

EE3114 Systems and Control

4. Transient and Steady-State Response Analysis

Semester B, 2017-18

Recall: An example from Lecture-3:

$$Y(s) = \frac{3(s+2)}{s(s^2+2s+10)} \Rightarrow sY(s) = \frac{3(s+2)}{s^2+2s+10}$$

Impulse response

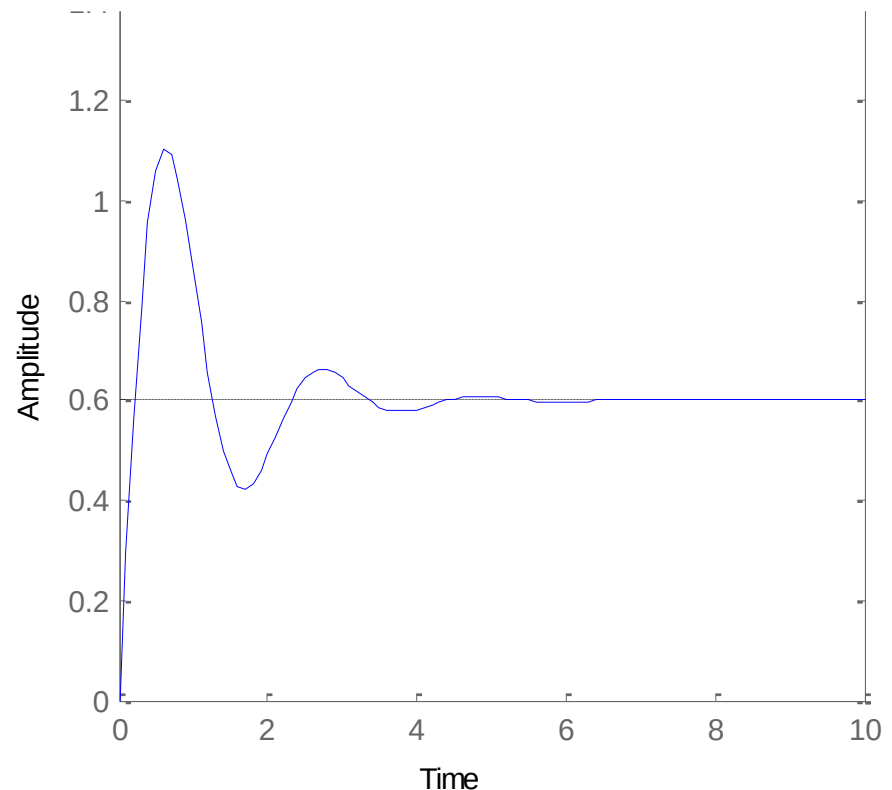
Step response

```
%% MatLab code  
num=[3 6];  
den=[1 2 10 0];  
Ys = tf(num,den);  
impulse(Ys)
```

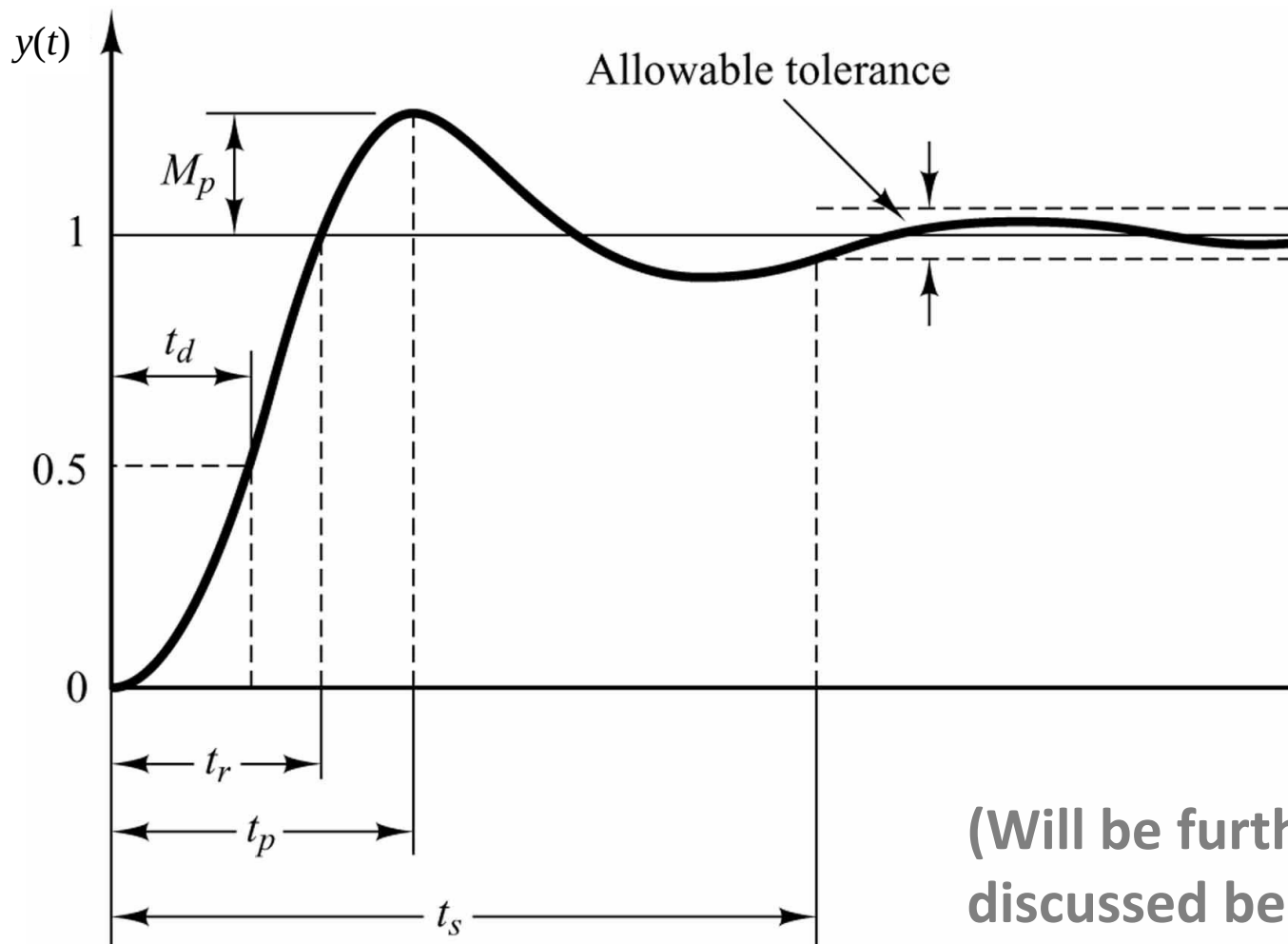
```
%% OR  
num=[3 6];  
den=[1 2 10];  
Ys = tf(num,den);  
step(Ys)
```

$$\frac{3(s+2)}{s(s^2+2s+10)}$$

$$\frac{3(s+2)}{s^2+2s+10}$$

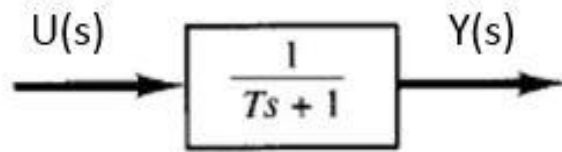


In general:

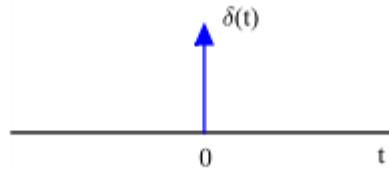


(Will be further discussed below)

Recall: A First-Order LTI System from Lecture-3



Unit **Impulse** Input:



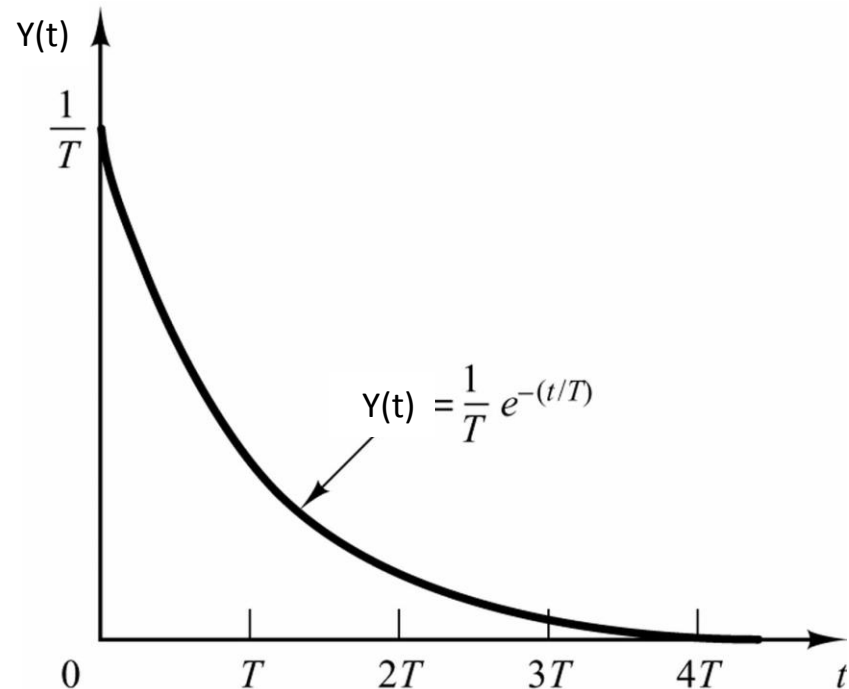
Output: $Y(s) = \frac{1}{Ts+1}$

Inverse Laplace Transform

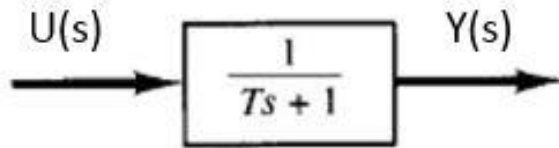


Unit Impulse Response

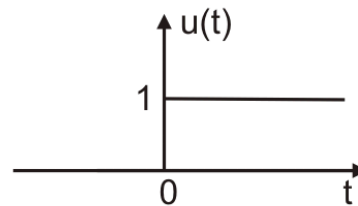
```
%MatLab code  
num = [1];  
den = [2 1]; %T=2  
impulse(num,den);
```



Unit Step Input:



Unit Step Input:



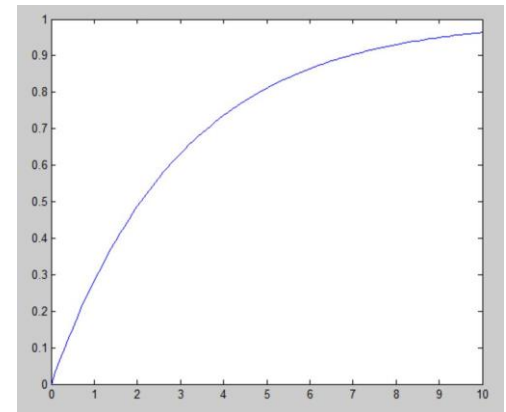
Output:

$$Y(s) = \frac{1}{Ts + 1} \frac{1}{s} = \frac{1}{s} - \frac{T}{Ts + 1} = \frac{1}{s} - \frac{1}{s + (1/T)}$$

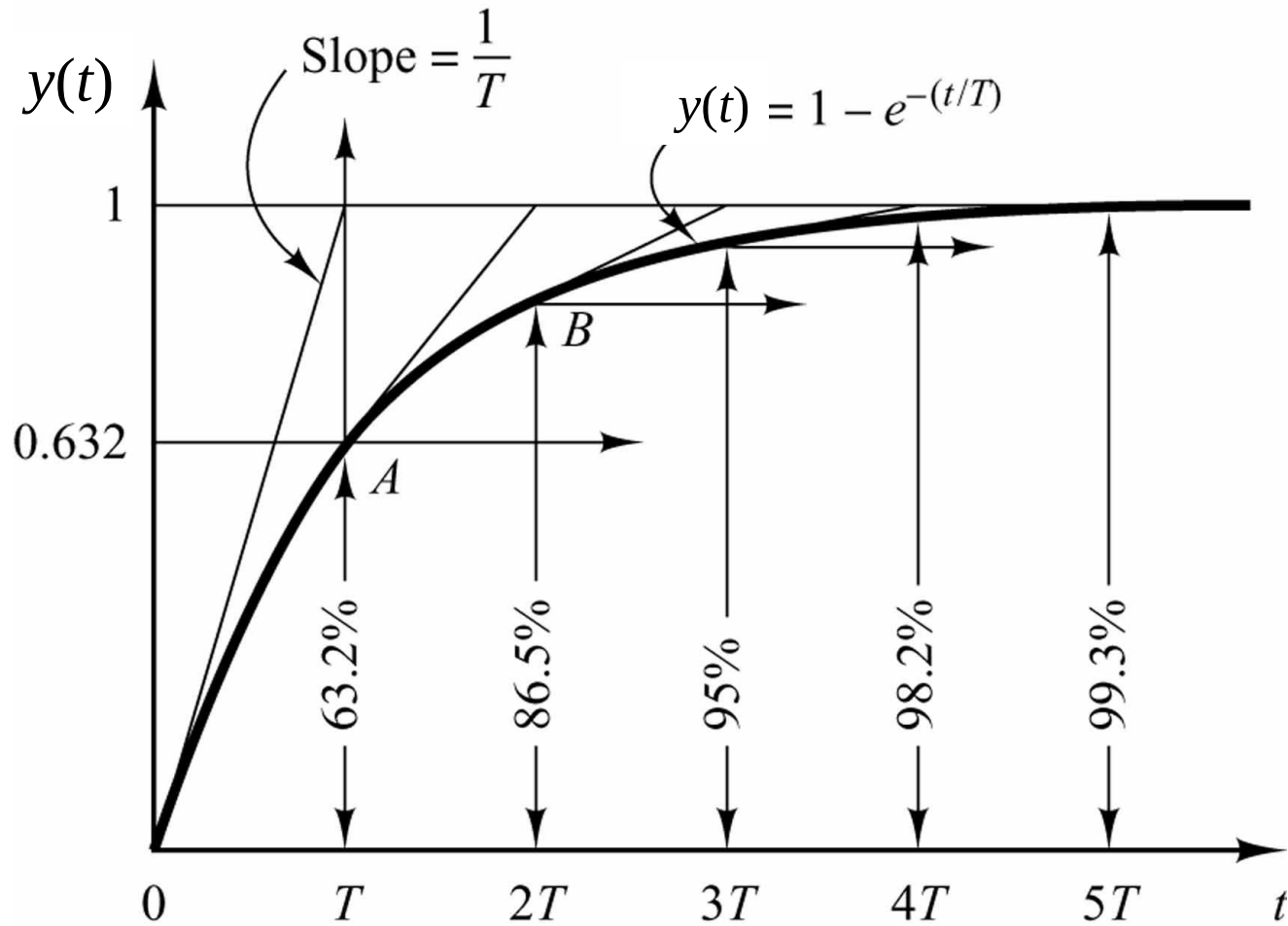
Inverse Laplace Transform → Unit Step Response

$$y(t) = 1 - e^{-t/T}, \quad \text{for } t \geq 0$$

```
%% MatLab code,  
Hs = tf(1,[3 1]); %T = 3  
step(Hs);  
[y,t] = step(Hs,10); %Time [0,10]  
plot (t, y);
```



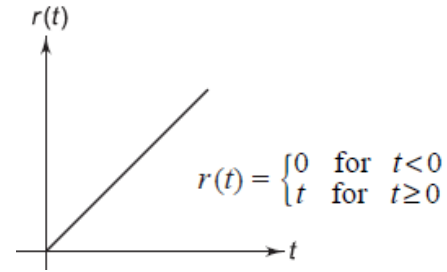
Unit-Step Response: more precisely



Unit-Ramp Response

Input: $u(t) = t \ (t \geq 0)$

$$U = \frac{1}{s^2}$$



$$Y(s) = \frac{1}{s^2(Ts+1)} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$y(t) = t - T + Te^{-\frac{t}{T}} \quad \text{for } t \geq 0$$

