

Test 1

Time allowed: 60 minutes

Instructions to candidates:

This paper has **EIGHT** pages (including this cover page).

A brief table of integrals is shown on page 2.

This paper has **TWO** questions.

The questions will begin on page 3.

Write your answers in the space provided under each question.

Answer ALL questions. Show ALL steps in order to get full marks.

Question		Marks
1	(a)	14
	(b)	20
	(c)	15
2	(a)	10
	(b)	20
	(c)	20
Total mark:		99

NOT TO BE TAKEN AWAY

Brief Table of Integrals

$\int du = u + C$	$\int u^p du = \frac{u^{p+1}}{p+1} + C, \quad p \neq -1$
$\int e^u du = e^u + C$	$\int \frac{1}{u} du = \ln u + C$
$\int \cos u du = \sin u + C$	$\int \sin u du = -\cos u + C$
$\int \sec^2 u du = \tan u + C$	$\int \csc^2 u du = -\cot u + C$
$\int \sec u \tan u du = \sec u + C$	$\int \csc u \cot u du = -\csc u + C$
$\int \sec u du = \ln \sec u + \tan u + C$	$\int \csc u du = -\ln \csc u + \cot u + C$
$\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + C$	$\int \frac{1}{1+u^2} du = \tan^{-1} u + C$

Question 1

Solve the following first order ordinary differential equations:

(a) $\frac{dy}{dx} = xy^3 \cos(x^2)$ given the condition $y(0) = 1$.

[14 marks]

$$y^{-3} dy = x \cos(x^2) dx$$

$$\int y^{-3} dy = \int x \cos x^2 dx$$

$$-\frac{1}{2} y^{-2} = \frac{1}{2} \sin(x^2) + C$$

$$u = x^2 \\ du = 2x dx$$

$$y(0) = 1$$

$$-\frac{1}{2} = \frac{1}{2} \sin(0) + C$$

$$C = -\frac{1}{2}$$

$$\therefore -\frac{1}{2} y^{-2} = \frac{1}{2} \sin(x^2) - \frac{1}{2}$$

$$y^{-2} = -\sin(x^2) + 1$$

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(b) $\frac{dy}{dx} = \frac{y^2 + x^2}{xy - 3x^2}$

[20 marks]

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^2 + 1}{\left(\frac{y}{x}\right) - 3}$$

Let $v = \frac{y}{x}$
 $vx = y$ ✓
 $\frac{dv}{dx}x + v = \frac{dy}{dx}$ ✓

$$\frac{dv}{dx}x + v = \frac{v^2 + 1}{v - 3} \quad \checkmark$$

$$\frac{dv}{dx}x = \frac{v^2 + 1 - v^2 + 3v}{v - 3}$$

$$\frac{dv}{dx}x = \frac{1 + 3v}{v - 3} \quad \checkmark$$

$$\int \frac{\frac{1}{3}(3v+1) - \frac{10}{3}}{3v+1} dv = \int \frac{1}{x} dx$$

$$\begin{aligned} & \frac{1}{3}(3v+1) - \frac{10}{3} \\ &= v + \frac{1}{3} - \frac{10}{3} \end{aligned}$$

$$\frac{1}{3}v - \frac{10}{9} \ln|3v+1| = \ln|x| + C \quad \checkmark$$

$$\frac{1}{3}\left(\frac{y}{x}\right) - \frac{10}{9} \ln\left|3\left(\frac{y}{x}\right) + 1\right| = \ln|x| + C \quad \checkmark$$

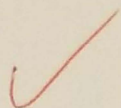
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(c) $x \frac{dy}{dx} + 2y = e^{3x}$

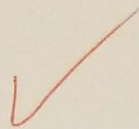
$$\frac{dy}{dx} + P(x)y = Q(x)$$

[16 marks]

$$\frac{dy}{dx} + 2yx^{-1} = x^{-1}e^{3x}$$



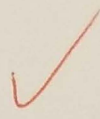
$$\text{Let } I(x) = e^{\int 2x^{-1} dx} = e^{\ln x^2} = x^2$$



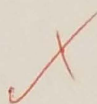
$$x^2 \frac{dy}{dx} + 2yx = xe^{3x}$$

$$\frac{d}{dx}[x^2 y] = xe^{3x}$$

$$x^2 y = \int xe^{3x} dx$$

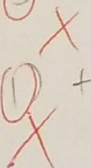


$$x^2 y = \frac{1}{3}e^{3x}x - \frac{1}{3}\int e^{3x} dx$$



$$x^2 y = \frac{1}{3}e^{3x}x - \frac{1}{3}e^{3x} + C$$

$$x^2 y = \frac{1}{3}e^{3x}(x - 1) + C$$



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Question 2

Solve the following higher order ordinary differential equations:

