

EE3109 Applied Electromagnetics
2024/25 Semester B
Test 1

Name: _____ **Solution** _____

Student #: _____

Date: _____

1. **(11 Marks)** In a region in free-space, the time-domain electric flux density is

$$\mathbf{D}(\mathbf{r}, t) = (x^2 + y^2) \cos(\omega t - \phi) \hat{\mathbf{z}}.$$

(a) Find the charge density $\rho(\mathbf{r}, t)$ (3)

(b) Find the time-domain magnetic field $\mathbf{H}(\mathbf{r}, t)$ (5)

(c) The magnetic field phasor $\mathbf{H}(\mathbf{r})$ (3)

Solution:

(a)

$$\rho(\mathbf{r}, t) = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial z} (x^2 + y^2) \cos(\omega t - \phi) = 0 \text{ [C/m}^3\text{]}$$

(b) We start from Faraday's Law and isolate $\mathbf{H}(\mathbf{r}, t)$:

$$\begin{aligned} \nabla \times \left(\frac{\mathbf{D}(\mathbf{r}, t)}{\epsilon_0} \right) &= -\frac{\partial}{\partial t} (\mu_0 \mathbf{H}(\mathbf{r}, t)) \\ \Rightarrow \mathbf{H}(\mathbf{r}, t) &= -\frac{1}{\mu_0 \epsilon_0} \int \nabla \times \mathbf{D}(\mathbf{r}, t) dt. \end{aligned}$$

Now,

$$\begin{aligned} \nabla \times \mathbf{D}(\mathbf{r}, t) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (0, 0, (x^2 + y^2) \cos(\omega t - \phi)) \\ &= \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}, 0 \right) (x^2 + y^2) \cos(\omega t - \phi) \\ &= (2y, -2x, 0) \cos(\omega t - \phi). \end{aligned}$$

Substituting into the equation to find $\mathbf{H}(\mathbf{r}, t)$:

$$\mathbf{H}(\mathbf{r}, t) = -\frac{1}{\mu_0 \epsilon_0} \int [(2y, -2x, 0) \cos(\omega t - \phi)] dt = \frac{2 \sin(\omega t - \phi)}{\omega \mu_0 \epsilon_0} (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}}).$$

(c) We first write the sinusoidal function as a cosine, then convert into a phasor:

$$\begin{aligned} \mathbf{H}(\mathbf{r}, t) &= \frac{2 \cos\left(\omega t - \phi - \frac{\pi}{2}\right)}{\omega \mu_0 \epsilon_0} (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}}); \\ \Rightarrow \mathbf{H}(\mathbf{r}) &= \frac{2}{\omega \mu_0 \epsilon_0} e^{-j\left(\phi + \frac{\pi}{2}\right)} (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}}). \end{aligned}$$

2. **(23 Marks)** A linearly-polarized plane wave travels in seawater at an elevation angle of $\theta = 60^\circ$, as shown in the diagram below. The material property of seawater is ($\epsilon_r = 81, \mu_r = 1, \sigma = 4$ [S/m]). The frequency of the wave is 5 MHz. The magnetic field phasor at the origin, at time $t = 0$, is -15 [mA/m] $\hat{\mathbf{y}}$.
- What is the wavelength of this plane wave? (3)
 - What is the phase velocity of this plane wave? (3)
 - What is the propagation direction unit vector ($\hat{\mathbf{k}}$) of this plane wave? (3)
 - What are the propagation constant, attenuation constant and phase constant of this wave? (5)
 - Write the magnetic field phasor of this wave. Substitute in as many known variables as possible. (6)
 - What is the amplitude of the electric field at 0.5 m below the origin (i.e. at $(0, 0, -0.5$ m)? (3)

Solution:

First we find out whether this material is a good conductor, good dielectric, or neither.

$$\frac{\sigma}{\omega\epsilon} = \frac{4}{(2\pi \times 5 \times 10^6)(8.854 \times 10^{-12})(81)} = 177.5 \gg 1 \Rightarrow \text{Good conductor.}$$

- (d) Using relationships for a good conductor, the attenuation constant is

$$\alpha = \sqrt{\pi f \mu_0 \sigma} = \sqrt{\pi(5 \times 10^6)(4\pi \times 10^{-7})(4)} = 8.886 \text{ [Np/m]},$$

the phase constant is

$$\beta = \alpha = 8.886 \text{ [rad/m]},$$

and the propagation constant is

$$\Rightarrow \gamma = \alpha + j\beta = 8.886(1 + j) \text{ [1/m]}.$$

- (a)

