \in (-\infty, 0]}{\operatorname{argmax}} \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{x} - \mathbf{1}\theta_0\|^2\right) \\ &= \underset{\theta_0 \in (-\infty, 0]}{\operatorname{argmin}} \underbrace{\|\mathbf{x} - \mathbf{1}\theta_0\|^2}_{f(\theta_0) = \|\mathbf{x}\|^2 - 2\mathbf{x}^T \mathbf{1}\theta_0 + \theta_0^2 N}. \end{aligned}$$

Minimization can be attempted to be found by

$$\frac{\partial f(\theta_0)}{\partial \theta_0} = -2\mathbf{x}^T \mathbf{1} + 2\theta_0 N = 0 \Rightarrow \hat{\theta}_0 = \frac{1}{N} \mathbf{x}^T \mathbf{1} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \quad \text{when} \quad \frac{1}{N} \sum_{n=0}^{N-1} x[n] \leq 0$$

Notice that **if** $\frac{1}{N} \sum_{n=0}^{N-1} x[n] > 0$, $\frac{\partial f(\theta_0)}{\partial \theta_0} = 0$ **can not be found**. Then, since $f(\theta_0)$ is convex, minimizer is a boundary 0. Therefore,

$$\hat{\theta}_0 = \min\left(0, \frac{1}{N} \sum_{n=0}^{N-1} x[n]\right).$$

Similarly,

$$\hat{\theta}_1 = \underset{\theta_1 \in [0 \ \infty)}{\operatorname{argmin}} \underbrace{\|\mathbf{x} - \mathbf{1}\theta_1\|^2}_{f(\theta_1)}$$

and with the similar reason, we have

$$\hat{\theta}_1 = \max \left(0, \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right).$$

Using both

$$\begin{aligned} L_G(\mathbf{x}) &= \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp \left(-\frac{1}{2\sigma^2} \|\mathbf{x} - \mathbf{1}\hat{\theta}_0\|^2 \right)}{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp \left(-\frac{1}{2\sigma^2} \|\mathbf{x} - \mathbf{1}\hat{\theta}_1\|^2 \right)} \\ &= \exp \left(-\frac{1}{2\sigma^2} \left(\|\mathbf{x} - \mathbf{1}\hat{\theta}_1\|^2 - \|\mathbf{x} - \mathbf{1}\hat{\theta}_0\|^2 \right) \right). \end{aligned}$$

$$\begin{aligned} \ln(L_G(\mathbf{x})) &= -\frac{1}{2\sigma^2} \left(\|\mathbf{x} - \mathbf{1}\hat{\theta}_1\|^2 - \|\mathbf{x} - \mathbf{1}\hat{\theta}_0\|^2 \right) \\ &= -\frac{1}{2\sigma^2} \left(-2\hat{\theta}_1 \mathbf{x}^T \mathbf{1} + 2\hat{\theta}_0 \mathbf{x}^T \mathbf{1} + \hat{\theta}_1^2 N - \hat{\theta}_0^2 N \right) \\ &= (\hat{\theta}_1 - \hat{\theta}_0) \frac{\mathbf{x}^T \mathbf{1}}{\sigma^2} + (\hat{\theta}_0^2 - \hat{\theta}_1^2) \frac{N}{2\sigma^2} \\ &= \left(\max \left(0, \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right) - \min \left(0, \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right) \right) \frac{\mathbf{x}^T \mathbf{1}}{\sigma^2} \\ &\quad + \left(\min \left(0, \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right)^2 - \max \left(0, \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right)^2 \right) \frac{N}{2\sigma^2} \\ &= \left| \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right| \frac{\mathbf{x}^T \mathbf{1}}{\sigma^2} - \operatorname{sgn} \left(\frac{1}{N} \sum_{n=0}^{N-1} x[n] \right) \left(\frac{1}{N} \sum_{n=0}^{N-1} x[n] \right)^2 \frac{N}{2\sigma^2} \geqslant_0^1 \gamma' \end{aligned}$$

Can solve for γ' to give specific P_{FA} . ■

Example 3: Unknown Deterministic Signal

Detecting totally unknown signal vector $\mathbf{s} \in \mathbb{R}^N$. The hypotheses are

$$\begin{aligned} \mathcal{H}_0 : \mathbf{x} &= \mathbf{w} \\ \mathcal{H}_1 : \mathbf{x} &= \underbrace{\mathbf{s}}_{\text{unknown}} + \mathbf{w} . \end{aligned}$$

where $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$.