(a) $\frac{d^5 y}{dx^5} + 2\frac{d^4 y}{dx^4} + 5\frac{d^3 y}{dx^3} = 0$

[10 marks]

$$\alpha^5 + 2\alpha^4 + 5\alpha^3 = 0$$

$$\alpha^3(\alpha^2 + 2\alpha + 5) = 0$$

$$\alpha^3 = 0$$

$$\alpha^2 + 2\alpha + 5 = 0$$

$$\alpha = 0, 0, 0$$

$$\alpha^2 + 2\alpha + 1 + 4 = 0$$

$$(\alpha + 1)^2 = -4$$

$$\alpha + 1 = \pm 2i$$

$$\alpha = -1 \pm 2i$$

$$\therefore y = e^{x(0)} (Ax^2 + Bx + C) + e^{-x} (D \cos(2x) + E \sin(2x))$$

$$= Ax^2 + Bx + C + e^{-x} (D \cos(2x) + E \sin(2x))$$

where A, B, C, D and E are arbitrary constants.

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(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = xe^{3x}$ using the method of undetermined coefficients.

[20 marks]

$$\alpha^2 - 2\alpha - 3 = 0$$

$$\alpha = 3, -1$$

$$\therefore y_c = Ae^{3x} + Be^{-x}$$

$$\text{Let } y_p = x(ax+b)e^{3x}$$

$$y_p = ax^2e^{3x} + bxe^{3x}$$

$$\begin{aligned} y'_p &= 3ax^2e^{3x} + 2axe^{3x} + 3bxe^{3x} + be^{3x} \\ &= 3ax^2e^{3x} + (2a+3b)xe^{3x} + be^{3x} \end{aligned}$$

$$\begin{aligned} y''_p &= 9ax^2e^{3x} + 6axe^{3x} + (6a+9b)e^{3x}x + (2a+3b)e^{3x} + 3be^{3x} \\ &= 9ax^2e^{3x} + (12a+9b)xe^{3x} + (2a+6b)e^{3x} \end{aligned}$$

$$0 = 9ax^2e^{3x} + (12a+9b)xe^{3x} + (2a+6b)e^{3x} - 6ax^2e^{3x} - (4a+6b)xe^{3x} - 2be^{3x} - 3ax^2e^{3x} - 3bxe^{3x}$$

$$0 = (12a+9b-4a-6b-3b)x + (2a+6b-2b)$$

$$\begin{cases} 8a = 1 \\ 2a+4b = 0 \end{cases} \quad \begin{cases} a = \frac{1}{8} \\ b = -\frac{1}{16} \end{cases}$$

$$\therefore y_p = \frac{1}{8}x^2e^{3x} - \frac{1}{16}xe^{3x}$$

$$y = y_c + y_p$$

$$= Ae^{3x} + Be^{-x} + \frac{1}{8}x^2e^{3x} - \frac{1}{16}xe^{3x}$$

where A and B are any arbitrary constants

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(c) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = \frac{e^{2x}}{x^5}$ using the method of variation of parameters.

[20 marks]

$$\alpha^2 - 4\alpha + 4 = 0$$

$$\alpha = 2, 2$$

$$\therefore y_c = e^{2x} (Ax + B)$$

$$= Ae^{2x}x + Be^{2x}$$

$$\text{Let } y_p = h_1(x)e^{2x}x + h_2(x)e^{2x}$$

$$\begin{cases} h_1'(x)e^{2x}x + h_2'(x)e^{2x} = 0 & (1) \end{cases}$$

$$\begin{cases} h_1'(x)(2e^{2x}x + e^{2x}) + h_2'(x)2e^{2x} = \frac{e^{2x}}{x^5} & (2) \end{cases}$$

$$(2) - 2(1)$$

$$h_1'(x)(2e^{2x}x + e^{2x} - 2e^{2x}x) = \frac{e^{2x}}{x^5}$$

$$h_1'(x)e^{2x} = \frac{e^{2x}}{x^5}$$

$$h_1'(x) = x^{-5}$$

$$h_1(x) = \int x^{-5} dx$$

$$= -\frac{1}{4}x^{-4} + C_1$$

From (1)

$$(x^{-5})e^{2x}x + h_2'(x)e^{2x} = 0$$

$$h_2'(x) = -e^{2x}x^{-4}(e^{-2x})$$

$$h_2'(x) = -x^{-4}$$

$$h_2(x) = -\int x^{-4} dx$$

$$= \frac{1}{3}x^{-3} + C_2$$

Put C_1 and $C_2 = 0$.

$$y_p = -\frac{1}{4}x^{-4}e^{2x}x + \frac{1}{3}x^{-3}e^{2x} = \frac{1}{12}x^{-3}e^{2x}$$

$$\therefore y = y_c + y_p = Ae^{2x}x + Be^{2x} + \frac{1}{12}x^{-3}e^{2x}$$

- End of Test -

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