

EE3114 Systems and Control

Tutorial - 9

Semester B, 2017-18

Discrete-Time LTI Systems

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t), & \mathbf{x}(0) &= \mathbf{x}_0 \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t)\end{aligned}$$



$$\dot{\mathbf{x}}(t) \approx \frac{\mathbf{x}((k+1)h) - \mathbf{x}(kh)}{h}$$



$$\begin{aligned}\mathbf{x}((k+1)h) &= hA\mathbf{x}(kh) + \mathbf{x}(kh) + hB\mathbf{u}(kh) \\ &= [hA + I]\mathbf{x}(kh) + [hB]\mathbf{u}(kh)\end{aligned}$$



$$\mathbf{x}_{k+1} = \tilde{A}\mathbf{x}_k + \tilde{B}\mathbf{u}_k$$

Easily:

$$\mathbf{y}_k = \tilde{C}\mathbf{x}_k + \tilde{D}\mathbf{u}_k = C\mathbf{x}_k + D\mathbf{u}_k$$

Difference Equations

$$\mathbf{x}_{k+1} = \tilde{A}\mathbf{x}_k + \tilde{B}\mathbf{u}_k$$

In one-dimensional setting, it is a **difference equation**:

$$x_{k+1} - ax_k = bu_k$$

Or, in general,

$$\begin{aligned} & x_{k+n} + a_1x_{k+n-1} + a_2x_{k+n-2} + \cdots + a_{n-1}x_{k+1} + a_nx_k \\ &= b_0u_{k+n} + b_1u_{k+n-1} + \cdots + b_{n-1}u_{k+1} + b_nu_k \end{aligned}$$

Recall: Z - Transform

$$Z\{x_k\} = X(z), \quad Z\{x_{k+n}\} = z^n X(z), \quad Z\{x_{k-n}\} = z^{-n} X(z)$$



$$z^n X(z) + a_1 z^{n-1} X(z) + \cdots + a_{n-1} z X(z) + a_n X(z) = b_0 z^n U(z) + \cdots + b_n U(z)$$



$$\text{Transfer Function: } G(z) = \frac{X(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}$$



$$\dot{x}(t) \approx \frac{x((k+1)h) - x(kh)}{h}$$

Exercise:

Convert the following one-dimensional differential equation to a difference equation:

$$\ddot{x}(t) + 2\dot{x}(t) - x(t) = u(t) \quad (10 \text{ minutes})$$



$$\dot{x}(t) \approx \frac{x((k+1)h) - x(kh)}{h}$$

Exercise:

Convert the following one-dimensional differential equation to a difference equation:

$$\ddot{x}(t) + 2\dot{x}(t) - x(t) = u(t)$$

Solution:

$$\begin{aligned} \ddot{x}(t) &\approx \frac{\dot{x}(t+h) - \dot{x}(t)}{h} \\ &\approx \frac{1}{h} \left[\frac{x((k+2)h) - x((k+1)h)}{h} - \frac{x((k+1)h) - x(kh)}{h} \right] \\ &= \frac{1}{h^2} [x((k+2)h) - 2x((k+1)h) + x(kh)] \\ &= \frac{1}{h^2} [x_{k+2} - 2x_{k+1} + x_k] \end{aligned}$$

Exercise: Consider the following difference equation:

$$v_{k+2} + v_{k+1} + v_k = u_k$$

By defining $\mathbf{x}_k = \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix}$ with $x_k^1 = v_k$ and $x_k^2 = v_{k+1}$,

write this difference equation in the state-space form. (5 minutes)

Exercise: Consider the following difference equation:

$$v_{k+2} + v_{k+1} + v_k = u_k$$

By defining $\mathbf{x}_k = \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix}$ with $x_k^1 = v_k$ and $x_k^2 = v_{k+1}$, write this difference equation in the state-space form.