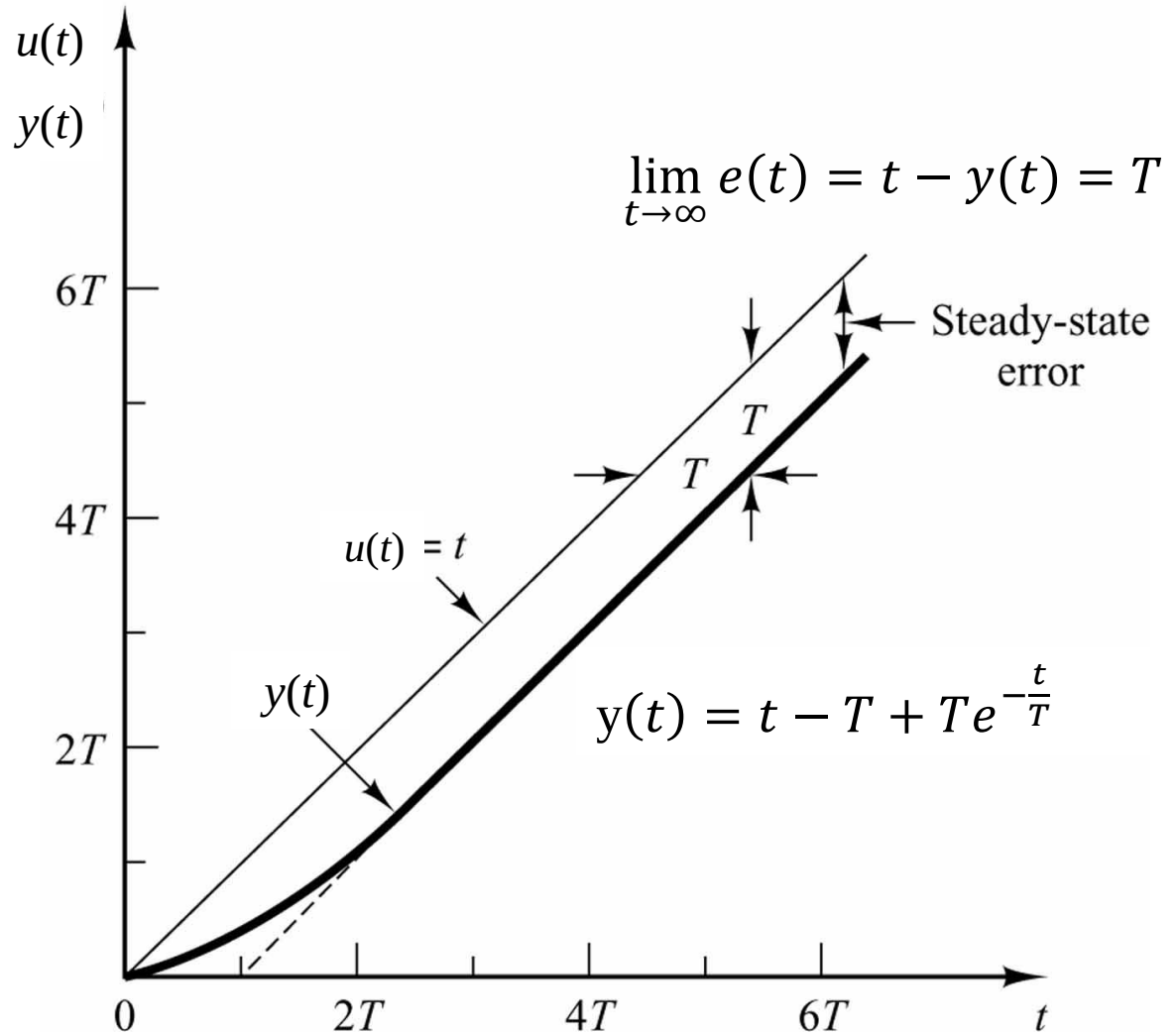
Error Signal: $e(t) = u(t) - y(t) = T(1 - e^{-\frac{t}{T}})$

$$\lim_{t \rightarrow \infty} e(t) = T$$

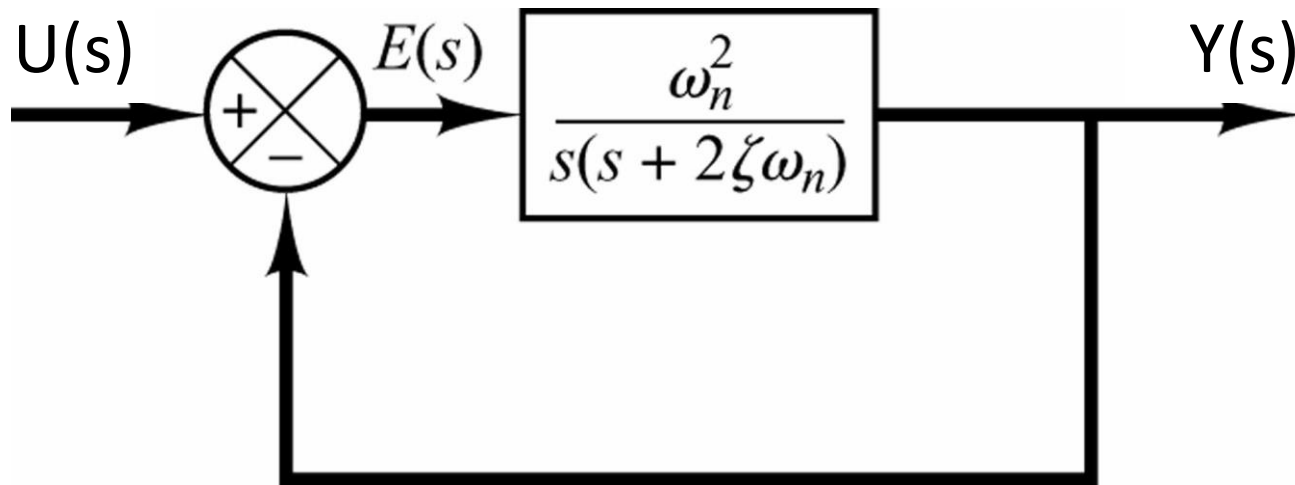
Unit Ramp Response

$$\frac{1}{s^2(Ts+1)}$$

```
%MatLab code
%lsim(num,den,u,t)
%for any input u
num = [1];
den = [1 1 0 0]; %T=1
t = 0:0.01:3;
u = t;
y = lsim(num,den,u,t);
plot(t,u,'-',t,y,'.')
```



Standard Second-Order System



$$\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Damping Ratio

Un-Damped Natural Frequency

$$\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Poles: $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

1. If $0 \leq \zeta < 1$, complex poles: $-\sigma \pm j\omega_d$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$



Damped Natural Frequency

2. If $\zeta = 1$, double poles: $-\omega_n$

3. If $\zeta > 1$, real poles: $-(\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$

Impulse Response

$$U(s) = 1 \quad \Rightarrow \quad Y = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

1. If $0 \leq \zeta < 1$, $y(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\sigma t} \sin \omega_d t$, for $t \geq 0$

Very low damping: $\zeta \sim 0$

$\Rightarrow \omega_d = \omega_n \sqrt{1 - \zeta^2} \sim \omega_n$, $\sigma = \zeta\omega_n \sim 0 \Rightarrow$ oscillatory

2. If $\zeta = 1$, $Y(s) = \frac{\omega_n^2}{(s + \zeta\omega_n)^2} \Rightarrow y(t) = t e^{-\zeta\omega_n t}$
 $\omega_d = \omega_n \sqrt{1 - \zeta^2} = 0$

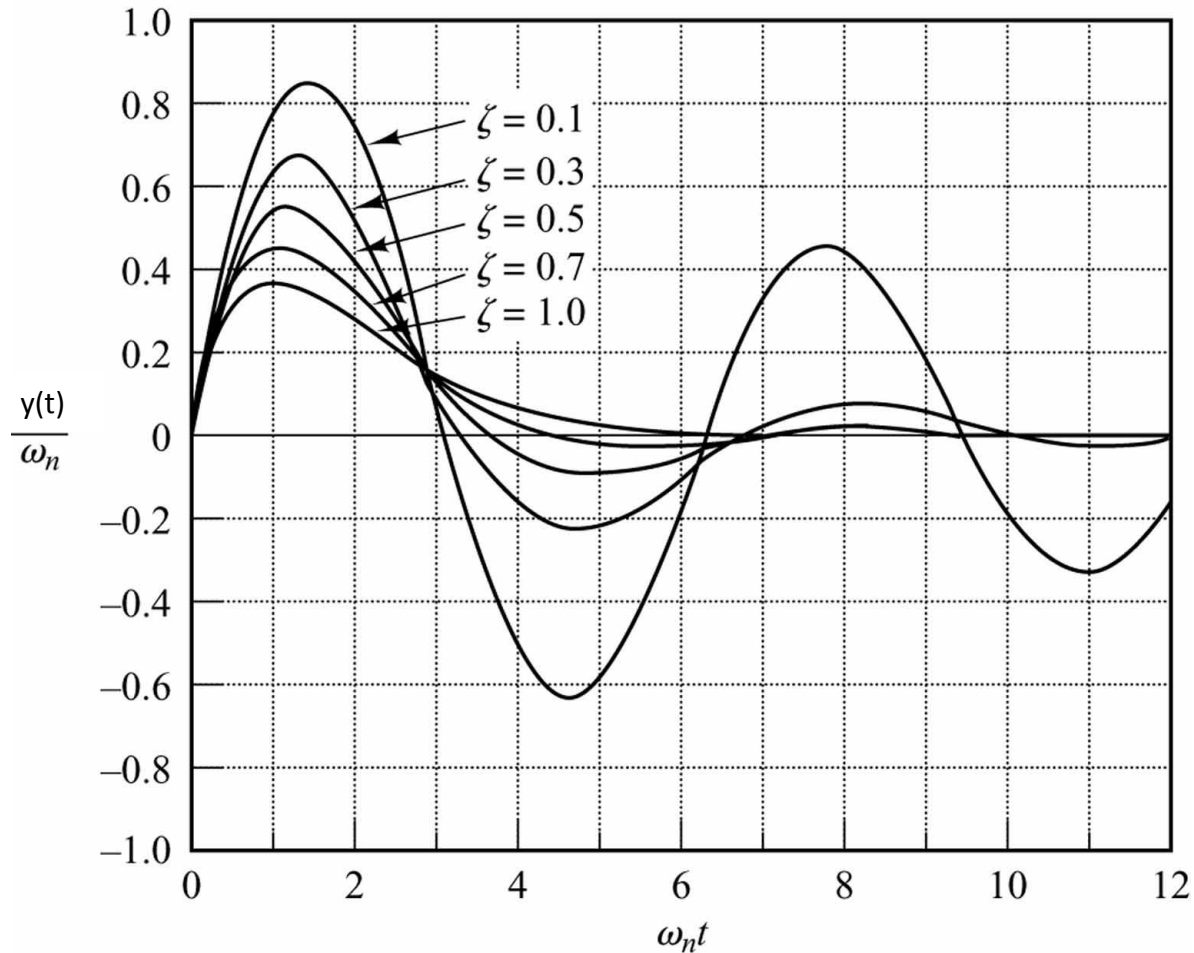
$\Rightarrow$ no oscillation

3. If $\zeta > 1$, over damped:

$$y(t) = \frac{1}{2\omega_n \sqrt{\zeta^2 - 1}} e^{-\zeta\omega_n t} \left(e^{\omega_n \sqrt{\zeta^2 - 1} t} - e^{-\omega_n \sqrt{\zeta^2 - 1} t} \right)$$

Impulse Response Curves

For $0 \leq \zeta < 1 \rightarrow$ oscillatory as $\zeta \rightarrow 0$



Step Response

1) Under Damped: $0 \leq \zeta < 1$

$$\begin{aligned} Y(s) &= \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \end{aligned}$$

→ **Response**

$$y(t) = 1 - e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right)$$

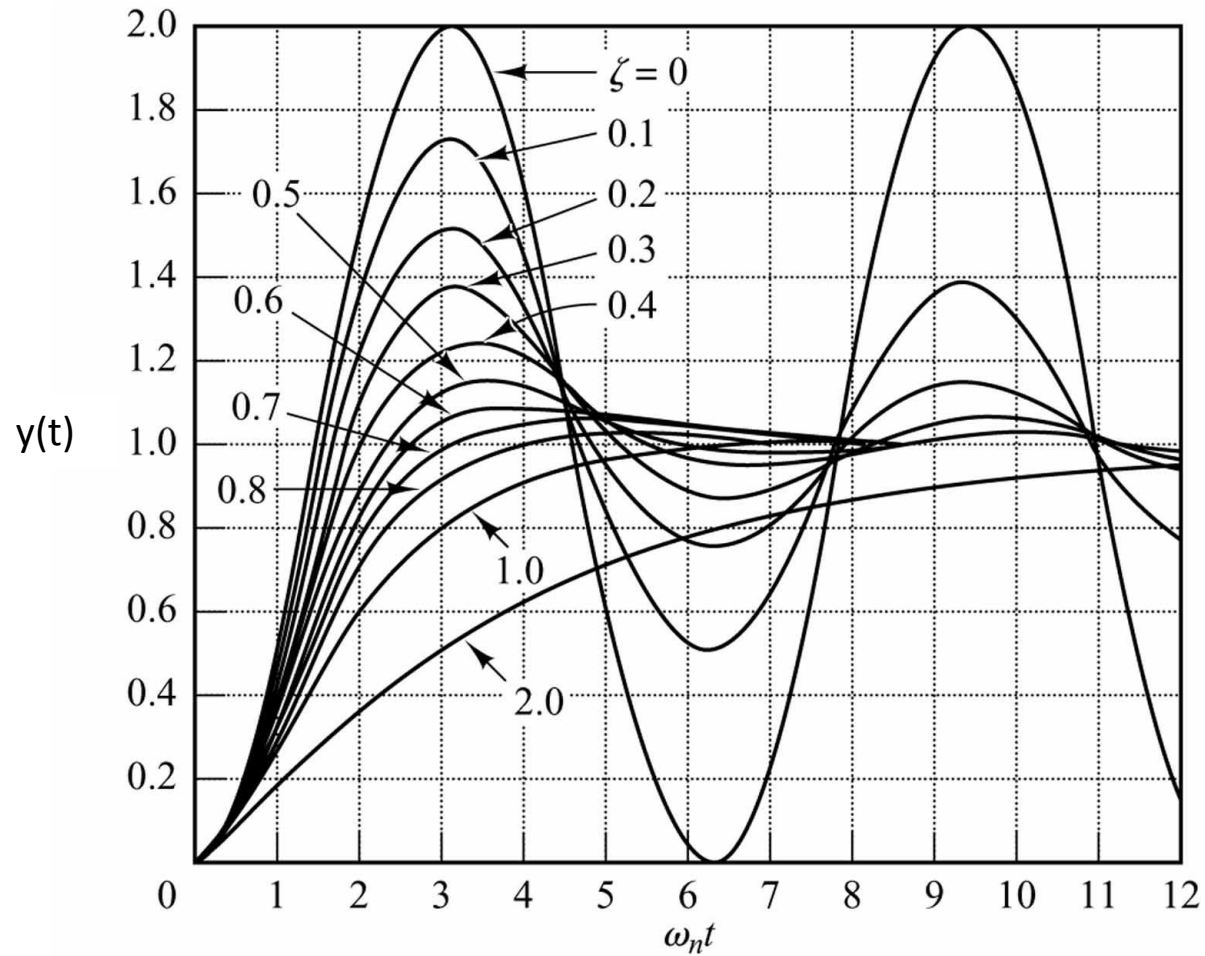
→ **Error** $e(t) = 1 - y(t)$

$$e(t) = e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right), \quad \text{for } t \geq 0$$

