

1. $U = [0, \infty)$, $A = [3, 7]$, $B = (5, 9]$, $D = \{1, 2, 3, 4, 5, 6\}$, $E = \{5, 6, 7, 8, 9, 10\}$

(a) $A \cup E = [3, 7] \cup \{8, 9, 10\}$

$$B^c = [0, \infty) \setminus (5, 9] = [0, 5] \cup (9, \infty)$$

$$\therefore (A \cup E) \cap B^c = [3, 5] \cup \{10\}$$

(b) $A^c = [0, \infty) \setminus [3, 7] = [0, 3) \cup (7, \infty)$

$$A^c \cap B = (7, 9]$$

$$(A^c \cap B)^c = [0, \infty) \setminus (7, 9] = [0, 7] \cup (9, \infty)$$

$$\therefore (A^c \cap B)^c \cup E = [0, 7] \cup \{8\} \cup [9, \infty)$$

(c) $A \setminus D = (3, 7] \setminus \{4, 5, 6\} = (3, 4) \cup (4, 5) \cup (5, 6) \cup (6, 7]$

$$B \setminus E = (5, 9] \setminus \{6, 7, 8\} = (5, 6) \cup (6, 7) \cup (7, 8) \cup (8, 9)$$

$$\therefore (A \setminus D) \cap (B \setminus E) = (5, 6) \cup (6, 7)$$

2. (a) $F(x) = \sqrt{\frac{4x+3}{2x-1}}$ is defined when (i) $\frac{4x+3}{2x-1} \geq 0$ and (ii) $2x - 1 \neq 0$.

	$x < -\frac{3}{4}$	$x = -\frac{3}{4}$	$-\frac{3}{4} < x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
Sign of $(4x + 3)$	−	0	+		+
Sign of $(2x - 1)$	−	−	−		+
Sign of $\frac{4x+3}{2x-1}$	+	0	−		+

$$\therefore \text{Dom}(F) = \left(-\infty, -\frac{3}{4}\right] \cup \left(\frac{1}{2}, \infty\right)$$

Let $y = \sqrt{\frac{4x+3}{2x-1}}$. Note that $y \geq 0$ (1).

$$\text{Then } y^2 = \frac{4x+3}{2x-1} \Rightarrow y^2(2x-1) = 4x+3$$

$$\Rightarrow x(2y^2 - 4) = 3 + y^2$$

$$\Rightarrow x = \frac{3+y^2}{2y^2-4}$$

which has real solutions $2y^2 - 4 \neq 0 \Rightarrow y^2 \neq 2 \Rightarrow y \neq \pm\sqrt{2}$ (2).

Combining (1) & (2),

$$\text{Ran}(F) = [0, \infty) \setminus \{\sqrt{2}\} \text{ or written as } [0, \sqrt{2}) \cup (\sqrt{2}, \infty).$$

- (b) $G(x) = \frac{2}{x^2-6x+8}$ is defined when $x^2 - 6x + 8 \neq 0 \Rightarrow (x-2)(x-4) \neq 0 \Rightarrow x \neq 2, 4$.
 $\therefore \text{Dom}(G) = \mathbb{R} \setminus \{2, 4\}$.

Let $y = \frac{2}{x^2-6x+8}$. Note that $y \neq 0 \dots\dots (1)$.

$$\text{Then } y(x^2 - 6x + 8) = 2 \Rightarrow yx^2 - 6yx + 8y - 2 = 0$$

$$\Rightarrow x = \frac{-(-6y) \pm \sqrt{(-6y)^2 - 4y(8y-2)}}{2y}$$

$$\text{which has real solutions } (-6y)^2 - 4y(8y-2) \geq 0 \Rightarrow 4y^2 + 8y \geq 0$$

$$\Rightarrow 4y(y+2) \geq 0$$

$$\Rightarrow y \leq -2 \text{ or } y > 0 \dots\dots (2).$$

Combining (1) & (2),

$$\text{Ran}(F) = (-\infty, -2] \cup (0, \infty).$$

3. Given that $f: [0, \infty) \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$, where

$$f(x) = \sqrt{x} - 1 \text{ and } g(x) = \begin{cases} x+2 & \text{for } x < 1 \\ x^2 & \text{for } x \geq 1 \end{cases}.$$

$$(a) (f \circ f)(x) = f(f(x)) = f(\sqrt{x} - 1) = \sqrt{\sqrt{x} - 1} - 1.$$

$$(f \circ f)(x) \text{ is defined when } \sqrt{x} - 1 \geq 0 \Rightarrow \sqrt{x} \geq 1 \Rightarrow x \geq 1.$$

$$\text{Dom}(f \circ f) = [1, \infty)$$

$$(b) \text{ For } 0 \leq x < 4, f(x) = \sqrt{x} - 1 < 1$$

$$\Rightarrow (g \circ f)(x) = g(f(x)) = g(\sqrt{x} - 1) = (\sqrt{x} - 1) + 2 = \sqrt{x} + 1.$$

