

Name: _____

Solutions

Student ID: _____

Answer ALL questions. (Full marks: 40)

Total Marks: _____

Q1: _____ Q2: _____ Q3: _____

Question 1 (10 marks)

Suppose $z^3 - 4yz + 4x^2 = 1$ determines the function $z = z(x, y)$ as a function of x, y locally at $(x, y, z) = (1, 1, 1)$.

- (i) Find the linear approximation of z at $(x, y, z) = (1, 1, 1)$. (6 marks)
 (ii) Find the quadratic surface approximation of z at $(x, y, z) = (1, 1, 1)$. (4 marks)

Solution: $z^3 - 4yz + 4x^2 = 1$

$$\Rightarrow \begin{cases} 3z^2 z_x - 4y z_x + 8x = 0 \\ 3z^2 z_y - 4z - 4y z_y = 0 \end{cases} \Rightarrow \begin{cases} z_x = \frac{-8x}{3z^2 - 4y} \\ z_y = \frac{4z}{3z^2 - 4y} \end{cases}$$

$$\Rightarrow \begin{cases} z_{xx} = \frac{-8(3z^2 - 4y) + 8x(6z \cdot z_x)}{(3z^2 - 4y)^2} \\ z_{xy} = \frac{8x(6z \cdot z_y - 4)}{(3z^2 - 4y)^2}, \quad z_{yx} = \frac{4z_x(3z^2 - 4y) - 4z \cdot (6z \cdot z_x)}{(3z^2 - 4y)^2} \\ z_{yy} = \frac{4z_y(3z^2 - 4y) - 4z \cdot (6z \cdot z_y)}{(3z^2 - 4y)^2} \end{cases}$$

(i) At $(x, y, z) = (1, 1, 1)$,

$$\begin{cases} z_x(1, 1) = 8 \\ z_y(1, 1) = -4 \end{cases}$$

$$\begin{aligned} \text{Hence, } z(x, y) &\approx z(1, 1) + z_x(1, 1)(x-1) + z_y(1, 1)(y-1) \\ &= 1 + 8(x-1) - 4(y-1) \\ &= -3 + 8x - 4y. \end{aligned}$$

(ii) At $(x, y, z) = (1, 1, 1)$

$$z_{xx}(1, 1) = \frac{-8 \times (-1) + 8 \times 6 \times 8}{(-1)^2} = 392$$

$$z_{xy}(1, 1) = \frac{8 \times (6 \times (-4) - 4)}{(-1)^2} = -224, \quad z_{yx}(1, 1) = \frac{4 \times 8 \times (-1) - 4 \times 6 \times 8}{(-1)^2} = -224$$

$$z_{yy}(1, 1) = \frac{4 \times (-4) \times (-1) - 4 \times (6 \times (-4) - 4)}{(-1)^2} = 128$$

$$\begin{aligned} \text{Hence } z(x, y) &\approx z(1, 1) + z_x(1, 1)(x-1) + z_y(1, 1)(y-1) + \frac{1}{2!} [z_{xx}(1, 1)(x-1)^2 + 2z_{xy}(1, 1)(x-1)(y-1) + z_{yy}(1, 1)(y-1)^2] \\ &= -3 + 8x - 4y + \frac{1}{2} [392(x-1)^2 - 448(x-1)(y-1) + 128(y-1)^2] \\ &= -3 + 8x - 4y + 196(x-1)^2 - 224(x-1)(y-1) + 64(y-1)^2. \end{aligned}$$

Question 2 (15 marks)

Evaluate the integral

$$\int_1^2 \int_{1/y}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy$$

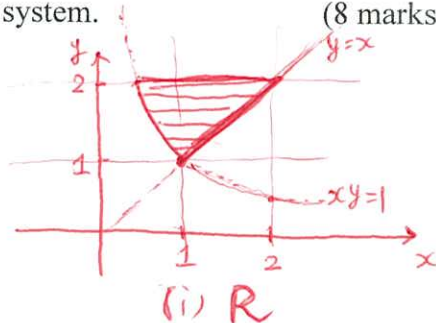
by the following steps:

- Sketch the region R of integration. (2 marks)
- Let $u = \sqrt{xy}$ and $v = \sqrt{\frac{y}{x}}$, calculate the Jacobian $J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$. (3 marks)
- Sketch the new region R^* in the new coordinate system (u, v) . (2 marks)
- Find the integration under the new coordinate system. (8 marks)

