

EF2452: Mathematics for Economics and Finance

Lecture 7

Part II of II

Department of Economics and Finance
City University of Hong Kong
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Main Topics

- Using the Lagrange multiplier method to solve the problem of maximizing a consumer's utility subject to the budget constraint
- Price ratio, marginal rate of substitution, demand function, indirect utility

Utility Maximization

- The consumer's optimization problem is to maximize utility subject to a budget constraint:

$$\text{maximize } u(x_1, x_2) \quad \text{subject to } p_1x_1 + p_2x_2 \leq M$$

- x_1 and x_2 are units of good 1 and good 2 consumed by the individual, with market prices being p_1 and p_2 for goods 1 and 2, respectively. The consumer has an income of M to spend on the two goods.
- It is often assumed that “More is Better” so that higher amount of a good is always preferred, the consumer spends all his income on the two goods. So, the budget constraint holds with equality:

$$p_1x_1 + p_2x_2 = M$$

Utility Maximization

- Suppose there are two goods with prices $p_1 = 2$ and $p_2 = 5$, the income is $M = 40$ and the utility function is

$$u(x_1, x_2) = x_1^{1/3} x_2^{1/2}$$

- The budget constraint, therefore, is

$$2x_1 + 5x_2 = 40$$

- The Lagrangean is

$$\mathcal{L} = x_1^{1/3} x_2^{1/2} - \lambda(2x_1 + 5x_2 - 40)$$

Utility Maximization

- The Lagrangean is

$$\mathcal{L} = x_1^{1/3} x_2^{1/2} - \lambda(2x_1 + 5x_2 - 40)$$

- The optimal conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \implies \frac{1}{3}x_1^{-2/3}x_2^{1/2} - 2\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 \implies \frac{1}{2}x_1^{1/3}x_2^{-1/2} - 5\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \implies 2x_1 + 5x_2 - 40 = 0$$

- There are three unknowns: x_1, x_2, λ .

Utility Maximization

- From the first two equations, we get

$$\frac{\frac{1}{3}x_1^{-2/3}x_2^{1/2}}{\frac{1}{2}x_1^{1/3}x_2^{-1/2}} = \frac{2}{5}$$

$$\implies \frac{2x_2}{3x_1} = \frac{2}{5} \implies x_2 = \frac{3}{5}x_1$$

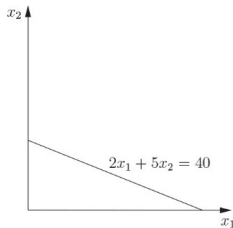
- Substituting the above in the budget constraint, we have

$$2x_1 + 3x_1 = 40 \implies x_1^* = 8$$

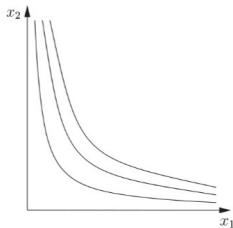
$$x_2^* = \frac{24}{5}, \quad \lambda^* = \frac{1}{6}x_1^{-2/3}x_2^{1/2} = \frac{1}{\sqrt{120}}$$

Tangency Condition

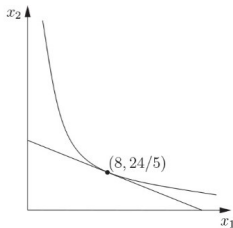
- The budget constraint tells us that we are looking for a solution that lies on the line $2x_1 + 5x_2 = 40$, which is drawn in Fig. (a).
- Fig. (b) shows some indifference curves $u(x_1, x_2) = \text{constant}$. As the constant increases, this curve moves in a 'north-easterly' direction away from the origin.



(a)



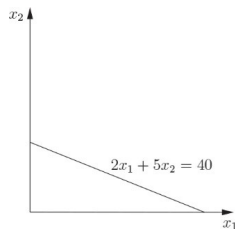
(b)



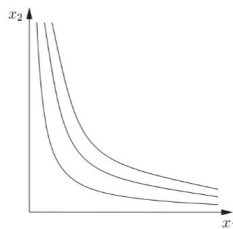
(c)

Tangency Condition

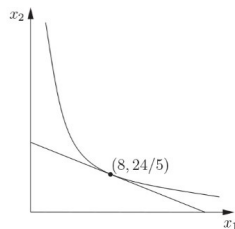
- To maximize u , subject to the budget constraint, we want the curve that is furthest from the origin and which intersects the budget line. This curve will be the one that is tangent to the budget line at the optimum point.