Solution:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

$$v_k = [1 \quad 0] \mathbf{x}_k$$

Next, consider the following general difference equation:

$$\begin{aligned} &v_{k+n} + a_1 v_{k+n-1} + a_2 v_{k+n-2} + \cdots + a_{n-1} v_{k+1} + a_n v_k \\ &= b_0 u_{k+n} + b_1 u_{k+n-1} + \cdots + b_{n-1} u_{k+1} + b_n u_k \end{aligned}$$

General difference equation:

$$v_{k+n} + a_1 v_{k+n-1} + a_2 v_{k+n-2} + \cdots + a_{n-1} v_{k+1} + a_n v_k \\ = b_0 u_{k+n} + b_1 u_{k+n-1} + \cdots + b_{n-1} u_{k+1} + b_n u_k$$

State-Space Canonical form:

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & 1 & \vdots \\ 0 & & \ddots & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \quad C = [1 \ 0 \ \cdots \ 0], \quad D = [\beta_0]$$

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1 \beta_0$$

$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$$

$$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0$$

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$$\beta_{n-1} = b_{n-1} - a_1 \beta_{n-2} - \cdots - a_{n-2} \beta_1 - a_{n-1} \beta_0$$

$$\beta_n = b_n - a_1 \beta_{n-1} - \cdots - a_{n-1} \beta_1 - a_n \beta_0$$

General difference equation:

$$v_{k+n} + a_1 v_{k+n-1} + a_2 v_{k+n-2} + \cdots + a_{n-1} v_{k+1} + a_n v_k \\ = b_0 u_{k+n} + b_1 u_{k+n-1} + \cdots + b_{n-1} u_{k+1} + b_n u_k$$

State-Space Canonical form:

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & 1 & \vdots \\ 0 & & \ddots & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \quad C = [1 \ 0 \ \cdots \ 0], \quad D = [\beta_0]$$

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1 \beta_0$$

$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$$

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$$\beta_{n-1} = b_{n-1} - a_1 \beta_{n-2} - \cdots - a_{n-2} \beta_1 - a_{n-1} \beta_0$$

$$\beta_n = b_n - a_1 \beta_{n-1} - \cdots - a_{n-1} \beta_1 - a_n \beta_0$$

Exercise:

$$v_{k+3} + 2v_{k+2} - 3v_{k+1} + 10v_k \\ = 2u_{k+1} - u_k$$

Find: A, B, C, D (5 minutes)

General difference equation:

$$v_{k+n} + a_1 v_{k+n-1} + a_2 v_{k+n-2} + \cdots + a_{n-1} v_{k+1} + a_n v_k \\ = b_0 u_{k+n} + b_1 u_{k+n-1} + \cdots + b_{n-1} u_{k+1} + b_n u_k$$

State-Space Canonical form:

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & 1 & \vdots \\ 0 & & \ddots & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \quad C = [1 \ 0 \ \cdots \ 0], \quad D = [\beta_0]$$

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1 \beta_0$$

$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$$

$$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0$$

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$$\beta_{n-1} = b_{n-1} - a_1 \beta_{n-2} - \cdots - a_{n-2} \beta_1 - a_{n-1} \beta_0$$

$$\beta_n = b_n - a_1 \beta_{n-1} - \cdots - a_{n-1} \beta_1 - a_n \beta_0$$

Exercise:

$$v_{k+3} + 2v_{k+2} - 3v_{k+1} + 10v_k \\ = 2u_{k+1} - u_k$$

Solution:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix},$$

$$C = [1 \ 0 \ 0], \quad D = [0]$$

Consider SISO LTI system:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + Bu_k$$

$$y_k = C\mathbf{x}_k + Du_k$$

Z - Transform →

$$zX(z) = AX(z) + BU(z)$$

$$Y(z) = CX(z) + DU(z)$$

Exercise: Find Transfer Function $G_1(z) = \frac{X(z)}{U(z)} = ?$

and Transfer Function $G_2(z) = \frac{Y(z)}{U(z)} = ?$ (5 minutes)

$$zX(z) = AX(z) + BU(z)$$

$$Y(z) = CX(z) + DU(z)$$

Solution:

$$\text{Transfer Function } G_1(z) = \frac{X(z)}{U(z)} = (zI - A)^{-1}B$$

$$\text{Transfer Function } G_2(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D$$



Stability:

Continuous-time LTI systems: Poles are **inside LHP** in s-plane

Discrete-time LTI systems: Poles are **inside unit circle** in z-plane

Recall:

$$v_{k+2} + v_{k+1} + v_k = u_k$$

Defining $\mathbf{x}_k = \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix}$ with $x_k^1 = v_k$ and $x_k^2 = v_{k+1}$



$$\mathbf{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

$$v_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_k$$

Exercise: Is this system stable? (10 minutes)