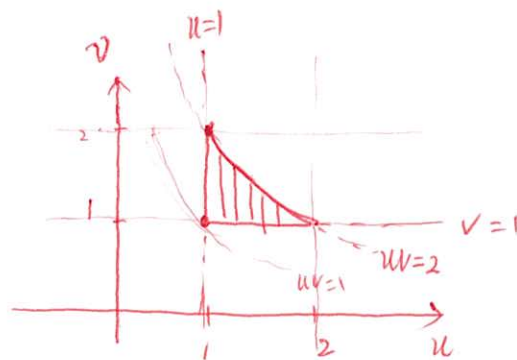
Solution: (ii) $\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases} \Rightarrow \begin{cases} x = \frac{u}{v} \\ y = uv \end{cases}$

$$\Rightarrow J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$$

$$= \frac{u}{v} + \frac{u}{v} = \frac{2u}{v}$$



(iii) $\begin{cases} y=1 \Rightarrow uv=1 \\ y=2 \Rightarrow uv=2 \\ x=y \Rightarrow v^2=1 \Rightarrow v=1 \\ xy=1 \Rightarrow u^2=1 \Rightarrow u=1 \end{cases} \Rightarrow$



(iv) $\iint_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy = \iint_{R^*} v e^u \cdot \frac{2u}{v} du dv$

$$= \iint_{R^*} 2u e^u du dv$$

$$= \int_1^2 \int_1^{\frac{2}{v}} 2u e^u dv du$$

$$= \int_1^2 2u e^u \left(\frac{2}{u} - 1 \right) du = 2 \int_1^2 (2e^u - u e^u) du$$

$$= 4 \int_1^2 e^u du - 2 \int_1^2 u e^u du$$

$$= 4e^u \Big|_1^2 - 2 \int_1^2 u de^u = 4(e^2 - e) - 2u e^u \Big|_1^2 + 2 \int_1^2 e^u du$$

$$= 4(e^2 - e) - 2(2e^2 - e) + 2e^u \Big|_1^2 = 4e^2 - 4e - 4e^2 + 2e + 2e^2 - 2e$$

$$= \underline{\underline{2e^2 - 4e}}$$

$$= 2e^2 - 4e = \underline{\underline{2e(e-2)}}$$

Question 3 (15 marks)

- (i) Show that, under the spherical polar coordinate, $\rho = 2\cos \varphi$ is the sphere $x^2 + y^2 + (z - 1)^2 = 1$, and $\varphi = \frac{\pi}{4}$ is the cone $z = \sqrt{x^2 + y^2}$. (5 marks)
- (ii) Use spherical polar coordinate to calculate the volume of the solid V that is bounded below by the sphere $x^2 + y^2 + (z - 1)^2 = 1$ and above by the cone $z = \sqrt{x^2 + y^2}$. (10 marks)

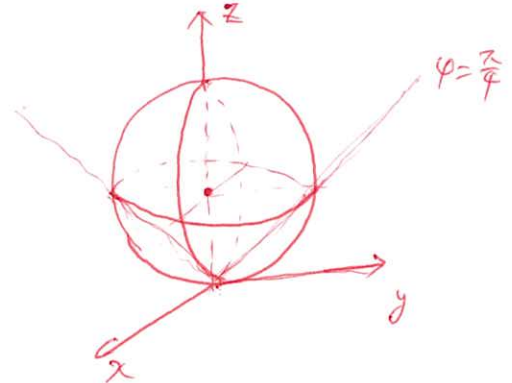
Solution. (i) $\begin{cases} x = \rho \sin \varphi \cos \theta \\ y = \rho \sin \varphi \sin \theta \\ z = \rho \cos \varphi \end{cases} \Rightarrow x^2 + y^2 + (z - 1)^2 = 1$ is equivalent to

$$\rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta + (\rho \cos \varphi - 1)^2 = 1 \quad \dots (x)$$

Simplifying (x), we get $\rho(\rho - 2\cos \varphi) = 0 \Rightarrow \rho = 2\cos \varphi$ must be the sphere.

$z = \sqrt{x^2 + y^2} \Rightarrow \frac{z}{\sqrt{x^2 + y^2}} = 1 \Rightarrow \frac{\rho \cos \varphi}{\rho \sin \varphi} = 1 \Rightarrow \cot \varphi = 1, 0 \leq \varphi \leq \pi.$

$\Rightarrow \varphi = \frac{\pi}{4}.$



(ii)
$$\iiint_V dx dy dz$$

$$= \iiint_{V^*} \rho^2 \sin \varphi d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos \varphi} \rho^2 \sin \varphi d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \times \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin \varphi \int_0^{2\cos \varphi} \rho^2 d\rho d\varphi$$

$$= 2\pi \times \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin \varphi \times \frac{\rho^3}{3} \Big|_0^{2\cos \varphi} d\varphi$$

$$= 2\pi \times \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin \varphi \cdot \frac{(2\cos \varphi)^3}{3} d\varphi = 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\frac{(2\cos \varphi)^3}{3} d(\cos \varphi)$$

$$= 2\pi \times (-1) \times \frac{8}{3} \times \frac{(\cos \varphi)^4}{4} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{4\pi}{3} \left((\cos \frac{\pi}{2})^4 - (\cos \frac{\pi}{4})^4 \right)$$

$$= -\frac{4\pi}{3} \times (-1) \times \left(\frac{\sqrt{2}}{2} \right)^4 = \frac{\pi}{3}$$