For $\zeta = 0$: $\omega_d = \omega_n \sqrt{1 - \zeta^2} = \omega_n$

$$y(t) = 1 - \cos \omega_n t, \quad \text{for } t \geq 0$$

Unit Step Response Curves

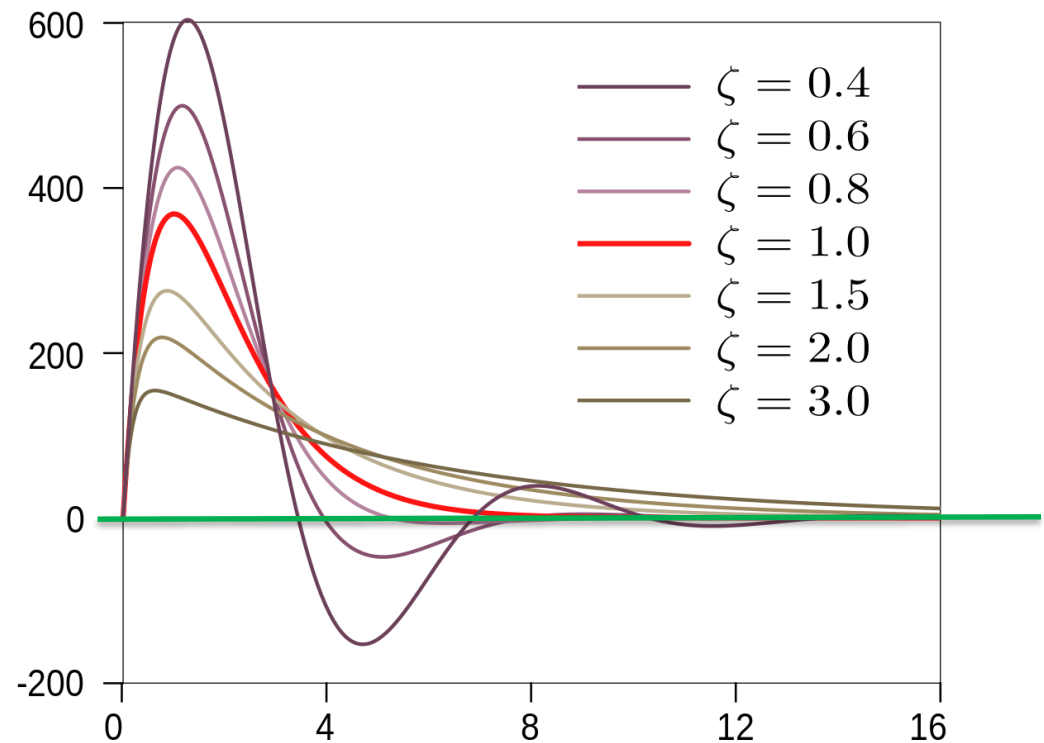


Step Response

2) Critically Damped: $\zeta = 1$

$$Y(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$



Step Response

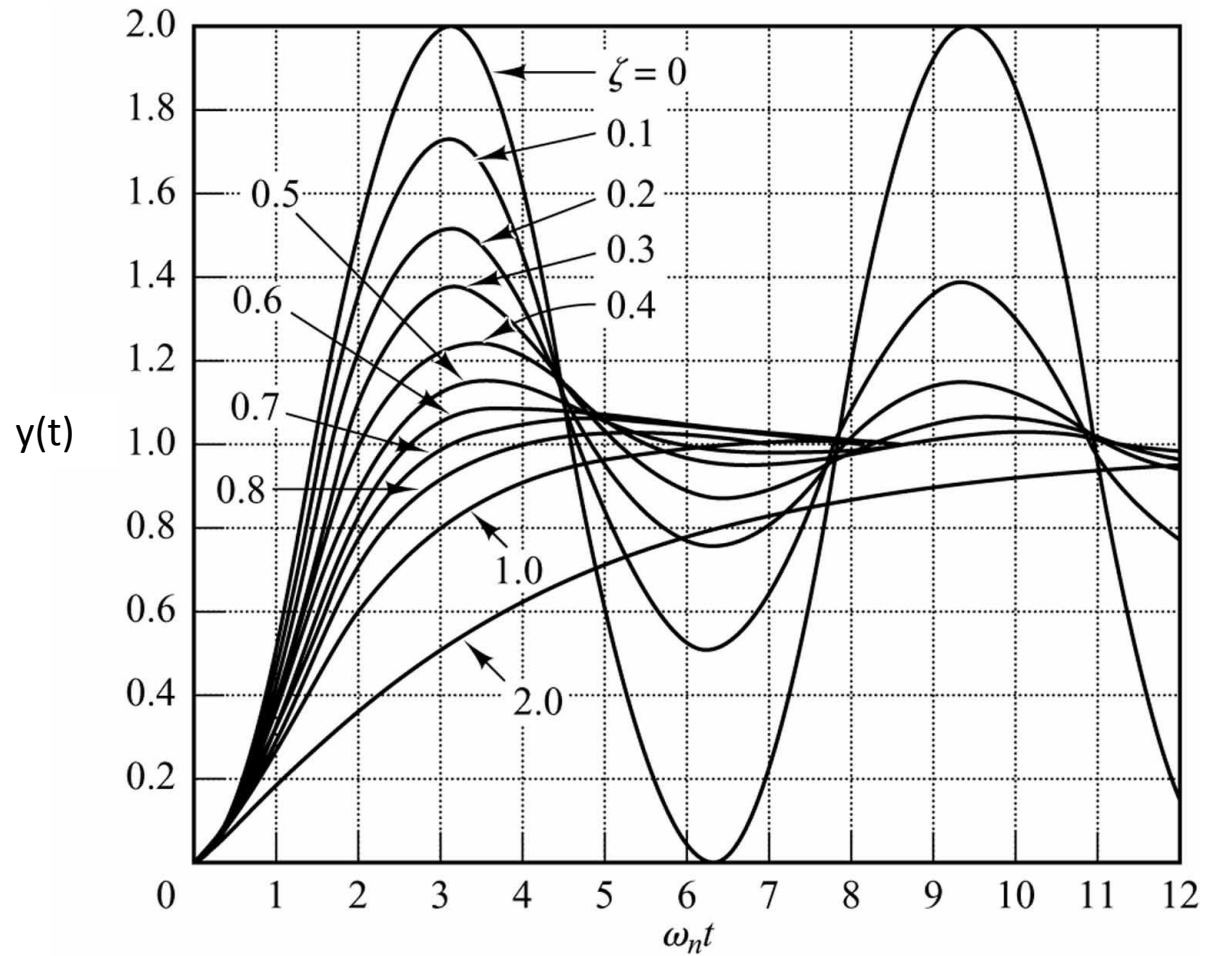
3) Over Damped: $\zeta > 1$

$$Y(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})}$$

$$\begin{aligned} y(t) &= 1 + \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta + \sqrt{\zeta^2 - 1})} e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t} \\ &\quad - \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta - \sqrt{\zeta^2 - 1})} e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \\ &= 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right), \end{aligned}$$

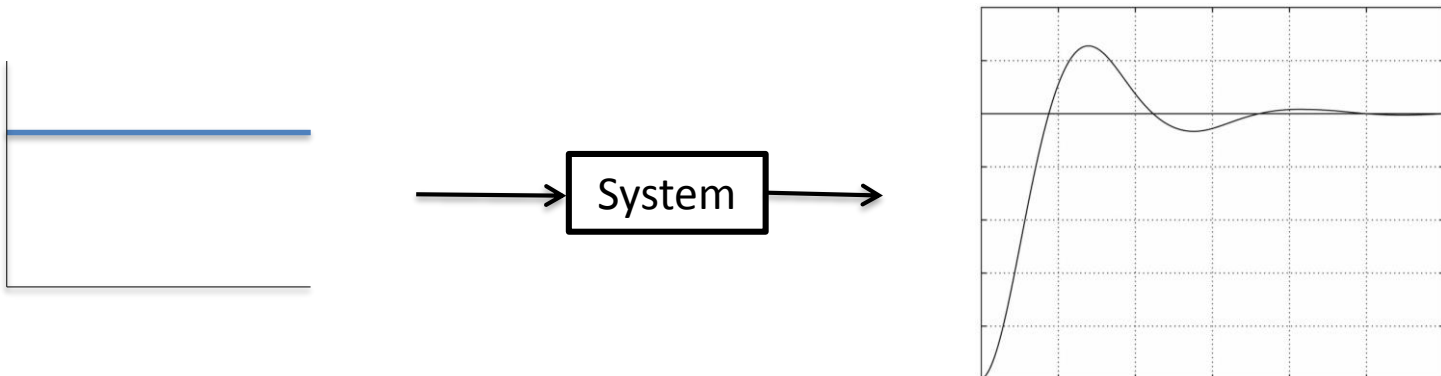
where $s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$

Unit Step Response Curves

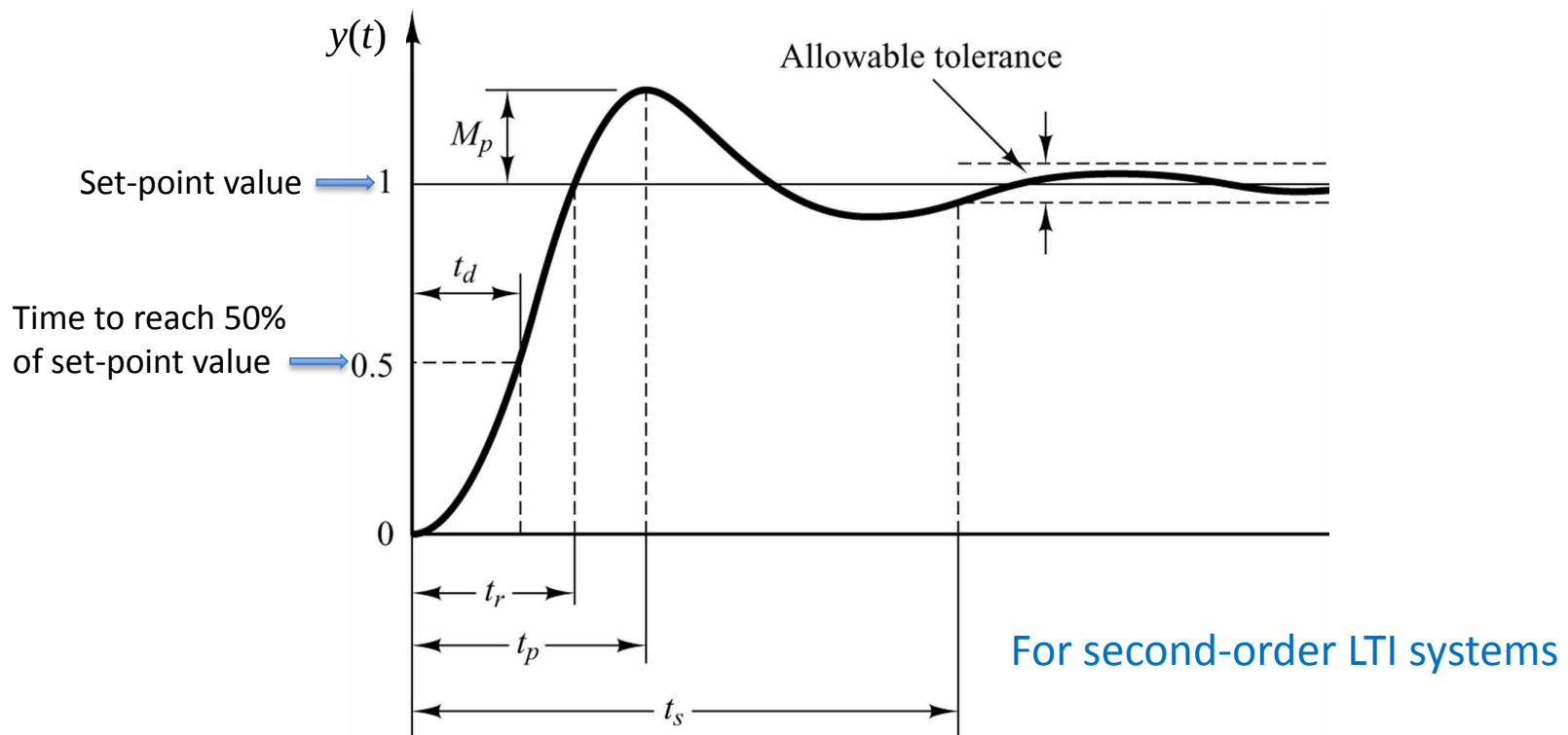


Time Response Specifications

- Objective of Control
 - Desired output
 - Define “Specifications” for system output from common input (impulse, step, ramp)
- Focus: Specifications for *Step Response*
(Similar procedures can be applied to other inputs)

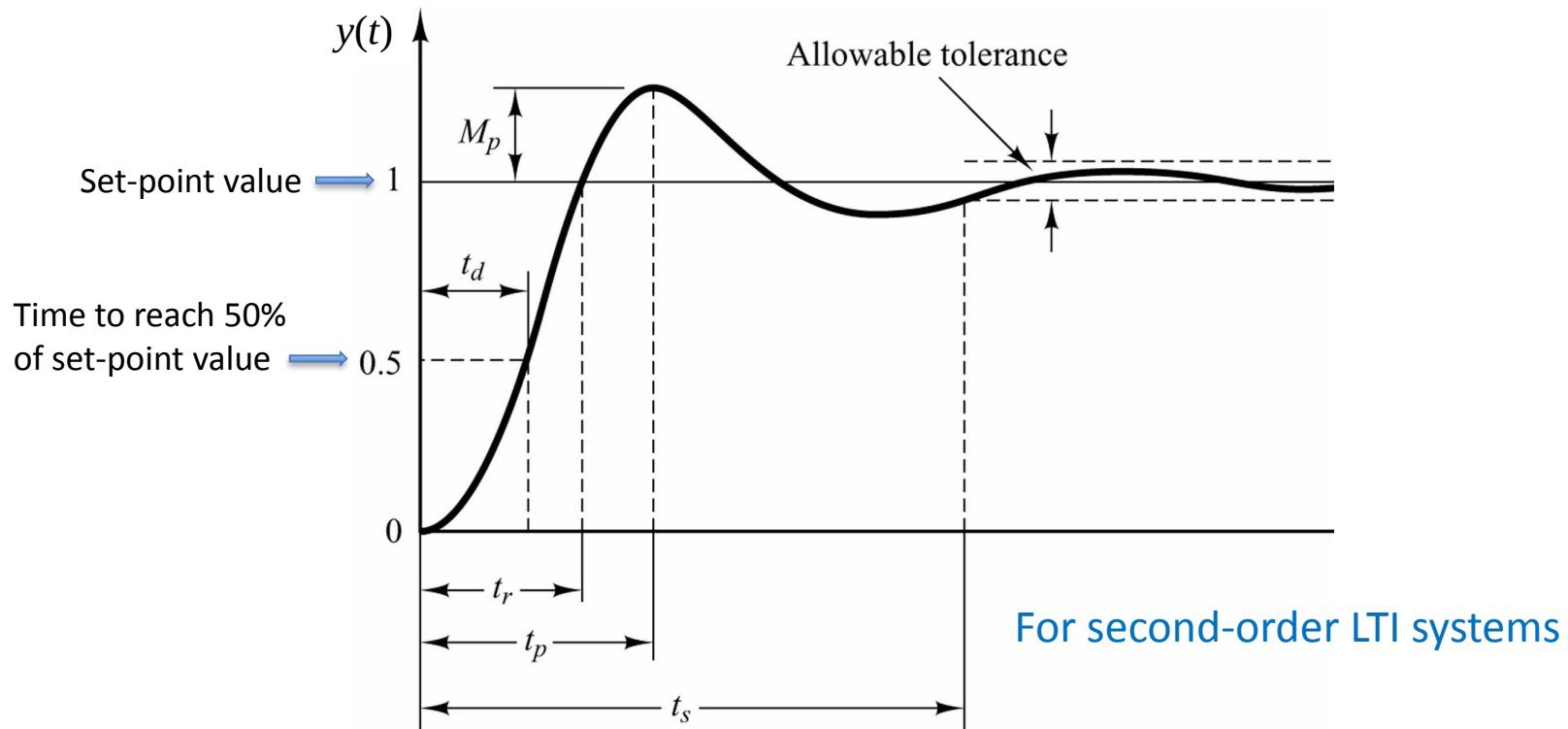


Transient Response Specifications



- **Rise time** t_r :
time required for response to rise from 0% to 100% of set-point value for under-damped systems;
time required for response to rise from 10% to 90% of set-point value for over-damped systems;
- **Settling time** t_s : time for output to settle within $\pm\delta\%$ of set-point value

Transient Response Specifications

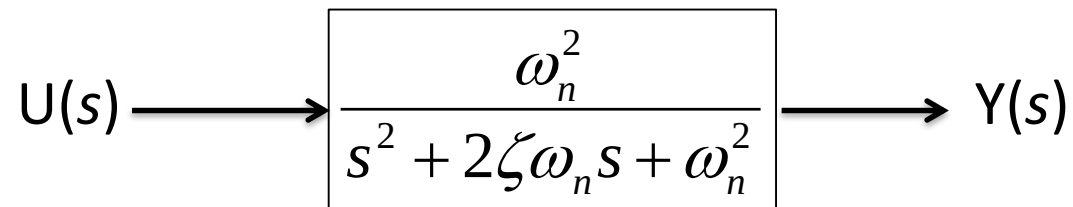


- **Overshoot** M_p : maximum % the output overshoots the set-point value
- **Peak time** t_p : time for output to reach max value
- t_r, t_s, M_p, t_p measure the system output against the step function
- “Fastness” measured by (t_r, t_p) versus “Nearness” measured by (M_p, t_s)