Let's use **GLRT**

$$L_G(\mathbf{x}) = \frac{\max_{\mathbf{s}} p(\mathbf{x}; \mathbf{s}, \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} \gtrless_0^1 \gamma.$$

Need to solve

$$\begin{aligned} \hat{\mathbf{s}} &= \operatorname{argmax}_{\mathbf{s} \in \mathbb{R}^N} \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{x} - \mathbf{s}\|^2\right) \\ &= \operatorname{argmin}_{\mathbf{s} \in \mathbb{R}^N} \underbrace{\|\mathbf{x} - \mathbf{s}\|^2}_{f(\mathbf{s})} \end{aligned}$$

Solve to find the minimizer of $f(\mathbf{s})$

$$\begin{aligned} \Rightarrow \quad & \frac{\partial f(\mathbf{s})}{\partial \mathbf{s}} = 0 \\ \Rightarrow \quad & \hat{\mathbf{s}} = \mathbf{x} \end{aligned}$$

Plugging in $L_G(\mathbf{x})$,

$$L_G(\mathbf{x}) = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{x} - \mathbf{x}\|^2\right)}{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{x}\|^2\right)} = \exp\left(\frac{1}{2\sigma^2} \|\mathbf{x}\|^2\right)$$

Right now

$$\begin{aligned} \exp\left(\frac{1}{2\sigma^2} \|\mathbf{x}\|^2\right) &\gtrless_0^1 \gamma \\ \Rightarrow \|\mathbf{x}\|^2 &\gtrless_0^1 \gamma' \end{aligned} \tag{2}$$

Pick γ' such that $\Pr(\|\mathbf{x}\|^2 > \gamma'; \mathcal{H}_0) = \alpha$.

Example 4: Unknown Signal Amplitude

$$\begin{aligned} \mathcal{H}_0 : \quad & \mathbf{x} = \mathbf{w} \\ \mathcal{H}_1 : \quad & \mathbf{x} = \mathbf{s} \underbrace{A}_{\text{unknown}} + \mathbf{w} \end{aligned}$$

where $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ and A is unknown.

Note that as seen from Example 1, **for either cases of $A > 0$ or $A \leq 0$ one can design UMP detector**. If the only restriction is $A \in \mathbb{R}$, we must rely on GLRT

$$\begin{aligned} p(\mathbf{x}; \mathcal{H}_0) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{x}\|^2\right) \\ p(\mathbf{x}; A, \mathcal{H}_1) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{x} - \mathbf{s}A\|^2\right). \end{aligned}$$

Solve for

$$\begin{aligned}\hat{A} &= \underset{A}{\operatorname{argmax}} p(\mathbf{x}; A, \mathcal{H}_1) \\ &= \underset{A}{\operatorname{argmin}} \underbrace{\|\mathbf{x} - \mathbf{s}A\|^2}_{f(A)}.\end{aligned}$$

Since the objective function is convex, try to apply the first order condition

$$\frac{\partial f(A)}{\partial A} = -2\mathbf{x}^T \mathbf{s} + 2A \|\mathbf{s}\| = 0 \Rightarrow \hat{A} = \frac{\mathbf{x}^T \mathbf{s}}{\|\mathbf{s}\|^2}.$$

Thus,

$$\begin{aligned}L_G(\mathbf{x}) &= \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \left\|\mathbf{x} - \frac{\mathbf{x}^T \mathbf{s}}{\|\mathbf{s}\|^2} \mathbf{s}\right\|^2\right)}{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \|\mathbf{x}\|^2\right)} \\ &= \exp\left(-\frac{1}{2\sigma^2} \left(-\frac{(\mathbf{x}^T \mathbf{s})^2}{\|\mathbf{s}\|^2}\right)\right)\end{aligned}$$

Then,

$$(\mathbf{x}^T \mathbf{s})^2 \gtrless_0^1 \gamma'.$$

Let $T(\mathbf{x}) = \mathbf{x}^T \mathbf{s}$ and the decision is

- If $\mathbf{x}^T \mathbf{s} > \sqrt{\gamma'}$ or $\mathbf{x}^T \mathbf{s} < -\sqrt{\gamma'}$, declare $\mathcal{H}_1$
- If $-\sqrt{\gamma'} \leq \mathbf{x}^T \mathbf{s} \leq \sqrt{\gamma'}$, declare $\mathcal{H}_0$