(a)



(b)



(c)

Slope of the budget line: price ratio

- Given a budget line $p_1x_1 + p_2x_2 = M$, differentiating both sides w.r.t. x_1 , we get

$$p_1 \frac{dx_1}{dx_1} + p_2 \frac{dx_2}{dx_1} = 0$$

$$\implies p_1 + p_2 \frac{dx_2}{dx_1} = 0$$

$$\implies \frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$$

- The absolute value of the slope of the budget line $\left(\frac{p_1}{p_2}\right)$ is referred to as the **price ratio**.

Slope of the indifference curve: *MRS*

- Given an indifference curve $u(x_1, x_2) = c$, differentiating both sides w.r.t. x_1 , we get

$$\frac{\partial u}{\partial x_1} \frac{dx_1}{dx_1} + \frac{\partial u}{\partial x_2} \frac{dx_2}{dx_1} = 0$$

$$\implies \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{dx_2}{dx_1} = 0$$

$$\implies \frac{dx_2}{dx_1} = -\frac{\partial u / \partial x_1}{\partial u / \partial x_2}$$

- The absolute value of the slope of the indifference curve at point (x_1, x_2) (given by $\frac{\partial u / \partial x_1}{\partial u / \partial x_2}$) is referred to as the **marginal rate of substitution (MRS)**. It gives the amount of x_2 a consumer is willing to sacrifice if she increases the consumption of x_1 by a unit, keeping herself at the same utility level.

Tangency Condition

- At the optimal point, the slope of the budget line and the indifference curve are equal:

$$-\frac{p_1}{p_2} = -\frac{\partial u / \partial x_1}{\partial u / \partial x_2}$$

- Price ratio equals the marginal rate of substitution at the point where utility is maximized.

Demand Function

- The solution to the consumer's utility maximization problem

$$\text{maximize } u(x_1, x_2) \quad \text{subject to } p_1x_1 + p_2x_2 = M$$

gives the optimal values of x_1^* and x_2^* as a function of prices and income, i.e.,

$$x_1^* = q_1(p_1, p_2, M) \quad x_2^* = q_2(p_1, p_2, M)$$

- The functions $q_1(p_1, p_2, M)$ and $q_2(p_1, p_2, M)$ are the consumer's demand functions. They are also known as the Marshallian demand functions.

Demand Function: Example

- For example, if $u(x_1, x_2) = x_1^{1/3} x_2^{1/2}$, then we have the following problem to solve:

$$\text{maximize } x_1^{1/3} x_2^{1/2} \quad \text{subject to } p_1 x_1 + p_2 x_2 = M$$

- Optimality requires the tangency condition to be satisfied:

$$\frac{p_1}{p_2} = \frac{\frac{1}{3} x_1^{-2/3} x_2^{1/2}}{\frac{1}{2} x_1^{1/3} x_2^{-1/2}} = \frac{2x_2}{3x_1}$$

- Substituting for x_2 in the budget constraint, we obtain the demand functions (i.e., optimal values of x_i s as a function of prices and income)

$$p_1 x_1 + p_2 \left(\frac{3p_1}{2p_2} x_1 \right) = M \implies x_1 = \frac{2M}{5p_1}$$

$$x_2 = \frac{3M}{5p_2}$$

The Indirect Utility Function

- The utility of the optimum bundle (x_1^*, x_2^*) is $u(x_1^*, x_2^*)$.
- Since x_1^* and x_2^* are functions of (p_1, p_2, M) , we can write

$$u(x_1^*, x_2^*) = u(q_1(p_1, p_2, M), q_2(p_1, p_2, M)) = V(p_1, p_2, M)$$

- The function V is called the indirect utility function: it specifies the individual consumer's optimal utility when the prices are p_1 and p_2 , and the income is M .

The Indirect Utility Function: Example

- For the example discussed earlier, we obtained

$$x_1 = q_1(p_1, p_2, M) = \frac{2M}{5p_1}, \quad x_2 = q_2(p_1, p_2, M) = \frac{3M}{5p_2}$$

- The indirect utility function is therefore

$$\begin{aligned} V(p_1, p_2, M) &= u(q_1(p_1, p_2, M), q_2(p_1, p_2, M)) \\ &= u\left(\frac{2M}{5p_1}, \frac{3M}{5p_2}\right) \\ &= \left(\frac{2M}{5p_1}\right)^{1/3} \left(\frac{3M}{5p_2}\right)^{1/2} \\ &= \frac{2^{1/3} 3^{1/2}}{5^{5/6}} \frac{M^{5/6}}{p_1^{1/3} p_2^{1/2}} \end{aligned}$$

References

Following chapters of the textbook:

22 Lagrangeans and the Consumer