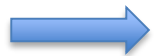
For a second-order system, given ω_n and ζ

Find: t_r , t_s , M_p , t_p

Standard form for second-order systems



$$U(s) = \frac{1}{s} \text{ (unit step input)}$$

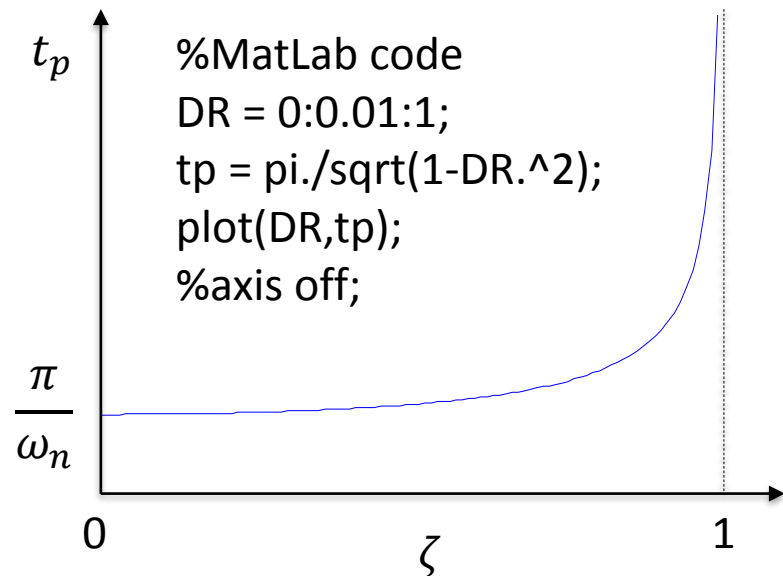
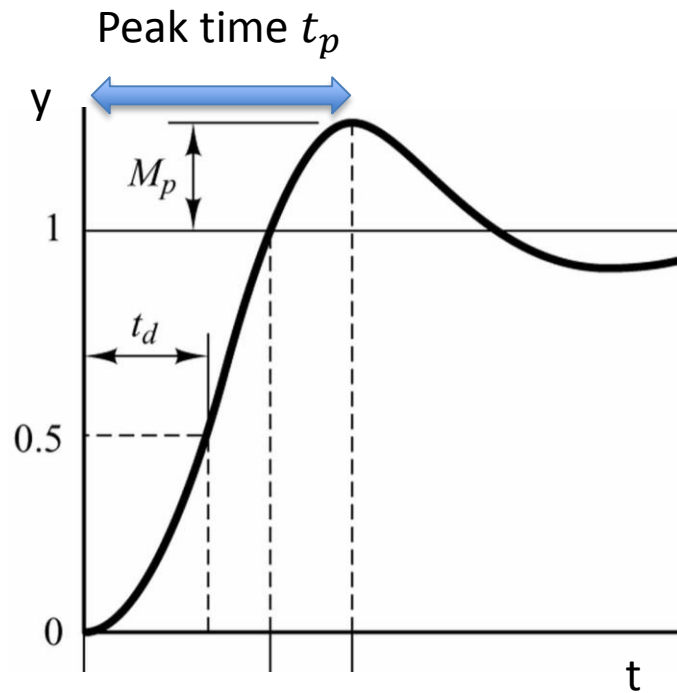


$$y(t) = 1 - e^{-\zeta\omega_n t} \left[\cos(\omega_n \sqrt{1 - \zeta^2} t) + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\omega_n \sqrt{1 - \zeta^2} t) \right]$$

Find **Peak Time** t_p by $\left. \frac{dy}{dt} \right|_{t=t_p} = 0$ (Maximum via Calculus)

$$\sin\left(\omega_n \sqrt{1 - \zeta^2} t_p\right) = 0 \rightarrow \omega_n \sqrt{1 - \zeta^2} t_p = \pi, 3\pi, 5\pi \dots$$

First maximum $\rightarrow$ Peak time $t_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$

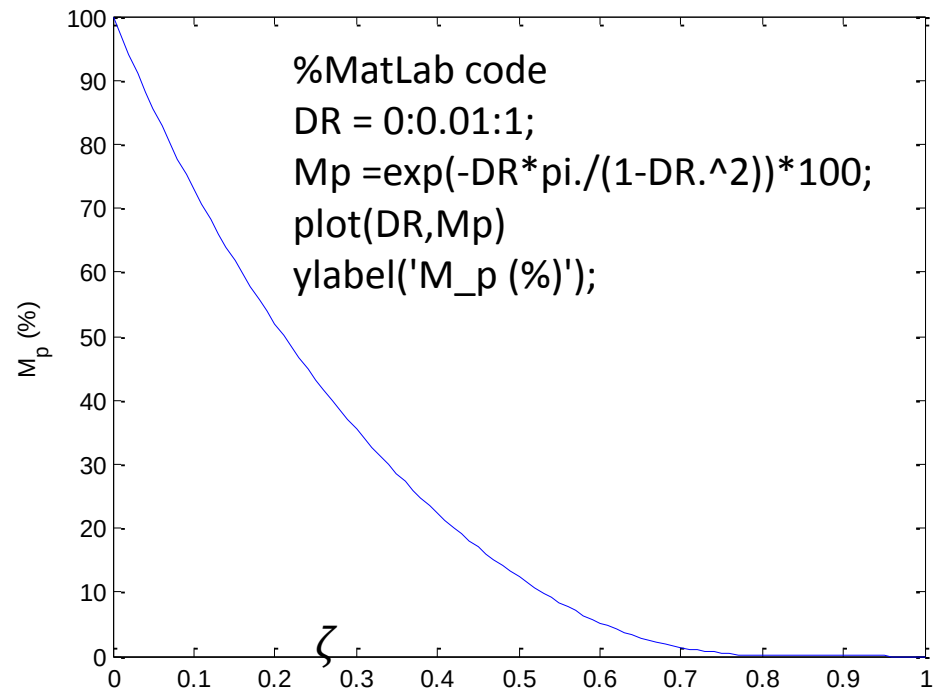
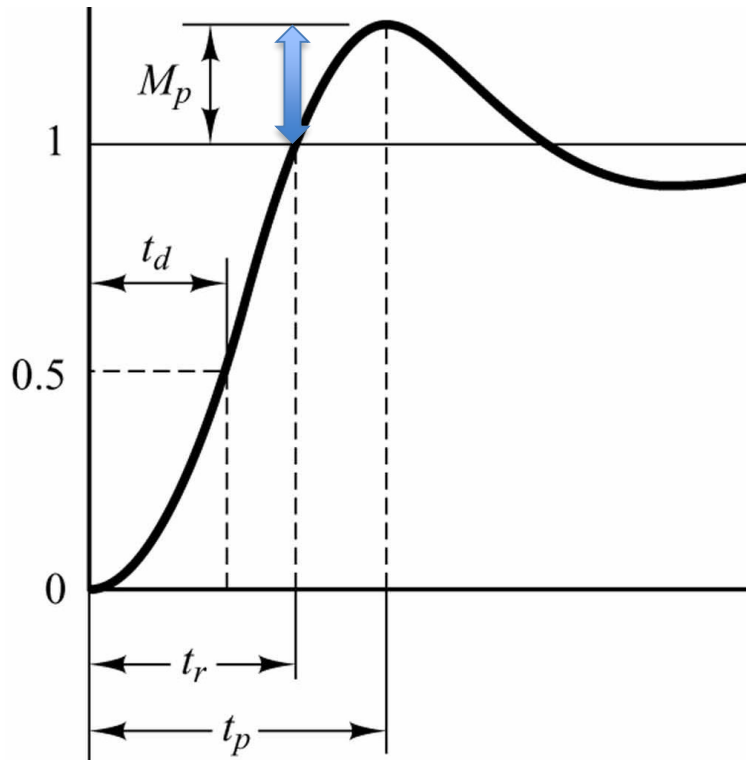


Find % Overshoot M_p : $M_p = \frac{(y(t_p)-1)}{1} \times 100\%$

where $y(t_p) = 1 + e^{-\zeta\omega_n t_p}$

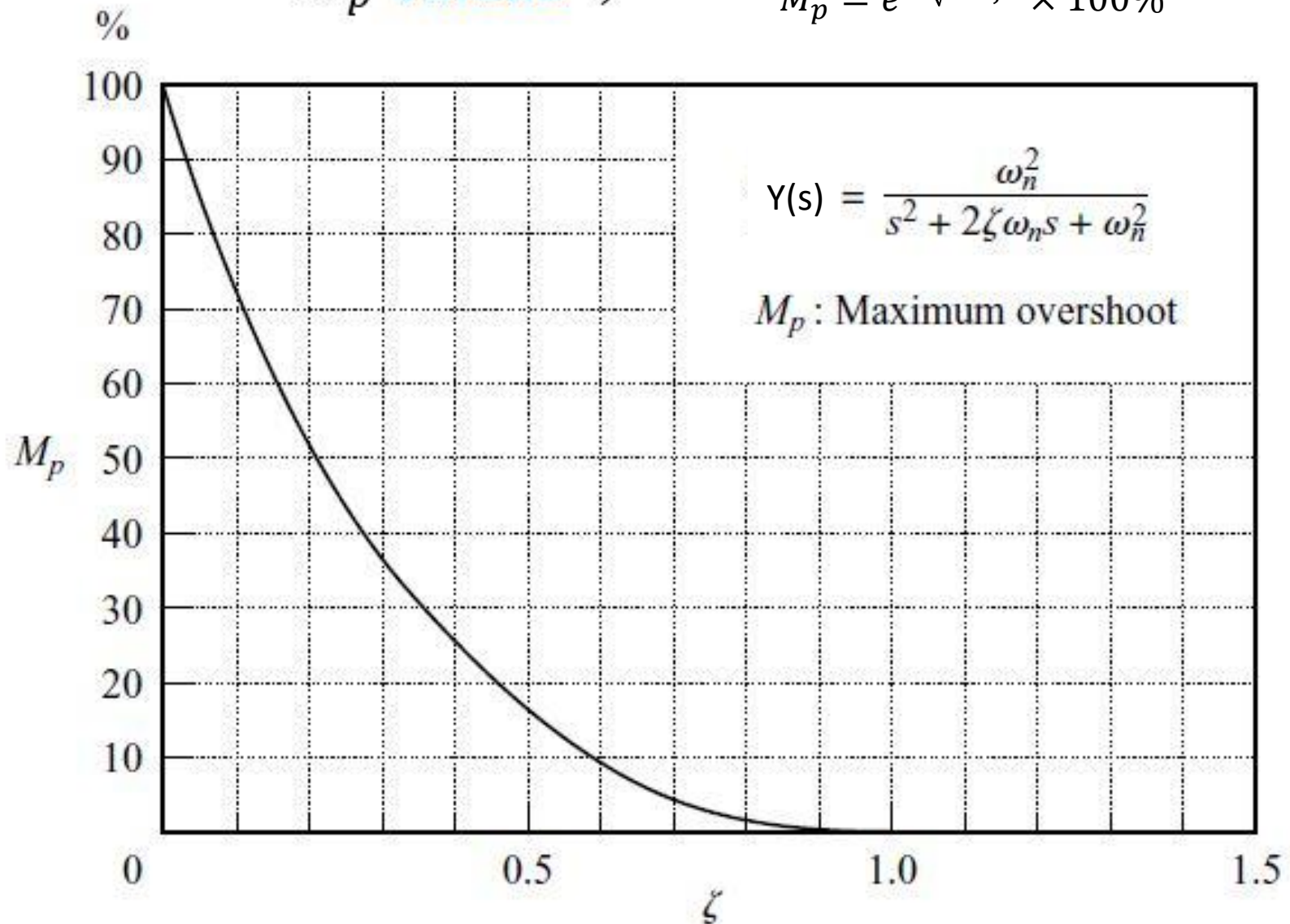
Percentage of Overshoot:

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

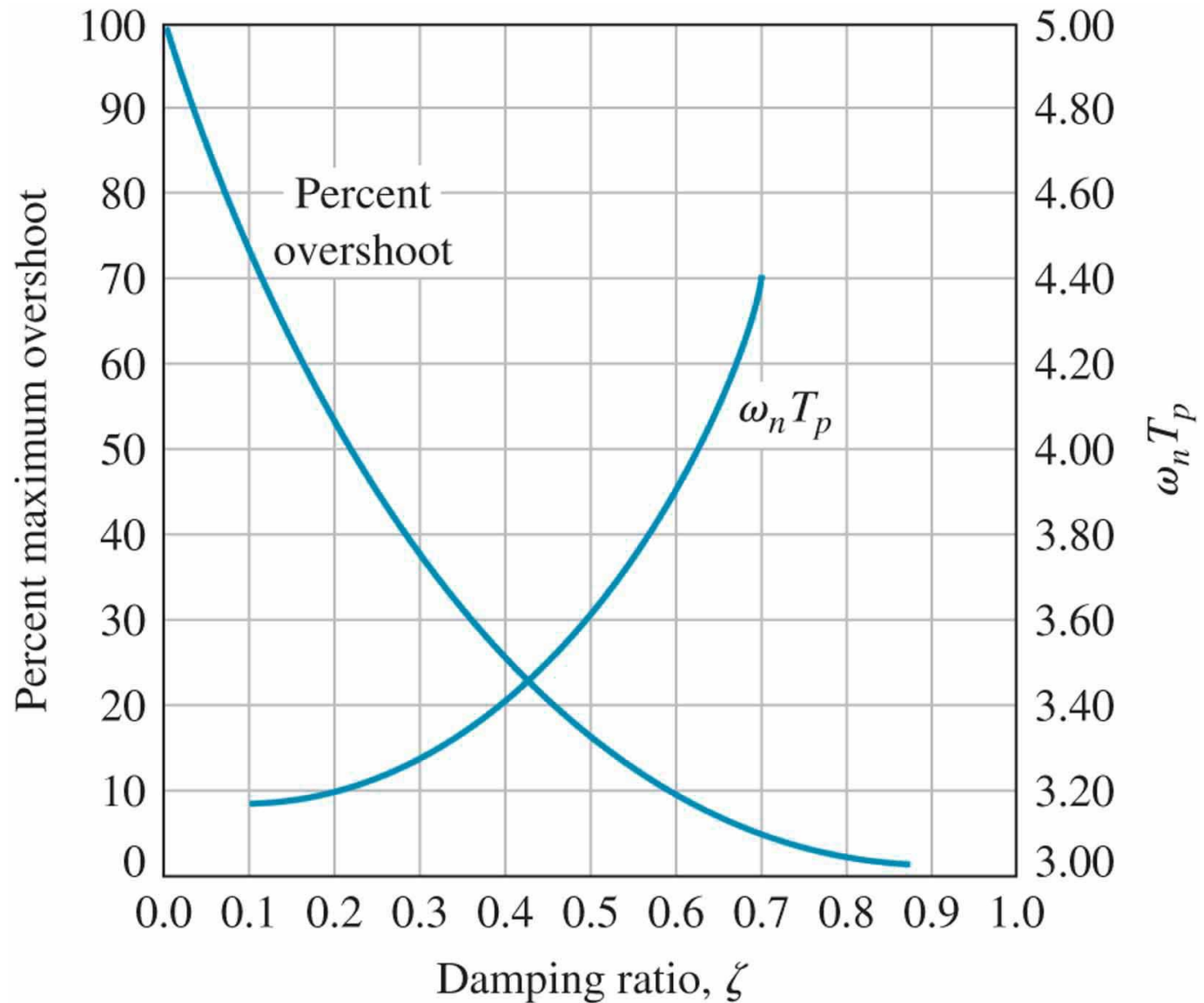


M_p versus ζ

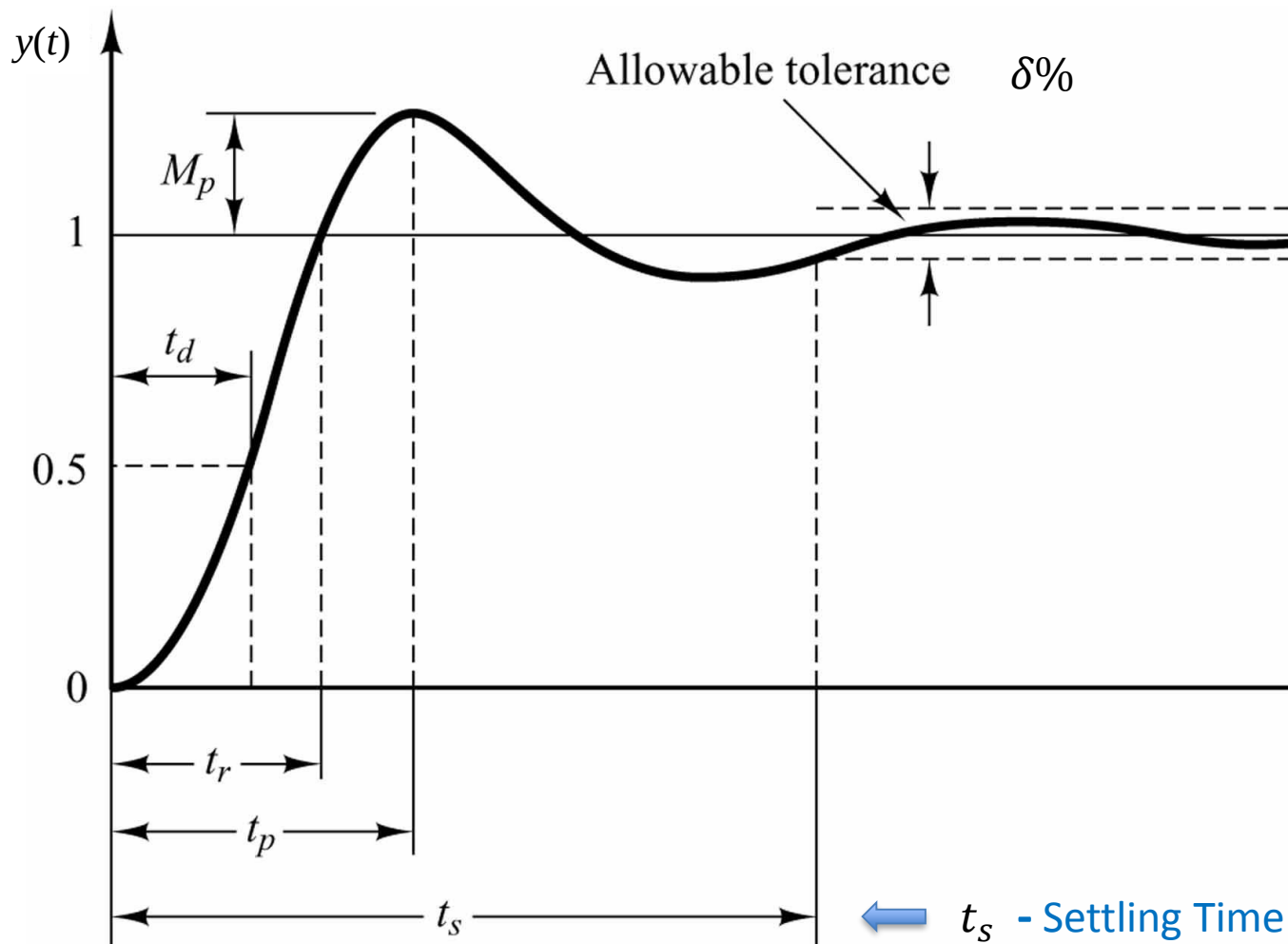
$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$



Trade-off between Response Fastness and Allowable Overshoot



Recall:



Recall:

$$y(t) = 1 - e^{-\zeta\omega_n t} \left[\cos(\omega_n \sqrt{1-\zeta^2} t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t) \right]$$

To estimate **Settling Time** t_s from envelop of $y(t)$:

- Deviation of $y(t)$ from 1 is the product of
 1. Decaying exponential $e^{-\zeta\omega_n t}$
 2. Sinusoidal function (cosine + sine)
- Approximation $\rightarrow e^{-\zeta\omega_n t} \Big|_{t=t_s} = \delta\%$
- For $\delta\% = 1\% = 0.01 \rightarrow \zeta\omega_n t = -\ln 0.01 = 4.6$
- 1% **Settling Time**: $t_s = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{\sigma}$

Sometimes:

- 2% Settling Time: $t_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma}$
- 5% Settling Time: $t_s = \frac{3}{\zeta\omega_n} = \frac{3}{\sigma}$

Note:

- Settling Time is inversely proportional to the product of damping ratio ζ and un-damped natural frequency ω_n
- Since ζ is usually determined from the requirement of maximum overshoot M_p permitted, the settling time is determined primarily by ω_n
- Duration of the *transient period*, i.e., $[0, t_s]$, may be varied by adjusting ω_n without changing M_p
- For a rapid response, ω_n must be large

To estimate **Rise Time** t_r set

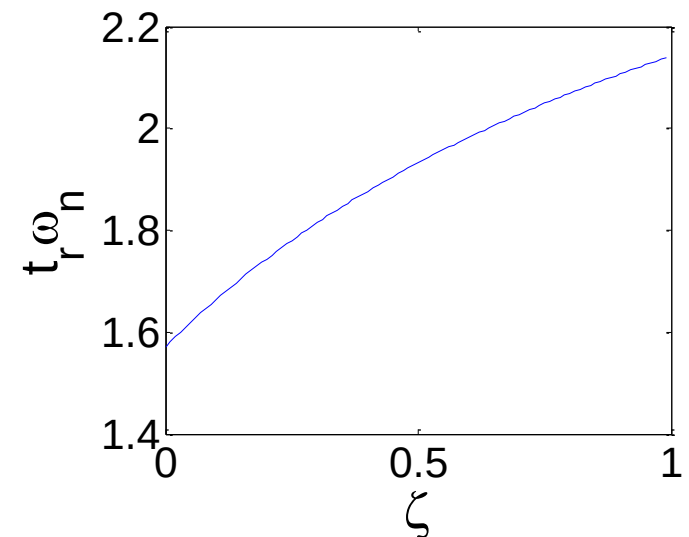
$$y(t_r) = 1 - e^{-\zeta\omega_n t_r} \left[\cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r) \right] = 1$$

Since $e^{-\zeta\omega_n t_r} \neq 0 \rightarrow \cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r) = 0$

$$\rightarrow \tan(\omega_d t_r) = -\frac{\sqrt{1-\zeta^2}}{\zeta} \rightarrow t_r = \frac{\pi - \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n \sqrt{1-\zeta^2}}$$

%MatLab code

```
DR = 0:0.01:1;  
plot(DR, pi - atan(sqrt(1-DR.^2)./DR)./sqrt(1-DR.^2))  
ylabel('t_r\omega_n')  
xlabel('\zeta')
```



To estimate Rise Time:

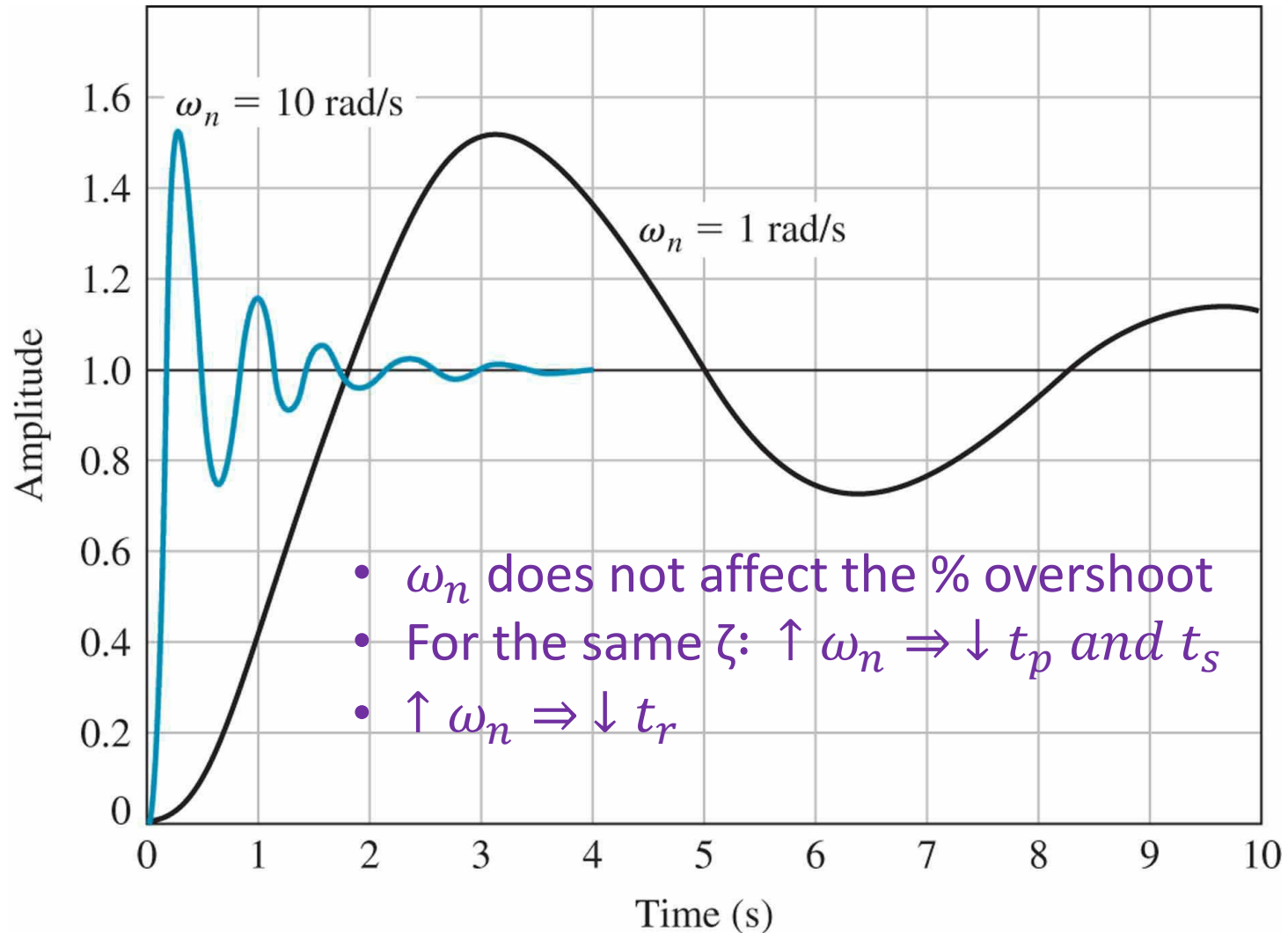
$$t_r = \frac{\pi - \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}}{\omega_n \sqrt{1 - \zeta^2}}$$

Rise Time is typically defined as the time to rise from 10% to 90%

It is approximated as $t_r \cong \frac{1.8}{\omega_n}$

Clearly, to have a small t_r , the ω_n must be large

Step Response: with $\zeta = 0.2$ for $\omega_n = 1$ and $\omega_n = 10$



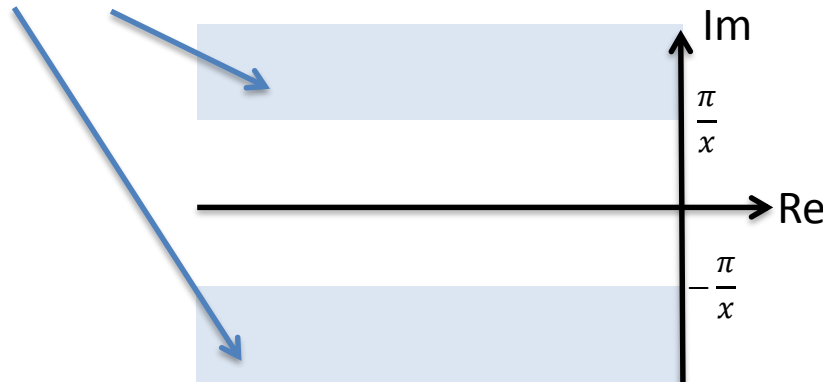
Allowable Region on s-Plane

- **Recall:** $p = -\sigma \pm j\omega_d$ (Cartesian representation)

$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \quad ((\omega_n, \zeta) \text{ representation})$$

- **Peak Time:** $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

- Depending only on *imaginary* part of complex poles
- Specification: $t_p \leq x \rightarrow \omega_n\sqrt{1-\zeta^2} \geq \frac{\pi}{x}$
- Pole pair **Allowable Region** to satisfy $t_p \leq x$



Percent Overshoot

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

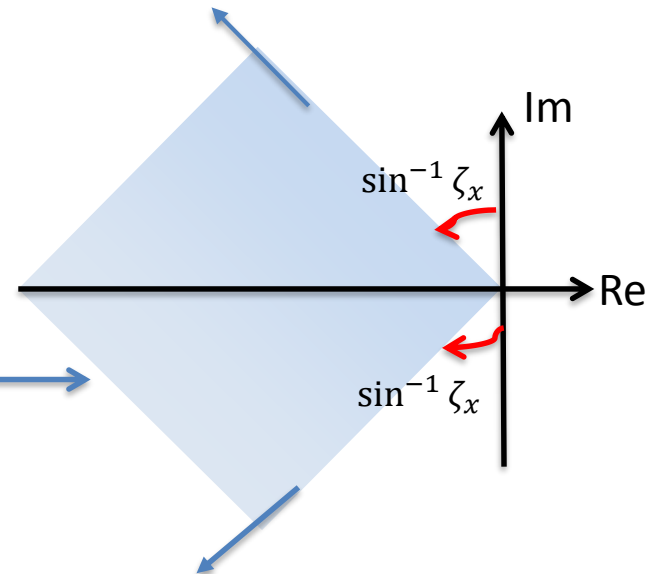
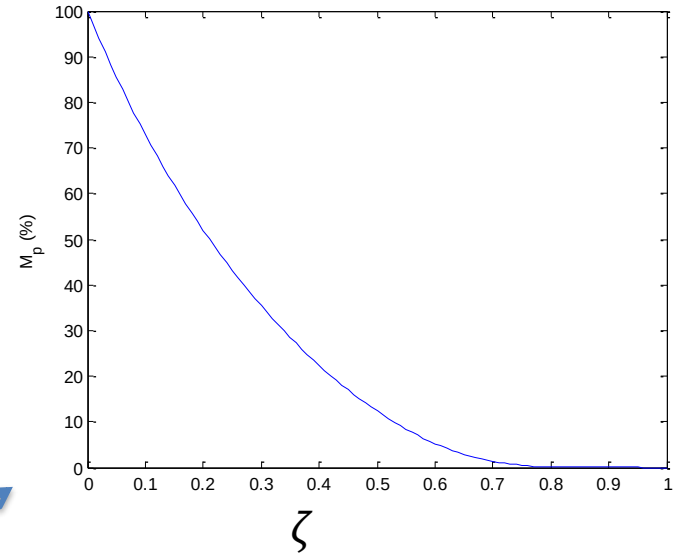
- Depending only on ζ
- Specification: $M_p < x\%$

$$\rightarrow \zeta \geq \zeta_x \rightarrow \theta \geq \sin^{-1} \zeta_x$$

- ζ_x obtained from figure
- Or, by solving equation

$$-\frac{\zeta_x\pi}{\sqrt{1-\zeta_x^2}} < \ln(x\%)$$

- Pole pair **Allowable Region** to satisfy $M_p < x\%$

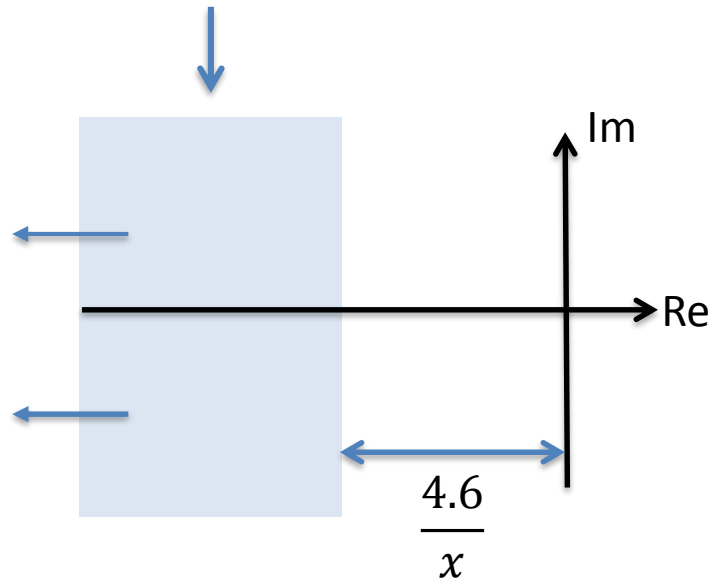


- 1% **Settling Time**:
$$t_s = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{\sigma}$$