Under $\mathcal{H}_0$, $\mathbf{x}^T \mathbf{s} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \|\mathbf{s}\|^2)$,

$$\begin{aligned}P_{FA} &= \int_{\sqrt{\gamma'}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 \|\mathbf{s}\|^2}} \exp\left(-\frac{1}{2\sigma^2 \|\mathbf{s}\|^2} t^2\right) dt + \int_{-\infty}^{-\sqrt{\gamma'}} \frac{1}{\sqrt{2\pi\sigma^2 \|\mathbf{s}\|^2}} \exp\left(-\frac{1}{2\sigma^2 \|\mathbf{s}\|^2} t^2\right) dt \\ &= 2 \int_{\sqrt{\gamma'}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 \|\mathbf{s}\|^2}} \exp\left(-\frac{1}{2\sigma^2 \|\mathbf{s}\|^2} t^2\right) dt \\ &= 2Q\left(\frac{\sqrt{\gamma'}}{\sqrt{\sigma^2 \|\mathbf{s}\|^2}}\right) = \alpha.\end{aligned}$$

The threshold is $\gamma' = \sigma^2 \|\mathbf{s}\|^2 Q^{-1}(\alpha/2)^2$.

Under $\mathcal{H}_1$, $\mathbf{x}^T \mathbf{s} = A \mathbf{s}^T \mathbf{s} + \mathbf{w}^T \mathbf{s} \sim \mathcal{N}(A \|\mathbf{s}\|^2, \sigma^2 \|\mathbf{s}\|^2)$,

$$\begin{aligned}
P_D &= \int_{\sqrt{\gamma'}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 \|\mathbf{s}\|^2}} \exp\left(-\frac{1}{2\sigma^2 \|\mathbf{s}\|^2}(t - A \|\mathbf{s}\|^2)^2\right) dt + \int_{-\infty}^{-\sqrt{\gamma'}} \frac{1}{\sqrt{2\pi\sigma^2 \|\mathbf{s}\|^2}} \exp\left(-\frac{1}{2\sigma^2 \|\mathbf{s}\|^2}(t - A \|\mathbf{s}\|^2)^2\right) dt \\
&= \int_{\sqrt{\gamma'}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 \|\mathbf{s}\|^2}} \exp\left(-\frac{1}{2\sigma^2 \|\mathbf{s}\|^2}(t - A \|\mathbf{s}\|^2)^2\right) dt + \int_{\sqrt{\gamma'}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 \|\mathbf{s}\|^2}} \exp\left(-\frac{1}{2\sigma^2 \|\mathbf{s}\|^2}(t + A \|\mathbf{s}\|^2)^2\right) dt \\
&= Q\left(\frac{\sqrt{\gamma'} - A \|\mathbf{s}\|^2}{\sqrt{\sigma^2 \|\mathbf{s}\|^2}}\right) + Q\left(\frac{\sqrt{\gamma'} + A \|\mathbf{s}\|^2}{\sqrt{\sigma^2 \|\mathbf{s}\|^2}}\right)
\end{aligned}$$

3 Composite Hypothesis Testing: Bayesian Approach

Previously, we assume the **unknown deterministic signal**.

In many practical scenarios, the **unknown signal can be modeled as random variable/process** and we can, in this case, employ **Bayesian Approach** to design the detector.

Consider detecting a presence of a random signal.

$$\begin{aligned}
\mathcal{H}_0 : \quad & \mathbf{x} = \mathbf{w} \\
\mathcal{H}_1 : \quad & \mathbf{x} = \mathbf{s} + \mathbf{w}
\end{aligned}$$

where $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w)$, $\mathbf{s} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{C}_s)$, and $\mathbf{w}$ and $\mathbf{s}$ are independent.

Under $\mathcal{H}_0$,

$$p(\mathbf{x}; \mathcal{H}_0) = \frac{1}{(2\pi)^{N/2} \sqrt{\det(\mathbf{C}_w)}} \exp\left(-\frac{1}{2} \mathbf{x}^T \mathbf{C}_w^{-1} \mathbf{x}\right).$$

Under $\mathcal{H}_1$, $\mathbf{x}$ is also Gaussian with $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{C}_s + \mathbf{C}_w)$.

$$p(\mathbf{x}|\mathcal{H}_1) = \frac{1}{(2\pi)^{N/2} \sqrt{\det(\mathbf{C}_s + \mathbf{C}_w)}} \exp\left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} (\mathbf{x} - \boldsymbol{\mu})\right).$$