Solution:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

$$v_k = [1 \ 0] \mathbf{x}_k$$



Transfer Function:

$$G(z) = C[zI - A]^{-1}B = [1 \ 0] \begin{bmatrix} z & -1 \\ 1 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{z^2 + z + 1}$$

Or $v_{k+2} + v_{k+1} + v_k = u_k$

$$\rightarrow z^2 V + zV + V = U \rightarrow \frac{V}{U} = \frac{1}{z^2 + z + 1}$$

$$\text{Poles: } p_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} \rightarrow |p_{1,2}| > 1 \rightarrow \text{unstable}$$

Note: For continuous-time LTI systems, these are **stable poles** !

Consider:

$$X(z) = (zI - A)^{-1}BU(z)$$

$$Y(z) = [C(zI - A)^{-1}B + D]U(z)$$

Inverse z -Transform

(with zero initial conditions)

→

$$\mathbf{x}_k = \{\Phi_k\} * \{\mathbf{u}_k\} = \sum_{i=1}^k \Phi_{k-i} B \mathbf{u}_{i-1} = \sum_{i=1}^k A^{k-i} B \mathbf{u}_{i-1}$$

(convolution)

(with non-zero initial condition $\mathbf{x}_0 \neq 0$)

→

$$\mathbf{x}_k = A^k \mathbf{x}_0 + \sum_{i=1}^k A^{k-i} B \mathbf{u}_{i-1}$$

In fact:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$$



$$\mathbf{x}_1 = A\mathbf{x}_0 + B\mathbf{u}_0$$

$$\mathbf{x}_2 = A\mathbf{x}_1 + B\mathbf{u}_1 = A[A\mathbf{x}_0 + B\mathbf{u}_0] + B\mathbf{u}_1 = A^2\mathbf{x}_0 + (B\mathbf{u}_1 + AB\mathbf{u}_0)$$

$$\mathbf{x}_3 = A\mathbf{x}_2 + B\mathbf{u}_2 = \dots = A^3\mathbf{x}_0 + (B\mathbf{u}_2 + AB\mathbf{u}_1 + A^2B\mathbf{u}_0)$$

.....

$$\mathbf{x}_k = A^k\mathbf{x}_0 + \sum_{i=1}^k A^{k-i}B\mathbf{u}_{i-1}$$

Note:

$$[B \ AB \ A^2B \ \dots \ A^{n-1}B] \begin{bmatrix} \mathbf{u}_{n-1} \\ \vdots \\ \mathbf{u}_1 \\ \mathbf{u}_0 \end{bmatrix}$$



Controllability
condition

%MatLab

co = ctrb (A,B)

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$$

$$\mathbf{y}_k = C\mathbf{x}_k \quad (D = 0)$$

Similarly:

$$\mathbf{y}_0 = C\mathbf{x}_0$$

$$\mathbf{y}_1 = C\mathbf{x}_1 = CA\mathbf{x}_0 + CB\mathbf{u}_0$$

$$\mathbf{y}_2 = C\mathbf{x}_2 = CA^2\mathbf{x}_0 + \dots$$

$$\vdots$$

$$\mathbf{y}_{n-1} = C\mathbf{x}_{n-1} = CA^{n-1}\mathbf{x}_0 + \dots$$



$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \mathbf{x}_0 + \dots$$



Observability
condition

```
%MatLab
ob = obsv (A,C);
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Exercise:

Verify the controllability and observability of the following discrete-time LTI system: (10 minutes)

$$v_{k+3} + 2v_{k+2} - 3v_{k+1} + 10v_k = 2u_{k+1} - u_k$$

$$y_k = v_k$$

Exercise:

Verify the controllability and observability of the following discrete-time LTI system: (10 minutes)

$$v_{k+3} + 2v_{k+2} - 3v_{k+1} + 10v_k = 2u_{k+1} - u_k$$

$$y_k = v_k$$

Solution: Recall

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix}, \quad C = [1 \ 0 \ 0], \quad D = [0]$$



$$[B \ AB \ A^2B] = \begin{bmatrix} 0 & 2 & -5 \\ 2 & -5 & 16 \\ -5 & 16 & -67 \end{bmatrix} \rightarrow \text{Controllable}$$

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Observable}$$

END