$$\text{For } x \geq 4, f(x) = \sqrt{x} - 1 \geq 1$$

$$\Rightarrow (g \circ f)(x) = g(f(x)) = g(\sqrt{x} - 1) = (\sqrt{x} - 1)^2.$$

$$\therefore (g \circ f)(x) = \begin{cases} \sqrt{x} + 1 & \text{if } 0 \leq x < 4 \\ (\sqrt{x} - 1)^2 & \text{if } x \geq 4 \end{cases}$$

$$\text{Dom}(g \circ f) = [0, \infty)$$

$$(c) \text{ For } x < -1, g(x) = x + 2 < 1$$

$$\Rightarrow (g \circ g)(x) = g(g(x)) = g(x + 2) = (x + 2) + 2 = x + 4.$$

$$\text{For } -1 \leq x < 1, g(x) = x + 2 \geq 1$$

$$\Rightarrow (g \circ g)(x) = g(g(x)) = g(x + 2) = (x + 2)^2.$$

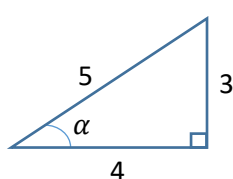
For $x \geq 1$, $g(x) = x^2 \geq 1$

$$\Rightarrow (g \circ g)(x) = g(g(x)) = g(x^2) = (x^2)^2 = x^4.$$

$$\therefore (g \circ g)(x) = \begin{cases} x+4 & \text{if } x < -1 \\ (x+2)^2 & \text{if } -1 \leq x < 1 \\ x^4 & \text{if } x \geq 1 \end{cases}$$

$$\text{Dom}(g \circ g) = \mathbb{R}$$

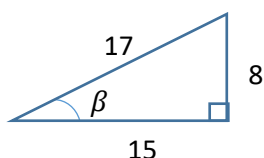
4. $\sin \alpha = -\frac{3}{5}$, where $\pi < \alpha < \frac{3\pi}{2}$ (Quadrant III)



$$\cos \alpha = -\frac{4}{5}$$

S	A
<u>T</u>	C

$\tan \beta = -\frac{8}{15}$, where $\frac{\pi}{2} < \beta < \pi$ (Quadrant II)



$$\sin \beta = \frac{8}{17}$$

$$\cos \beta = -\frac{15}{17}$$

<u>S</u>	A
T	C

$$\csc(\alpha - \beta) = \frac{1}{\sin(\alpha - \beta)} = \frac{1}{\sin \alpha \cos \beta - \cos \alpha \sin \beta} = \frac{1}{\left(-\frac{3}{5}\right)\left(-\frac{15}{17}\right) - \left(-\frac{4}{5}\right)\left(\frac{8}{17}\right)} = \frac{85}{77}$$

$\pi < \alpha < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$ (Quadrant II)

$$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1}{2}(1 + \cos \alpha) = \frac{1}{2}\left[1 + \left(-\frac{4}{5}\right)\right] = \frac{1}{10} \Rightarrow \cos \frac{\alpha}{2} = \pm \frac{1}{\sqrt{10}}$$

$$\therefore \cos \frac{\alpha}{2} = -\frac{1}{\sqrt{10}}$$

<u>S</u>	A
T	C

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1}{2}(1 - \cos \alpha) = \frac{1}{2}\left[1 - \left(-\frac{4}{5}\right)\right] = \frac{9}{10} \Rightarrow \sin \frac{\alpha}{2} = \pm \frac{3}{\sqrt{10}}$$

$$\therefore \sin \frac{\alpha}{2} = \frac{3}{\sqrt{10}}$$

$$\begin{aligned} \cos\left(\frac{\alpha}{2} + \beta\right) &= \cos \frac{\alpha}{2} \cos \beta - \sin \frac{\alpha}{2} \sin \beta = \left(-\frac{1}{\sqrt{10}}\right)\left(-\frac{15}{17}\right) - \left(\frac{3}{\sqrt{10}}\right)\left(\frac{8}{17}\right) = -\frac{9}{17\sqrt{10}} \\ &= -\frac{9\sqrt{10}}{170} \end{aligned}$$

$$\begin{aligned}
5. \quad (a) \quad LHS &= \sin x + 2 \sin 3x + \sin 5x \\
&= (\sin x + \sin 3x) + (\sin 3x + \sin 5x) \\
&= 2 \sin\left(\frac{x+3x}{2}\right) \cos\left(\frac{x-3x}{2}\right) + 2 \sin\left(\frac{3x+5x}{2}\right) \cos\left(\frac{3x-5x}{2}\right), \text{ by sum-to-product formula} \\
&= 2 \sin(2x) \cos(-x) + 2 \sin(4x) \cos(-x) \\
&= 2 \sin(2x) \cos x + 2 \sin(4x) \cos x, \text{ since } \cos x \text{ is an even function} \\
&= 2 \cos x [\sin(2x) + \sin(4x)] \\
&= 2 \cos x \cdot 2 \sin\left(\frac{2x+4x}{2}\right) \cos\left(\frac{2x-4x}{2}\right), \text{ by sum-to-product formula} \\
&= 2 \cos x \cdot 2 \sin(3x) \cos(-x) \\
&= 2 \cos x \cdot 2 \sin(3x) \cos x, \text{ since } \cos x \text{ is an even function} \\
&= 4 \cos^2 x \sin 3x = RHS \\
\therefore \sin x + 2 \sin 3x + \sin 5x &= 4 \cos^2 x \sin 3x.
\end{aligned}$$