Thus,

$$\begin{aligned}
L(\mathbf{x}) &= \frac{p(\mathbf{x}|\mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} \\
&= \frac{\frac{1}{(2\pi)^{N/2} \sqrt{\det(\mathbf{C}_s + \mathbf{C}_w)}} \exp\left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)}{\frac{1}{(2\pi)^{N/2} \sqrt{\det(\mathbf{C}_w)}} \exp\left(-\frac{1}{2} \mathbf{x}^T \mathbf{C}_w^{-1} \mathbf{x}\right)} \\
\ln(L(\mathbf{x})) &= \underbrace{-\left((\mathbf{x} - \boldsymbol{\mu})^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} (\mathbf{x} - \boldsymbol{\mu}) - \mathbf{x}^T \mathbf{C}_w^{-1} \mathbf{x}\right)}_{f(\mathbf{x})} \geqslant_0^1 \gamma
\end{aligned}$$

$$\begin{aligned}
f(\mathbf{x}) &= -\mathbf{x}^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} \mathbf{x} + 2\mathbf{x}^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} \boldsymbol{\mu} - \boldsymbol{\mu}^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} \boldsymbol{\mu} + \mathbf{x}^T \mathbf{C}_w^{-1} \mathbf{x} \\
&= \mathbf{x}^T (\mathbf{C}_w^{-1} - (\mathbf{C}_s + \mathbf{C}_w)^{-1}) \mathbf{x} + 2\mathbf{x}^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} \boldsymbol{\mu} - \boldsymbol{\mu}^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} \boldsymbol{\mu}
\end{aligned}$$

By the matrix inversion lemma $(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{DA}^{-1}\mathbf{B} + \mathbf{C}^{-1})^{-1}\mathbf{DA}^{-1}$, we have

$$\begin{aligned}
(\mathbf{C}_s + \mathbf{C}_w)^{-1} &= \mathbf{C}_w^{-1} - \mathbf{C}_w^{-1} (\mathbf{C}_w^{-1} + \mathbf{C}_s^{-1})^{-1} \mathbf{C}_w^{-1} \\
\Rightarrow \quad \mathbf{C}_w^{-1} (\mathbf{C}_w^{-1} + \mathbf{C}_s^{-1})^{-1} \mathbf{C}_w^{-1} &= \mathbf{C}_w^{-1} - (\mathbf{C}_s + \mathbf{C}_w)^{-1}
\end{aligned}$$

and

$$T(\mathbf{x}) = \mathbf{x}^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} \boldsymbol{\mu} + \frac{1}{2} \mathbf{x}^T \mathbf{C}_w^{-1} (\mathbf{C}_w^{-1} + \mathbf{C}_s^{-1})^{-1} \mathbf{C}_w^{-1} \mathbf{x} \geqslant_0^1 \gamma'. \quad (3)$$

Example 5: Bayesian Approach

$$\begin{aligned}
\mathcal{H}_0 : \quad \mathbf{x} &= \mathbf{w} \\
\mathcal{H}_1 : \quad \mathbf{x} &= \mathbf{s}A + \mathbf{w}
\end{aligned}$$

where $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$, $A \sim \mathcal{N}(\mu_A, \sigma_A^2)$, and $\mathbf{w}$ and A are independent. $\mathbf{s}$ is known.

Under $\mathcal{H}_1$,

$$\begin{aligned}
E[\mathbf{x}] &= \mathbf{s}\mu_A \\
\text{var}(\mathbf{x}) &= \text{var}(A)\mathbf{s}\mathbf{s}^T + \sigma^2 \mathbf{I} = \sigma_A^2 \mathbf{s}\mathbf{s}^T + \sigma^2 \mathbf{I}.
\end{aligned}$$

By substituting $\mathbf{C}_s = \sigma_A^2 \mathbf{s}\mathbf{s}^T$, $\mathbf{C}_w = \sigma^2 \mathbf{I}$, and $\boldsymbol{\mu} = \mathbf{s}\mu_A$ into (3),

$$T(\mathbf{x}) = \mathbf{x}^T (\sigma_A^2 \mathbf{s}\mathbf{s}^T + \sigma^2 \mathbf{I})^{-1} \mathbf{s}\mu_A + \frac{1}{2\sigma^4} \mathbf{x}^T \left(\frac{1}{\sigma^2} \mathbf{I} + \frac{1}{\sigma_A^2} \mathbf{s}\mathbf{s}^T \right)^{-1} P\mathbf{x} \geqslant_0^1 \gamma'$$