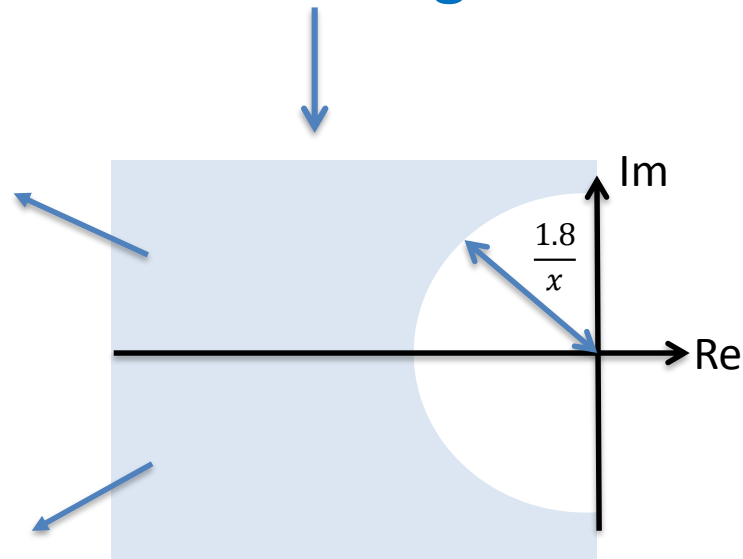
- Depending only on *real* part of complex poles

- Specification: $t_s \leq x \rightarrow \zeta\omega_n \geq \frac{4.6}{x}$

- Pole pair **Allowable Region** to satisfy $t_s \leq x$

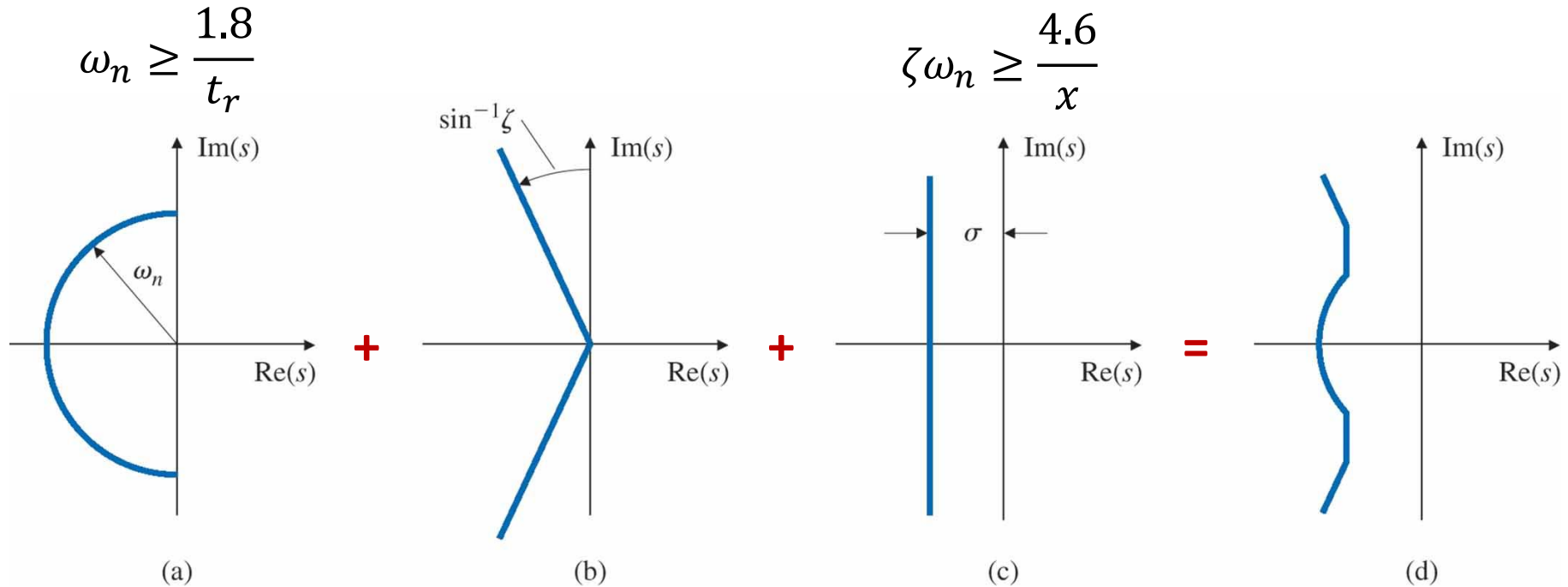


- Rise time $t_r \cong \frac{1.8}{\omega_n}$
 - Depending only on ω_n
 - Specification: $t_r \leq x \rightarrow \omega_n \geq \frac{1.8}{x}$
 - Pole pair **Allowable Region** to satisfy $t_r \leq x$



Allowable Region on s-Plane

(delineated all by transient requirements)



Rise Time

Overshoot

1% Settling Time

Composite of all
3 requirements

Example: Transformation of Specifications to the s-Plane

Find the Allowable Region in the s-plane for the poles of a Transfer Function of a system if its response requirements are:
 $t_r \leq 0.6 \text{ sec}$, $M_p \leq 10\%$ and 1% settling time $t_s \leq 3 \text{ sec}$

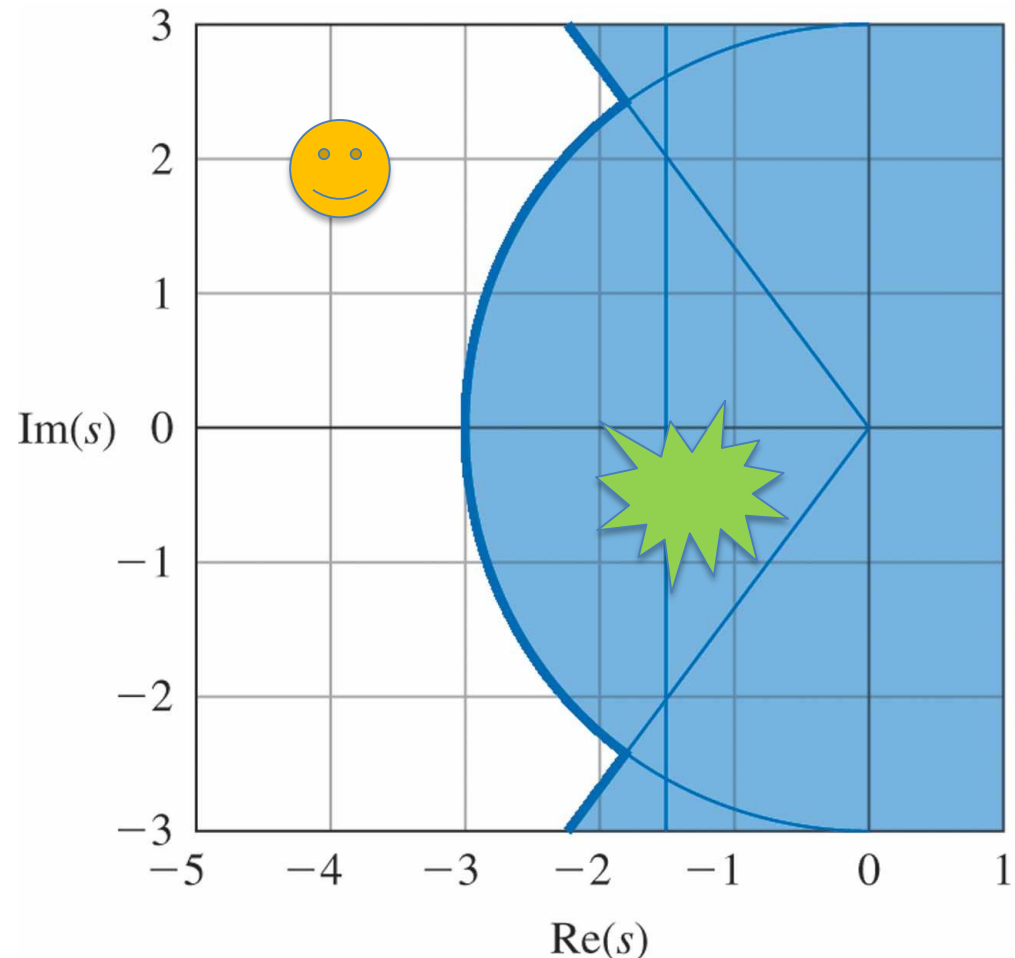
Solution:

$$\omega_n \geq \frac{1.8}{0.6} = 3 \text{ rad/sec}$$

From Figure (M_p versus ζ)

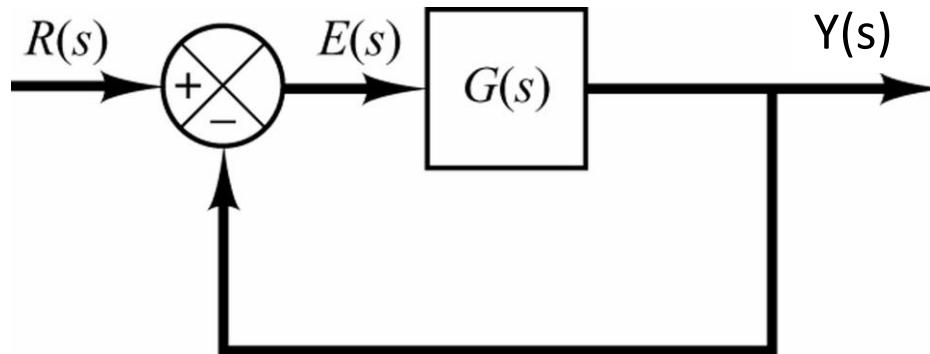
$$\zeta \geq 0.6$$

$$\sigma \geq \frac{4.6}{x} = 1.5 \text{ sec}$$



Steady-State Errors in Unity-Feedback Control Systems

- How well does the system output “follow” the input?
- To obtain some general ideas by applying step, ramp, and parabolic inputs to **unity** feedback system:



$$Y(s) = \frac{G(s)}{1 + G(s)} R(s)$$

First, classify systems into different types

- A *stable* feedback system is of Type- N , if

$$G(s) = \frac{K(T_a s + 1) \cdots (T_m s + 1)}{s^N (T_1 s + 1) \cdots (T_n s + 1)}$$

where T_i are distinct

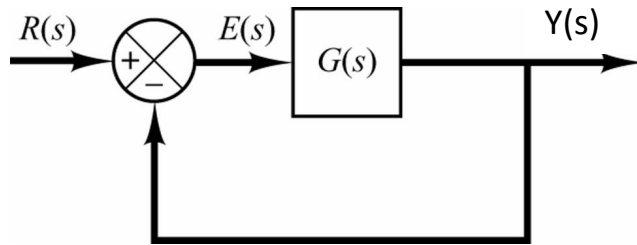
- **Example:**

$$G(s) = \frac{(s+1)(s+8)}{s^2(s+2)(s+4)} \rightarrow \text{Type-2}$$

- As N increases, accuracy is improved but stability is reduced

Stability – To be discussed below

Steady-State Errors



$$Y(s) = \frac{G(s)}{1 + G(s)} R(s)$$

- Consider error $e(t) = r(t) - y(t)$ for tracking performance
($e(t) = 0 \rightarrow$ perfect tracking)

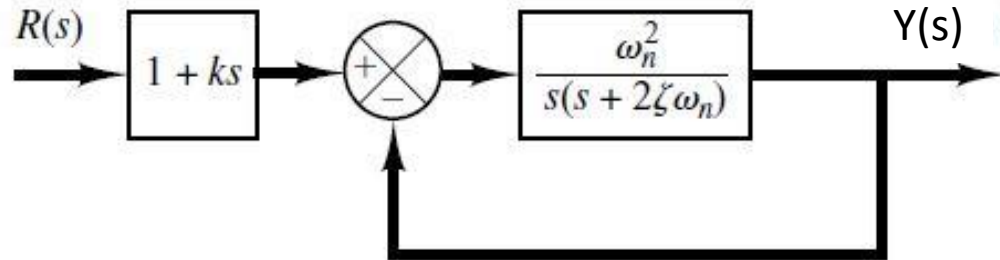
$$E(s) = R(s) - Y(s) = \frac{1}{1 + G(s)} R(s)$$

- Steady-state tracking error (by Final Value Theorem)

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

- Note:** Steady-state error analysis is meaningful only for stable systems (output of unstable systems will blow up)

Example:



If the input is a unit ramp, then the steady-state error is

$$\begin{aligned} e(\infty) &= r(\infty) - c(\infty) \\ &= \lim_{s \rightarrow 0} s \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 k s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \frac{1}{s^2} \\ &= \frac{2\zeta\omega_n - \omega_n^2 k}{\omega_n^2} \end{aligned}$$

Therefore, if k is chosen as

$$k = \frac{2\zeta}{\omega_n}$$

then the steady-state error for following a ramp input can be made equal to zero.]

1. Static Position Error Constant K_p

Static Position Error Constant: $K_p = \lim_{s \rightarrow 0} G(s)$

For step input: $e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1+G(s)} \frac{1}{s} = \frac{1}{1+K_p}$

- For Type-0 systems: $K_p = \lim_{s \rightarrow 0} \frac{K(T_a s + 1) \cdots (T_m s + 1)}{(T_1 s + 1) \cdots (T_n s + 1)} = K$

- For Type-1 or higher systems:

$$K_p = \lim_{s \rightarrow 0} \frac{K(T_a s + 1) \cdots (T_m s + 1)}{s^N (T_1 s + 1) \cdots (T_n s + 1)} \rightarrow \infty \quad (N \geq 1)$$

2. Static Velocity Error Constant K_v

Static Velocity Error Constant: $K_v = \lim_{s \rightarrow 0} sG(s)$

For ramp inputs: $e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1+G(s)} \frac{1}{s^2} = \lim_{s \rightarrow 0} \frac{1}{s+sG(s)} = \frac{1}{K_v}$

- For Type-0 systems: $K_v = \lim_{s \rightarrow 0} \frac{sK(T_a s+1) \cdots (T_m s+1)}{(T_1 s+1) \cdots (T_n s+1)} = 0$
- For Type-1 system: $K_v = \lim_{s \rightarrow 0} \frac{sK(T_a s+1) \cdots (T_m s+1)}{s(T_1 s+1) \cdots (T_n s+1)} = K$
- For Type-2 (or higher) systems:

$$K_p = \lim_{s \rightarrow 0} \frac{sK(T_a s+1) \cdots (T_m s+1)}{s^N (T_1 s+1) \cdots (T_n s+1)} \rightarrow \infty \quad (N \geq 2)$$

3. Static Acceleration Error Constant K_a

Static Acceleration Error Constant: $K_a = \lim_{s \rightarrow 0} s^2 G(s)$

For parabolic input (acceleration input):

$$r(t) = \frac{t^2}{2} \text{ for } t \geq 0$$

Since

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1 + G(s)} \frac{1}{s^3} = \lim_{s \rightarrow 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{K_a}$$

$$K_a = \lim_{s \rightarrow 0} \frac{s^2 K (T_a s + 1) \cdots (T_m s + 1)}{s^N (T_1 s + 1) \cdots (T_n s + 1)} = \begin{cases} 0 & \text{for } N = 0, 1 \\ K & \text{for } N = 2 \\ \infty & \text{for } N \geq 3 \end{cases}$$

Summary

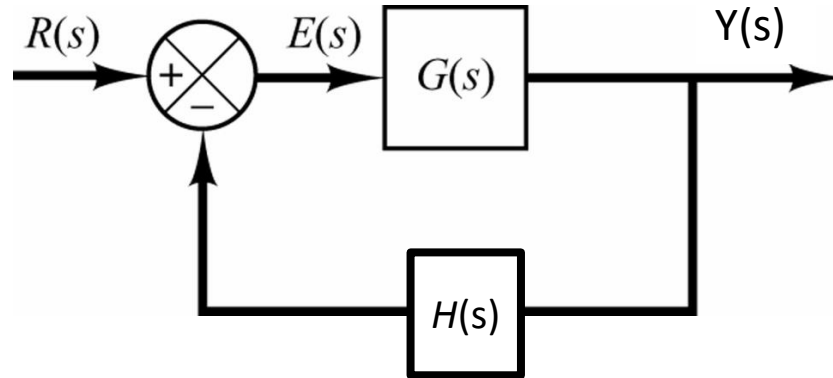
Steady-State Errors for Unity Feedback Systems