$$\begin{aligned}
(b) \quad \sin x + \sin 3x + \sin 5x = 0 &\Rightarrow 4 \cos^2 x \sin 3x - \sin 3x = 0 \quad (\text{from (a)}) \\
&\Rightarrow \sin 3x (4 \cos^2 x - 1) = 0 \\
&\Rightarrow \sin 3x = 0 \quad \text{or} \quad \cos^2 x = \frac{1}{4} \\
&\Rightarrow \sin 3x = 0 \quad \text{or} \quad \frac{1}{2}(1 + \cos 2x) = \frac{1}{4} \\
&\Rightarrow \sin 3x = 0 \quad \text{or} \quad \cos 2x = -\frac{1}{2} \\
\therefore 3x = n\pi + (-1)^n \cdot \sin^{-1}(0) = n\pi &\Rightarrow x = \frac{n\pi}{3} \\
\text{or } 2x = 2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right) = 2n\pi \pm \frac{2\pi}{3} &\Rightarrow x = n\pi \pm \frac{\pi}{3} \\
\therefore x = \frac{n\pi}{3} \quad \text{or} \quad x = n\pi \pm \frac{\pi}{3} &\text{ for } n \in \mathbb{Z}.
\end{aligned}$$

$$6. \quad S = \cos(2x) + \cos(4x) + \cos(6x) + \cdots + \cos(2nx)$$

Multiply both sides by $2 \sin x$:

$$\begin{aligned}
2 \sin x \cdot S &= 2 \cos(2x) \sin x + \cos(4x) \sin x + \cos(6x) \sin x + \cdots + \cos(2nx) \sin x \\
&= (-\sin x + \sin 3x) + (-\sin 3x + \sin 5x) + (-\sin 5x + \sin 7x) + \cdots \\
&\quad + \{-\sin[(2n-1)x] + \sin[(2n+1)x]\} \\
&= -\sin x + \sin[(2n+1)x] \\
\Rightarrow S &= \frac{\sin[(2n+1)x] - \sin x}{2 \sin x} = \frac{2 \cos\left[\frac{(2n+1)x+x}{2}\right] \sin\left[\frac{(2n+1)x-x}{2}\right]}{2 \sin x} \\
&= \frac{\cos[(n+1)x] \sin(nx)}{\sin x}
\end{aligned}$$

7. (a) $q(x) = x^4 + 4x^3 + 4x^2 - 9$

$q(1) = 0 \Rightarrow x - 1$ is a factor of $q(x)$.

$q(-2) = -9 \Rightarrow x + 2$ is not a factor of $q(x)$.

$q(-3) = 0 \Rightarrow x + 3$ is a factor of $q(x)$.

(b) From (a), $(x - 1)$ and $(x + 3)$ are factors of $q(x)$.

$\therefore q(x)$ is divisible by $(x - 1)(x + 3) = x^2 + 2x - 3$.

$$\begin{array}{r}
 \overline{) x^2 + 2x + 3} \\
 x^2 + 2x - 3 \overline{) x^4 + 4x^3 + 4x^2 - 9} \\
 \underline{x^4 + 2x^3 - 3x^2} \\
 2x^3 + 7x^2 \\
 \underline{2x^3 + 4x^2 - 6x} \\
 3x^2 + 6x - 9 \\
 \underline{3x^2 + 6x - 9} \\
 0
 \end{array}$$

By long division, $q(x) = (x^2 + 2x - 3)(x^2 + 2x + 3)$
 $= (x - 1)(x + 3)(x^2 + 2x + 3)$

$$\frac{2x^3 + 8x^2 - 4x + 18}{x^4 + 4x^3 + 4x^2 - 9} = \frac{2x^3 + 8x^2 - 4x + 18}{(x - 1)(x + 3)(x^2 + 2x + 3)} \equiv \frac{A}{x - 1} + \frac{B}{x + 3} + \frac{Cx + D}{x^2 + 2x + 3}$$

$$\begin{aligned}
 \therefore 2x^3 + 8x^2 - 4x + 18 &= A(x + 3)(x^2 + 2x + 3) + B(x - 1)(x^2 + 2x + 3) \\
 &\quad + (Cx + D)(x - 1)(x + 3)
 \end{aligned}$$

Put $x = 1$: $24 = A(4)(6) \Rightarrow A = 1$

Put $x = -3$: $48 = B(-4)(6) \Rightarrow B = -2$

Compare coefficient of x^3 : $2 = A + B + C \Rightarrow C = 3$

Compare constant term: $18 = 9A - 3B - 3D \Rightarrow D = -1$

$$\therefore \frac{2x^3 + 8x^2 - 4x + 18}{x^4 + 4x^3 + 4x^2 - 9} = \frac{1}{x - 1} - \frac{2}{x + 3} + \frac{3x - 1}{x^2 + 2x + 3}$$