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{8.886} = 0.7071 \text{ [m]}$$

- (b)

$$v_p = \lambda f = (0.7071)(5 \times 10^6) = 3.536 \times 10^6 \text{ [m/s]}$$

- (c)

$$\hat{\mathbf{k}} = (\cos 60^\circ, 0, \sin 60^\circ) = \frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{z}}$$

- (e) In general:

$$\mathbf{H}(\mathbf{r}) = H_0 \exp(-\alpha \hat{\mathbf{k}} \cdot \mathbf{r}) \exp(-j\beta \hat{\mathbf{k}} \cdot \mathbf{r}) \hat{\mathbf{h}}.$$

At the origin: $\mathbf{H}(\mathbf{r}) = H_0 \hat{\mathbf{h}} = -15 \text{ [mA/m]} \hat{\mathbf{y}} \Rightarrow H_0 = -15 \text{ [mA/m]}; \hat{\mathbf{h}} = \hat{\mathbf{y}}.$

$$\alpha \hat{\mathbf{k}} \cdot \mathbf{r} = \beta \hat{\mathbf{k}} \cdot \mathbf{r} = 8.886 \left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z \right) = 4.443x + 7.696z.$$

Substituting into the general equation above, we have:

$$\mathbf{H}(\mathbf{r}) = 15 \exp(-4.443x - 7.696z) \exp(-j(4.443x + 7.696z)) \hat{\mathbf{y}} \text{ [mA/m]}.$$

- (f) We first find the intrinsic impedance of the seawater:

$$\eta_c = (1 + j) \frac{\alpha}{\sigma} = \sqrt{2} \frac{8.886}{4} e^{j\frac{\pi}{4}} = 3.142 e^{j\frac{\pi}{4}} [\Omega] .$$

Then we write the amplitude of the magnetic and electric fields as follows:

$$\begin{aligned} & |\mathbf{H}(0, 0, -0.5\text{m})| \\ &= \left| 15 \exp(-4.443(0) - 7.696(-0.5)) \exp(-j(4.443(0) + 7.696(-0.5))) \right| \\ &= \left| 15 \exp(-7.696(-0.5)) \right| \\ &= 703.5 [\text{mA/m}] \end{aligned}$$

$$|\mathbf{E}(0, 0, -0.5\text{m})| = |\eta_c| |\mathbf{H}(0, 0, -0.5\text{m})| = (3.142)(0.7035) = 2.210 [\text{V/m}]$$

3. **(6 Marks + 3 Bonus)** What are the polarization of the following waves? Please justify your answer.

- (a) $\mathbf{E}(x, y, z, t) = 5 \cos(\omega t - kz) \hat{\mathbf{x}} + 5 \cos(\omega t - kz + 270^\circ) \hat{\mathbf{y}}$ (3)
 (b) $\mathbf{E}(x, y, z, t) = 5 \cos(\omega t - kz) \hat{\mathbf{x}} + 5 \cos(\omega t - kz + 45^\circ) \hat{\mathbf{y}}$ (3)
 (c) $\mathbf{E}(x, y, z, t) = 5 \cos(\omega t + kz) \hat{\mathbf{x}} + 5 \cos(\omega t + kz + 270^\circ) \hat{\mathbf{y}}$ (3)

Solution:

- (a) The wave travels in the +z-direction; $a_x = a_y$; $\delta = -90^\circ \Rightarrow \text{RHC}$.
 (b) The wave travels in the +z-direction; $\delta = 45^\circ \Rightarrow \text{LH Elliptical}$.
 (c) $\hat{\mathbf{k}} = -\hat{\mathbf{z}}$. To use the relationships we established in class, we transform using $(x, y, z) \rightarrow (y', x', -z')$

The wave then becomes:

$$\mathbf{E}(x', y', z, t) = 5 \cos(\omega t - kz) \hat{\mathbf{y}} + 5 \cos(\omega t - kz + 270^\circ) \hat{\mathbf{x}}$$

Since $a_x = a_y$; $\delta = -270^\circ = 90^\circ$, the wave is LHC polarized.