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1 + K_p}$	∞	∞
Type 1 system	0	$\frac{1}{K_v}$	∞
Type 2 system	0	0	$\frac{1}{K_a}$
Type 3 system	0	0	0

Higher N

All 0

Unity versus Non-Unity Feedback Systems



Example:

Consider a non-unity feedback system with plant transfer function $G(s) = \frac{1}{s(1+s\tau)}$, τ is constant, and feedback transfer function $H(s) = h$, $h > 0$

Determine the system response to the reference input

Solution:

System tracking error: $e(t) = r(t) - y(t)$

$$E(s) = R(s) - Y(s) = R(s) - \frac{G(s)}{1 + hG(s)} R(s) = \frac{1 + (h - 1)G(s)}{1 + hG(s)} R(s)$$

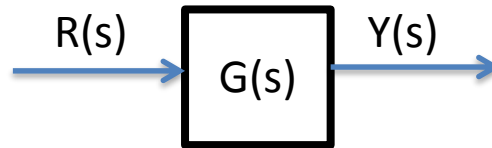
For step input, $R(s) = 1/s$, steady-state error:

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s(1+s\tau)+h-1}{s(1+s\tau)+h} = \frac{h-1}{h} \quad (\tau - \text{constant})$$

- Unity feedback ($h = 1$) $\rightarrow e_{ss} = 0 \rightarrow$ Type-1 system
- Non-unity feedback ($h \neq 1$) $\rightarrow e_{ss} \neq 0 < \infty \rightarrow$ Type-0 system

$\rightarrow$ Non-unity feedback may change the system type

Stability

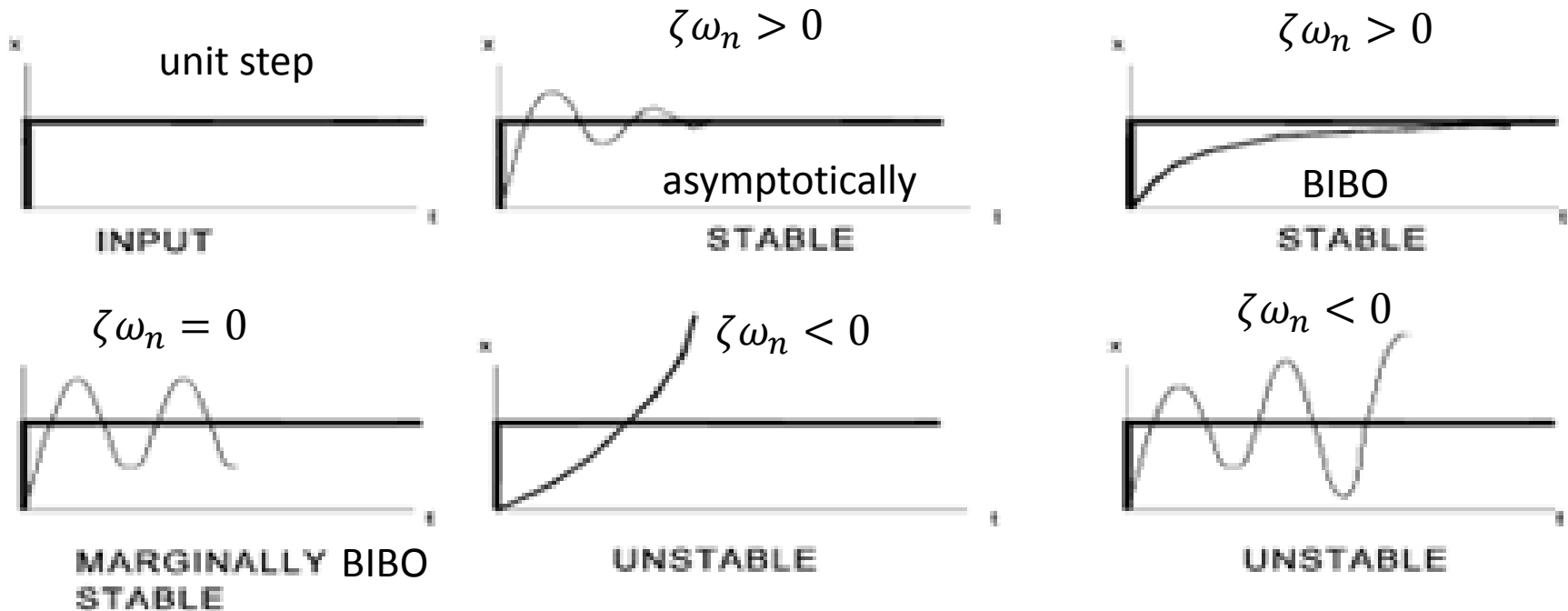


- A *stable* system is a dynamic system with a bounded response to a bounded input, namely, bounded input yields bounded output (BIBO), for all time $t \geq 0$
- An *asymptotically stable* system is a dynamical system with its output converges to zero as time $t \rightarrow \infty$
- Since $Y(s) = G(s)R(s) \rightarrow$
Poles of $Y(s) = \text{Poles of } G(s) + \text{Poles of } R(s)$

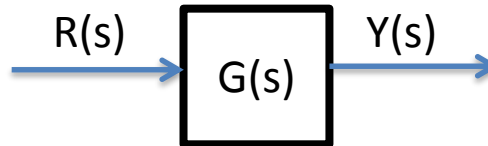
Recall: Step Response of Second-Order LTI systems

$$y(t) = 1 - e^{-\zeta\omega_n t} \left[\cos(\omega_n \sqrt{1-\zeta^2} t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t) \right]$$

Poles: $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$



Stability Condition



For bounded $r(t)$, one requires $G(s)$ poles are located *strictly inside* LHP (not including the boundary) to ensure bounded $y(t)$

System stable if and only if

all poles of $G(s)$ are located inside LHP

(all the poles of $G(s)$ have negative real parts)

- A system is **unstable** if any pole has a positive real part, or if there are any double poles on the imaginary axis
- A system is **marginally stable** if there is at least one root on the imaginary axis with other roots in LHP

Determining System Stability

How to determine if all roots of $D(s)$ that are inside LHP, without going through the great trouble of computing the roots?

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} \cdots + a_{n-1} s + a_n}$$

First Test:

- if some $a_1, a_2, \dots, a_n \leq 0$ then $D(s)$ has roots inside RHP
- **Proof:** For illustration, let $n = 3$ and $D(s)$ has roots p_1, p_2, p_3

$$\begin{aligned} D(s) &= (s - p_1)(s - p_2)(s - p_3) \\ &= s^3 - (p_1 + p_2 + p_3)s^2 + (p_1p_2 + p_2p_3 + p_1p_3)s \\ &\quad - p_1p_2p_3 \end{aligned}$$

So, if p_1, p_2, p_3 are *all inside* LHP, i.e. they have negative real parts, then coefficients of $D(s)$ must be *all positive*

First Test is only sufficient but not necessary:

First Test

→ If *not all* coefficients of $D(s)$ are positive, then *not all* poles are inside LHP, namely, there are some roots in RHP

Example:

$D(s) = s^3 + s^2 + 0.1s - 0.2 \rightarrow D(s)$ has some roots in RHP
(roots: $-0.675 + j 0.341$, $-0.675 - j 0.341$, 0.3497)

If p_1, p_2, p_3 are *all inside* LHP, namely, they have negative real parts, then coefficients of $D(s)$ must be *all positive*.

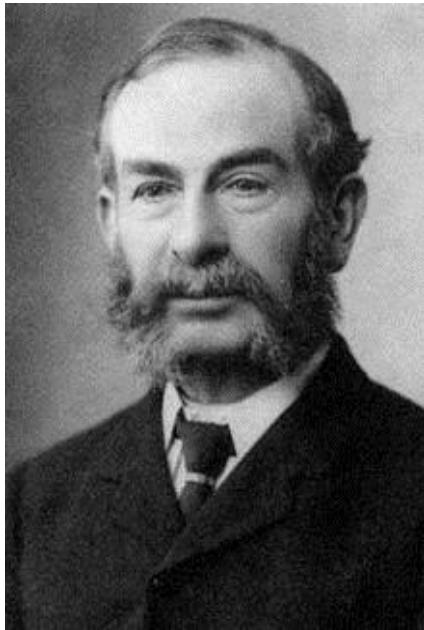
But the *converse may not be true*:

Example:

$D(s) = s^3 + s^2 + 0.1s + 0.2 \rightarrow D(s)$ has some roots in RHP
(roots: $0.0395 + j 0.429$, $0.0395 - j 0.429$, -1.0791)

Routh-Hurwitz Stability Criterion

A necessary and sufficient (if and only if) test



Edward Routh (1831 – 1907)



Adolf Hurwitz (1859-1919)

Recall: Quadratic Polynomial $ax^2 + bx + c$

Viete Theorem:

The roots x_1 and x_2 of $ax^2 + bx + c = 0$ satisfy:

$$\begin{cases} x_1 + x_2 = -\frac{b}{a} \\ x_1 x_2 = \frac{c}{a} \end{cases}$$

Roots and Stability

$x_1 > 0, x_2 > 0$ unstable

$x_1 > 0, x_2 < 0$ unstable

$x_1 < 0, x_2 > 0$ unstable

$x_1 < 0, x_2 < 0$ stable

$$ax^2 + bx + c \rightarrow \text{Viete Thm} \rightarrow \begin{cases} x_1 + x_2 = -\frac{b}{a} \\ x_1 x_2 = \frac{c}{a} \end{cases}$$

Put in Table:

a	c
b	0
$-\frac{\begin{vmatrix} a & c \\ b & 0 \end{vmatrix}}{b}$	0



a	c
b	0
c	0

$$a > 0$$

$$b > 0$$

$$c > 0$$

no sign change $\rightarrow$ Viete Thm $\rightarrow x_1 < 0, x_2 > 0 \rightarrow$ stable

$$a > 0$$

$$b > 0$$

$$c < 0$$

sign changes $\rightarrow$ Viete Thm $\rightarrow x_1 < 0, x_2 > 0 (x_2 < |x_1|)$
or $x_1 > 0, x_2 < 0 (x_1 < |x_2|)$
 $\rightarrow$ unstable

Routh-Hurwitz Stability Criterion

Procedure (Routh Test):

Given:

$$D(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-2} s^2 + a_{n-1} s + a_n$$

Form Array:

(Routh Array)

s^n	a_0	a_2	a_4	$\cdot$	$\cdot$
s^{n-1}	a_1	a_3	a_5	$\cdot$	$\cdot$
s^{n-2}	b_1	b_2	b_3	$\cdot$	$\cdot$
s^{n-3}	c_1	c_2	c_3	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
s^0	z_1	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Given:

$$D(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-2} s^2 + a_{n-1} s + a_n$$

Form the Routh Array:

Calculating:

s^n	a_0	a_2	a_4	.	.
s^{n-1}	a_1	a_3	a_5	.	.
s^{n-2}	b_1	b_2	b_3	.	.
s^{n-3}	c_1	c_2	c_3	.	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
s^0	z_1	.	.	.	.

$$b_1 = -\frac{1}{a_1} \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}; b_2 = -\frac{1}{a_1} \begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}; \dots$$

$$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}; c_2 = -\frac{1}{b_1} \begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}; \dots$$

- Row s^n and s^{n-1} are directly from coefficients of $D(s)$
- s^{n-2} computed from rows s^n and s^{n-1}

$$b_1 = -\frac{1}{a_1} \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}; b_2 = -\frac{1}{a_1} \begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}; \dots$$

- Row s^{n-3} computed from rows s^{n-1} and s^{n-2}

$$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}; c_2 = -\frac{1}{b_1} \begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}; \dots$$

- and so on, until row s^0

Check: first column of the array:

- If all elements are **positive**, then all roots of $D(s)$ are in LHP

→ System is stable

- If some elements are **negative**, then

→ System is unstable

of sign changes in column = # of roots of $D(s)$ in RHP

Example:

Given:

$$D(s) = s^3 + 6s^2 + 8s + 6$$

Solution: Routh Array is

$$\begin{array}{ccc} s^3 & 1 & 8 \\ s^2 & 6 & 6 \\ s^1 & -\frac{1 \times 6 - 8 \times 6}{6} = 7 & 0 \\ s^0 & -\frac{6 \times 0 - 7 \times 6}{7} = 6 & \end{array} \quad \begin{array}{l} b_1 = -\frac{1}{a_1} \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \\ c_1 = -\frac{1}{b_1} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} \end{array}$$

→ $D(s)$ has all roots in LHP → System is stable

Example:

Given:

$$D(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

Solution: Routh array is

s^6 :	1	3	1	4
s^5 :	4	2	4	0
s^4 :	$\frac{5}{2} = \frac{4 \cdot 3 - 1 \cdot 2}{4}$	$0 = \frac{4 \cdot 1 - 4 \cdot 1}{4}$	$4 = \frac{4 \cdot 4 - 1 \cdot 0}{4}$	
s^3 :	$2 = \frac{\frac{5}{2} \cdot 2 - 4 \cdot 0}{\frac{5}{2}}$	$-\frac{12}{5} = \frac{\frac{5}{2} \cdot 4 - 4 \cdot 4}{\frac{5}{2}}$	0	
s^2 :	$3 = \frac{2 \cdot 0 - \frac{5}{2}(-\frac{12}{5})}{2}$	$4 = \frac{2 \cdot 4 - (\frac{5}{2} \cdot 0)}{2}$		
sign change $\rightarrow$	s :	$-\frac{76}{15} = \frac{3(-\frac{12}{5}) - 8}{3}$	0	
$\rightarrow$	s^0 :	$4 = \frac{-\frac{76}{15} \cdot 4 - 0}{-\frac{76}{15}}$		

- $D(s)$ has roots in RHP $\rightarrow$ System is unstable
- There are 2 sign changes $\rightarrow$ There are 2 poles in RHP

% Verified by Matlab

D = [1 4 3 2 1 4 4];

roots(D)

% 6 roots: -0.8858, -3.2644, -0.6046 + 0.9935i, -0.6046 - 0.9935i,

% 0.6797 + 0.7488i, 0.6797 - 0.7488i (these 2 roots are in RHP)

% To compare step responses when $N(s) = 1$ and $N(s) = s+1$, respectively

N1 = 1;

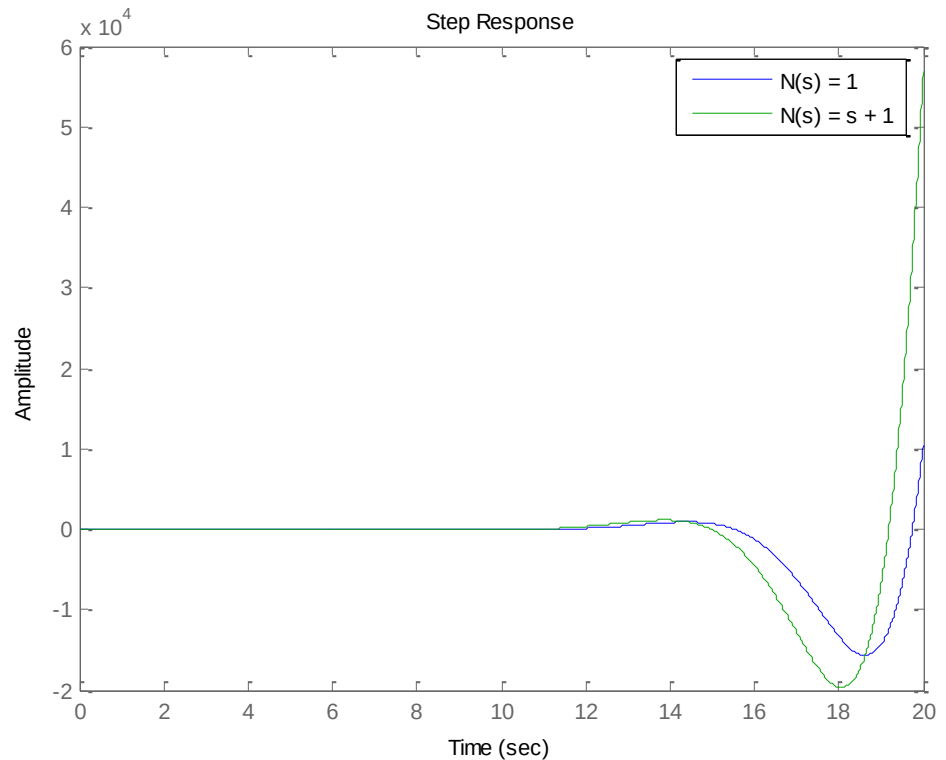
N2 = [1 1];

SYS1 = tf(N1,D)

SYS2 = tf(N2,D)

step(SYS1,SYS2,0:0.01:20)

legend('N(s) = 1','N(s) = s + 1')



Special Case I:

*A row has first element = zero
but not all elements are zero*

- $D(s)$ has roots on imaginary axis or inside RHP
- Procedure:
 1. Replace zero by $\varepsilon > 0$
 2. Complete array as before
 3. Let $\varepsilon \rightarrow 0^+$ and check the first column again:
 - No sign changes $\rightarrow$ some roots on imaginary axis
 - Sign changes $\rightarrow$ some roots in RHP

Example:

$$D(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9$$

Solution: Routh array is

	s^5	1		2	6	
	s^4	3		6	9	
	~~s^3~~	~~0~~		~~3~~	~~0~~	
NEW	s^3	ϵ		3	0	← Replace zero by ϵ
	s^2	$\frac{6\epsilon - 9}{\epsilon}$		9		$\frac{6\epsilon - 9}{\epsilon} \rightarrow -\infty$ as $\epsilon \rightarrow 0^+$
	s	$3 - \frac{3\epsilon^2}{2\epsilon - 3}$		0		
	s^0	9				

First column: 2 sign changes → 2 poles not in LHP

% Validation by Matlab,

D = [1 3 2 6 6 9];

roots(D)

% the roots are -2.9043, -0.7046 + 0.9929i, -0.7046 - 0.9929i

% 0.6567 + 1.2881i, 0.6567 - 1.2881i (2 roots in RHP)

%

% To compare step response when $N(s) = 1$ and $N(s) = s+1$,

N1 = 1;

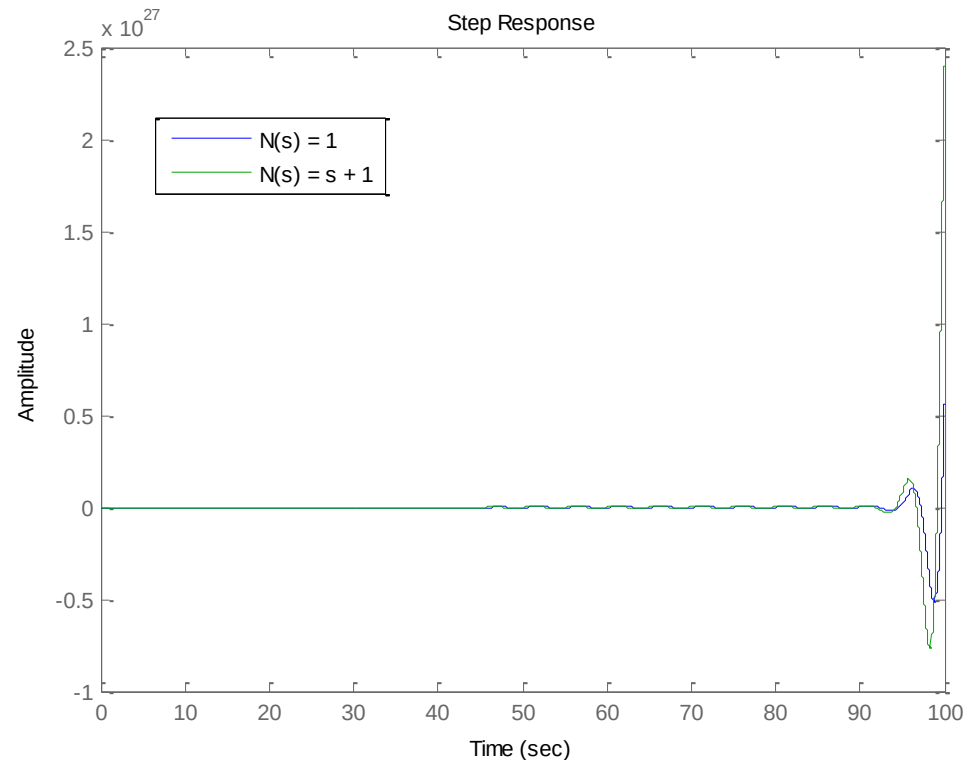
N2 = [1 1];

SYS1 = tf(N1,D)

SYS2 = tf(N2,D)

step(SYS1,SYS2,0:0.001:100)

legend('N(s) = 1','N(s) = s + 1')



Special Case II: *One row has all zero elements*

- $D(s)$ has some roots on imaginary axis or inside RHP
- Procedure:
 1. Form auxiliary polynomial $Q(s)$ with coefficients from previous row
 2. Use coefficients of $\frac{dQ(s)}{ds}$ to replace zero row
 3. Complete array and check first column again:
 - No sign changes $\rightarrow$ some roots on imaginary axis
 - Sign changes $\rightarrow$ some roots in RHP

Note:

- $D(s)$ can be factorized by $Q(s)$: $D(s) = Q(s)\tilde{D}(s)$
- Pair of roots of $Q(s)$ are purely real or purely imaginary

Example:

$$D(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12$$

Solution: Routh array is

s^5	1	11	28	
s^4	5	23	12	
s^3	6.4	25.6	0	
s^2	3	12		$\leftarrow Q(s) = 3s^2 + 12$
~~s~~	~~0~~	~~0~~		
NEW s	6	0		$\leftarrow \frac{dQ(s)}{ds} = 6s$
s^0	12			

No sign changes in first column $\rightarrow$ all roots have negative real parts except for a pair on the imaginary axis: roots of $Q(s) = \pm j2$

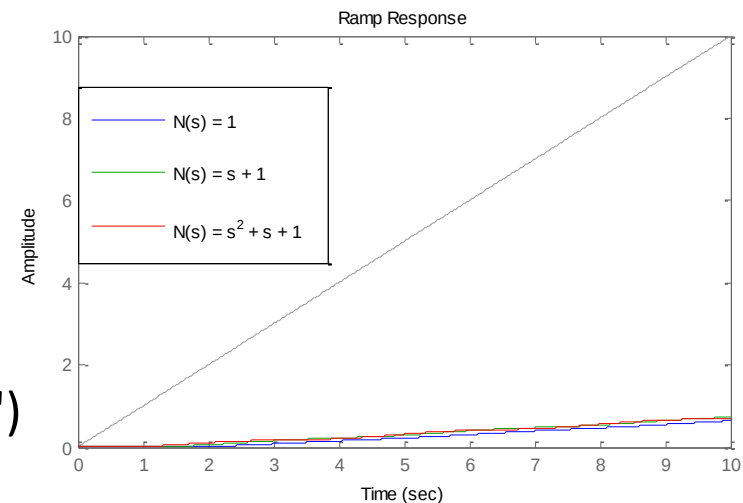
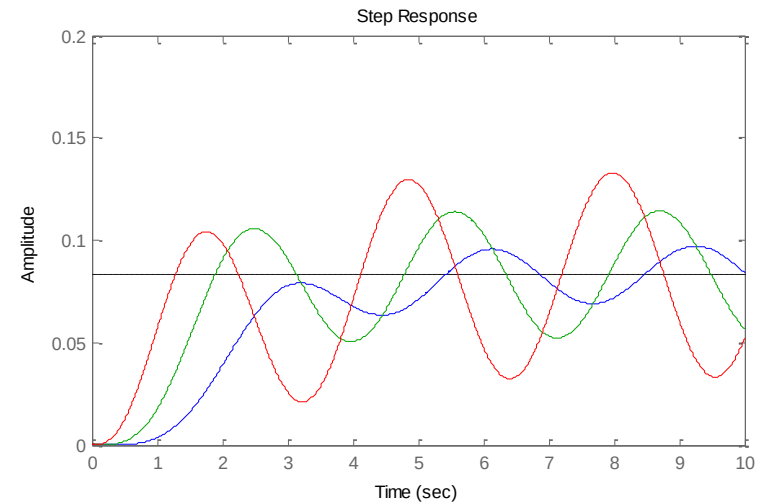
Note: original roots are at $-3, \pm j2, -1, -1$

$$\begin{aligned} D(s) &= Q(s)\tilde{D}(s) = (3s^2 + 12) \left(\frac{s^3}{3} + \frac{5s^2}{3} + \frac{7s}{3} + 1 \right) \\ &= (s^2 + 4)(s^3 + 5s^2 + 7s + 3) = (s + j2)(s - j2)(s + 3)(s + 1)^2 \end{aligned}$$

```

% To compare step and ramp responses when
%  $N(s) = 1$ ,  $N(s) = s+1$ , and  $N(s) = s^2+s+1$ 
D = [1 5 11 23 28 12];
N1 = 1;
N2 = [1 1];
N3 = [1 1 1];
SYS1 = tf(N1,D);
SYS2 = tf(N2,D);
SYS3 = tf(N3,D);
t = 0:0.01:10; % Simulation time
subplot(2,1,1);
step(SYS1,SYS2,SYS3,t)
subplot(2,1,2);
lsim(SYS1,SYS2,SYS3,t,t); % Ramp input u = t
legend('N(s) = 1','N(s) = s + 1', 'N(s) = s^2 + s + 1')
%title('Ramp Response')

```



Example:

$$D(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63,$$

Solution: Routh array is

s^5	1	4	3	
s^4	1	24	63	
s^3	-20	-60	0	
s^2	21	63		$\leftarrow Q(s) = 21s^2 + 63$
~~s~~	~~0~~	~~0~~		
NEW s	42	0		$\leftarrow \frac{dQ(s)}{ds} = 42s$
s^0	63			

- There are 2 sign changes in first column $\rightarrow$ $D(s)$ has 2 roots in RHP

$$Q(s) = 21s^2 + 63 = 21(s + j\sqrt{3})(s - j\sqrt{3})$$

$\rightarrow$ $D(s)$ has 2 imaginary roots $\pm j\sqrt{3}$

Note: the original roots are: $-3, 1 \pm j\sqrt{6}, \pm j\sqrt{3}$

$$\begin{aligned} D(s) &= Q(s)\tilde{D}(s) = (21s^2 + 63) \left(\frac{s^3}{21} + \frac{s^2}{21} + \frac{s}{21} + 1 \right) \\ &= (s^2 + 3)(s^3 + s^2 + s + 21) = (s^2 + 3)(s + 3)(s^2 - 2s + 7) \end{aligned}$$

Example:

$$D(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

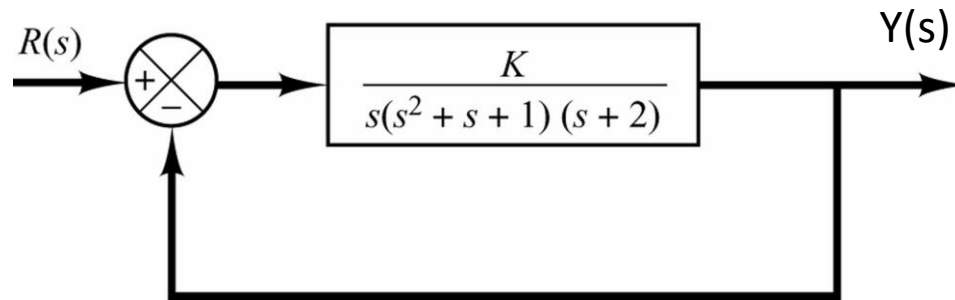
Routh Table

s^5	1	24	-25	
s^4	2	48	-50	
s^3	0	0	0	$Q = 2s^4 + 48s^2 - 50$
s^3	8	96		$Q' = 8s^3 + 96s$
s^2	24	-50		
s^1	112.7	0		
s^0	-50			

One sign change → unstable (one pole on RHP)

$$\begin{aligned} D(s) &= s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 \\ &= (2s^4 + 48s^2 - 50) \left(\frac{s}{2} + 1 \right) \end{aligned}$$

Application to Control System Analysis



Closed-loop Transfer Function: $\frac{Y(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$

Characteristic Equation:

$$D(s) = s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

Routh Array:

s^4	1	3	K
s^3	3	2	0
s^2	$\frac{7}{3}$	K	
s^1	$2 - \frac{9}{7}K$		
s^0	K		

For stability, all coefficients in first column must be positive $\rightarrow 14/9 > K > 0$
(when $K = 14/9$, system is marginally stable, so becomes oscillatory)

```
% Matlab plot
```

```
K = [1 14/9 2];
```

```
t = 0:0.001:50;
```

```
for i = 1:length(K);
```

```
    D0 = conv([1 1 1],[1 2]);
```

```
    SYS1 = tf(K(i),[D0 0]);
```

```
    SYS = feedback(SYS1,1);
```

```
    subplot(length(K),2,(i-1)*2+1);
```

```
    step(SYS,t);
```

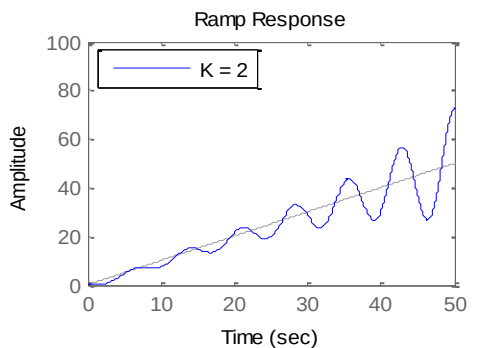
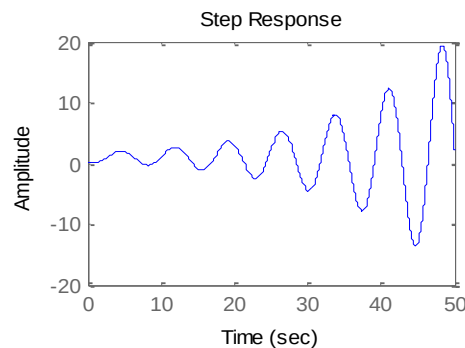
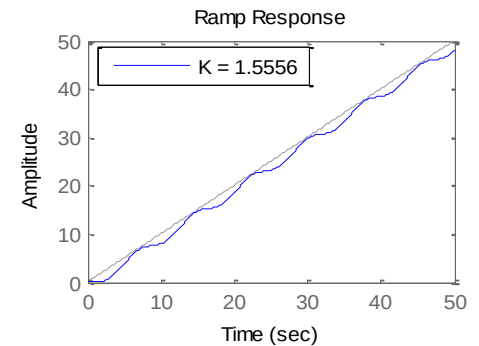
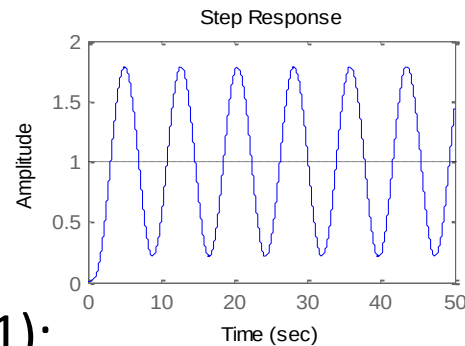
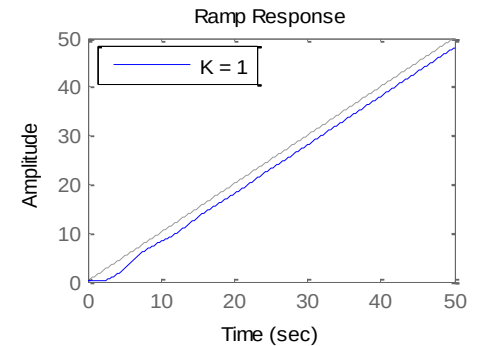
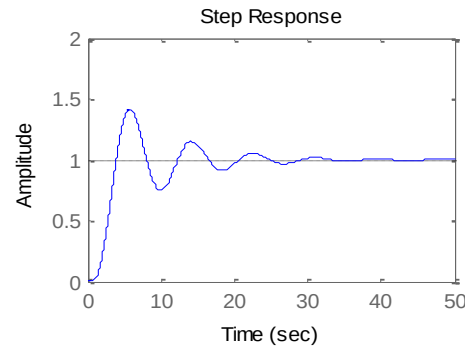
```
    subplot(length(K),2,i*2);
```

```
    lsim(SYS,t,t);
```

```
    %title('Ramp Response');
```

```
    legend(['K = ',num2str(K(i))],'location','NorthWest');
```

```
end
```



Stability Summary

- System type indicates the ability of a system to follow or reject particular time functions with no steady-state errors
- A stable closed-loop system has all poles strictly inside LHP
- A system is stable if and only if all elements in first column of the Routh array are positive

END